stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 191/79
(DEPARTMENT OF APPLIED MATHEMATICS)

I.G. SPRINKHUIZEN-KUYPER

A FRACTIONAL INTEGRAL OPERATOR CORRESPONDING TO NEGATIVE POWERS OF A SECOND ORDER PARTIAL DIFFERENTIAL OPERATOR

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS Subject classification scheme (1980): 26A33, 33A65, 42F10, 44A35

A fractional integral operator corresponding to negative powers of a second order partial differential operator*)

by

Ida Sprinkhuizen-Kuyper

ABSTRACT

A fractional integral operator is derived, for which the partial differential operator $\partial_{xx} - \partial_{yy} - \nu y^{-1}\partial_y$ plays a similar role as the ordinary differentiation d/dx does with respect to the fractional integral of Riemann-Liouville. An application is given to the Koornwinder polynomials (a class of orthogonal polynomials in two variables). Also, an operator is obtained which has the Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$ as its eigenfunctions.

KEY WORDS & PHRASES: *fractional calculus, generalized convolution, Riesz distributions, Koornwinder polynomials, Jacobi functions.*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

This paper deals with the theory of a fractional integral operator I_ν^μ corresponding to the second order partial differential operator

(1.1) $$D_\nu = \partial_{xx} - \partial_{yy} - \nu y^{-1}\partial_y$$

The operator I_ν^μ is a generalization of the fractional integral of Riemann-Liouville. The theory of the fractional integral operator of Riemann-Liouville is given in Section 2, together with an application to Jacobi polynomials.

The idea of fractional integration is the following. Suppose we have a differential operator Q and an initial value problem

(1.2) $$Q^k f(x) = g(x), \qquad x \in \mathbb{R}^n, \; k \in \mathbb{N},$$

with initial conditions on f such that (1.2) has a unique solution, which can be written as

(1.3) $$f(x) = I^k g(x),$$

where

(1.4) $$I^k g(x) = \int K(x,y)g(y)dy,$$

for some kernel K, depending on k. If (1.4) is defined for k in $\mathbb{C}$ (or $\mathbb{R}$), $\mathbb{R}\mathrm{e}\, k > k_0$ and such that

(1.5) $$I^{k_1} I^{k_2} g(x) = I^{k_1+k_2} g(x),$$

and

(1.6) $$Q\, I^{k+1} g(x) = I^k g(x),$$

then the operator I^k, $k \in \mathbb{C}$, is called a fractional integral operator corresponding to the operator Q. Here "fractional" means that k can be *non-integer*, and "integral" refers to the fact that I^k is the inverse of a *differential* operator Q (and that it can be written as an integral for $\mathrm{Re}\, k > k_0$). If g is such that $Q^\ell g$ exists and

$$I^k Q^\ell g = Q^\ell I^k g, \qquad \ell \in \mathbb{N},\ \mathrm{Re}\, k > k_0,$$

then $I^k g$ can be defined for all $k \in \mathbb{C}$:

$$(1.7) \qquad I^k g := I^{k+\ell} Q^\ell g, \qquad \ell \in \mathbb{N},\ \mathrm{Re}\,(k+\ell) > k_0.$$

Here $I^{k+\ell}$ is given in (1.4). The last step is necessary in the cases where Q is a hyperbolic operator (as is D_ν), since in that case (1.4) corresponds to a convergent integral only if $\mathrm{Re}\, k > k_0 > 1$. M. RIESZ [13] used the technique of fractional integration and analytical continuation in order to solve (1.2) with $Q = \Box$, where

$$(1.8) \qquad \Box = \partial_{x_0 x_0} - \partial_{x_1 x_1} - \ldots - \partial_{x_{n-1} x_{n-1}},$$

the wave operator in $\mathbb{R}^n$. In order to solve the problem with $Q = D_\nu$, where D_ν is given by (1.1), we will use the results of M. Riesz if ν is an integer. In that case D_ν corresponds to the wave operator in $\mathbb{R}^{\nu+2}$ acting on functions which are rotation invariant. It will appear that the results obtained in this way also hold for $\nu \in \mathbb{C}$, $\mathrm{Re}\, \nu > -1$.

The origin of the research in this paper is a differential recurrence relation for the Koornwinder polynomials, which are two-variable analogues of the Jacobi polynomials. This differential recurrence relation results in a formula of the form (1.2) with $x \in \mathbb{R}^2$ and $Q = D_\nu$. The problem for the Koornwinder polynomials is sketched in Section 3. The fractional integral operator I_ν^μ, $\nu \in \mathbb{N}$ is found from Riesz's theory in Section 4. In Section 5 a class of distributions and a generalized convolution structure is introduced such that $I_\nu^\mu f$ can be defined as a convolution of the distribution f with a distribution z_μ which is the analogue of the Riesz distribution. Then the fractional integral operator I_ν^μ is defined for $\nu \in \mathbb{C}$, $\mathrm{Re}\, \nu > 0$ (Section 6) and the action of I_ν^μ on the Koornwinder polynomials is given (section 7).

In Section 8 it is shown how the operator I^{μ}_{ν} results in an integral operator $\mathcal{J}^{\mu}_{\nu,\alpha}$ when we consider the action of I^{μ}_{ν} on functions of the type

$$(1.9) \qquad f(x,y) = r^{\alpha}F(t), \qquad x = r\,\mathrm{ch}\,t,\ y = r\,\mathrm{sh}\,t.$$

It is proved that the Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$ are eigenfunctions of $\mathcal{J}^{\mu}_{\nu,\alpha}$. This leads to the action of I^{μ}_{ν} on the James type zonal polynomials, which can be written in the form (1.9) with $F(t)$ a Jacobi function of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$. The expansion of the Koornwinder polynomials in a series of James type zonal polynomials completes the proof of the results in Section 7. The last section, Section 9, is meant to give some additional results. In this section the integral operator I^{μ}_{ν} is explicitly given in the form (1.4). Here the kernel is expressed by two different formulas containing hypergeometric functions. Both formulas hold for a different part of the region of integration. By means of this expression for I^{μ}_{ν} the definition of I^{μ}_{ν} is extended to $\mathrm{Re}\,\nu > -1$.

2. THE FRACTIONAL INTEGRAL OPERATOR OF RIEMANN-LIOUVILLE

The classical theory of fractional integration and its applications can be found in OLDHAM & SPANIER [11]. As a reference for fractional integrals of generalized functions (distributions) see ERDÉLYI [3], GELFAND & SHILOV [7, Ch. I, §3.5] and SPRINKHUIZEN-KUYPER [16, §7 and §8].

DEFINITION 2.1. If f is a continuous function on $\mathbb{R}$ which is zero for $x \le x_0$, and if $\mathrm{Re}\,\mu > 0$, then

$$(2.1) \qquad I^{\mu}f(x) := \begin{cases} \frac{1}{\Gamma(\mu)} \int_{x_0}^{x} f(t)(x-t)^{\mu-1}dt, & x > x_0, \\ 0, & x \le x_0 \end{cases}$$

The operator I^{μ} has the following properties:

(i) $I^{\mu_1}I^{\mu_2}f(x) = I^{\mu_1+\mu_2}f(x)$,

(ii) $\frac{d}{dx} I^{\mu+1}f(x) = I^{\mu}f(x)$,

(iii) $\lim_{\mu \downarrow 0} I^{\mu} f(x) = f(x)$,

(iv) $I^{\mu} f(x)$ is an analytical function of μ for $\mathrm{Re}\,\mu > 0$.

The definition of I^{μ} has a natural extension when we permit f to be a distribution with its support bounded to the left.

Let $\mathcal{D}(\mathbb{R})$ be $C_c^{\infty}(\mathbb{R})$ with the usual topology, and let $\mathcal{D}'(\mathbb{R})$ be its dual consisting of the distributions on $\mathcal{D}(\mathbb{R})$. The subset $\mathcal{D}'_+(\mathbb{R})$ of $\mathcal{D}'(\mathbb{R})$ consists of those distributions which have supports bounded to the left. We will use the notation (f,ϕ) for the action of the distribution f on the test function ϕ; $(\cdot,\cdot)$ is linear in both arguments. Each locally integrable function f will be identified with a distribution (for which we use the same symbol f) by

$$(2.2)\qquad (f,\phi) = \int f(x)\phi(x)dx.$$

Consider the distribution p_{μ} in $\mathcal{D}'_+(\mathbb{R})$ which is defined for $\mathrm{Re}\,\mu > 0$ by

$$(2.3)\qquad p_{\mu}(x) := \frac{1}{\Gamma(\mu)} x_+^{\mu-1},$$

$$(2.4)\qquad x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

and by

$$(2.5)\qquad (p_{\mu},\phi) := (-1)^k (p_{\mu+k},\phi^{(k)}),$$

if $\mathrm{Re}\,\mu > -k$, $\phi \in \mathcal{D}(\mathbb{R})$.

The function $\mu \to (p_{\mu},\phi)$ is an entire analytic function. The convolution of p_{μ} and a distribution f in $\mathcal{D}'_+(\mathbb{R})$ exists because

$$(2.6)\qquad \mathrm{supp}(f) \cap \mathrm{supp}(x \to p_{\mu}(y-x))$$

is bounded for any $y \in \mathbb{R}$. The convolution is defined by

$$(2.7)\qquad (f*p_{\mu},\phi) := (f(x),(p_{\mu}(y),\phi(x+y))).$$

The fractional integral operator I^μ acts on distributions f in $\mathcal{D}'_+(\mathbb{R})$ as

$$(2.8)\qquad I^\mu f := f * p_\mu .$$

<u>LEMMA 2.2</u>. *Let* $f \in \mathcal{D}'_+(\mathbb{R})$ *and* $\mu, \mu_1, \mu_2 \in \mathbb{C}$, *then*

(i) $I^{\mu_1} I^{\mu_2} f = I^{\mu_1+\mu_2} f$,

(ii) $\frac{d}{dx} I^{\mu+1} f = I^{\mu+1} \frac{d}{dx} f = I^\mu f$,

(iii) $I^0 f = f$.

<u>PROOF</u>. If $\mathrm{Re}\,\mu_i > 0$, $i = 1,2$, then

$$(2.9)\qquad p_{\mu_1} * p_{\mu_2}(x) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \int y_+^{\mu_1-1} (x-y)_+^{\mu_2-1}\, dy$$

$$= \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\, x_+^{\mu_1+\mu_2-1} \int_0^1 y^{\mu_1-1} (1-y)^{\mu_2-1}\, dy$$

$$= p_{\mu_1+\mu_2}(x).$$

Hence the first part of the lemma is a consequence of (2.9), the associativity of the convolution and the analytical continuation as given in (2.5).

Similarly the second part of the lemma results from

$$(2.10)\qquad \frac{d}{dx} p_{\mu+1}(x) = p_\mu(x), \qquad \forall \mu \in \mathbb{C},$$

and

$$(2.11)\qquad \frac{d}{dx}(f * p_{\mu+1}) = (\frac{d}{dx} f) * p_{\mu+1} = f * \frac{d}{dx} p_{\mu+1}.$$

In order to prove the third part of the lemma we use (2.7), where ϕ is allowed to be a test function in $\mathcal{D}_-(\mathbb{R})$:

$$\mathcal{D}_-(\mathbb{R}) = \{\psi \in C^\infty \mid \mathrm{supp}(\psi) \subset (-\infty, m] \text{ for some } m \in (-\infty,\infty)\},$$

with the usual topology (see [16]). Then

$$(I^{\mu}f,\phi) = (f * p_{\mu},\phi) = (f,J^{\mu}\phi),$$

where

$$J^{\mu}\phi(x) = (p_{\mu}(y),\phi(x+y)) = (p_{\mu}(y-x),\phi(y)).$$

Hence,

$$J^{\mu}\phi(x) = \begin{cases} \int p_{\mu}(y-x)\phi(y)dy, & \mathrm{Re}\,\mu > 0, \\ (-1)^{k}J^{\mu+k}\phi^{(k)}(x), & \mathrm{Re}\,\mu > -k. \end{cases}$$

If $\mu > 0$, then p_{μ} is a regular distribution corresponding to a positive continuous function. Thus

$$J^{\mu}\phi(x) \equiv 0 \quad \text{for some} \quad \mu > 0 \Rightarrow \phi(x) \equiv 0.$$

Application to $J^{0}\phi - \phi$ and use of the composition property of J^{μ} results in $J^{0}\phi = \phi$ and thus

$$(I^{0}f,\phi) = (f,J^{0}\phi) = (f,\phi), \qquad \forall\phi \in \mathcal{D}_{-}(\mathbb{R}) \supset \mathcal{D}(\mathbb{R}). \qquad \square$$

<u>COROLLARY 2.3</u>. *Let* $g \in \mathcal{D}'_{+}(\mathbb{R})$. *The unique solution* $f \in \mathcal{D}'_{+}(\mathbb{R})$ *of*

$$(2.12) \qquad \left(\frac{d}{dx}\right)^{n} f = g$$

is given by

$$(2.13) \qquad f = I^{n}g.$$

In the theory of special functions we often met relations which can be written as

$$(2.14) \qquad \frac{d}{dx} f_{\alpha+1}(x) = f_{\alpha}(x), \qquad x > 0,$$

where $f_\alpha(x)$ is jointly continuous in x and α, analytical in α, sufficiently often differentiable in x, and

$$(2.15) \qquad f_\alpha(0) = 0, \qquad \text{if } \mathbb{R}\mathrm{e}\,\alpha > \alpha_0,$$

for some $\alpha_0 \in \mathbb{R}$.

If α is such that f_α is locally integrable on $\{x \mid x \geq 0\}$, then the function f_α is identified with a distribution in $\mathcal{D}'_+(\mathbb{R})$ which is zero on $\{x \mid x < 0\}$ (cf. (2.2)). For those values of α (2.14), (2.15) and Corollary 2.3 yield:

$$(2.16) \qquad I^\mu f_\alpha(x) = f_{\alpha+\mu}(x), \qquad x > 0,$$

for $\mu \in \mathbb{N}$.

In many cases (2.16) also holds for $\mu \in \mathbb{C}$, $\mathbb{R}\mathrm{e}\,\mu > 0$. There are two techniques to prove (2.16) for $\mu \in \mathbb{C}$, $\mathbb{R}\mathrm{e}\,\mu > 0$:

(i) Give a direct proof by calculation, or

(ii) Use the theorem of Carlson for analytical continuation (see Theorem 2.4 below).

THEOREM 2.4. (Carlson, cf. TITCHMARSH [19, p. 186]). *Let* $f(z)$ *and* $g(z)$ *be analytical functions for* $\mathbb{R}\mathrm{e}\,z > 0$, *and let* $f(z) = g(z)$ *if* $z = 1,2,3,\ldots$. *If, for some* $M > 0$ *and* $\mu \in (0,\pi)$:

$$|f(z) - g(z)| \leq M e^{\mu|z|}, \qquad \mathbb{R}\mathrm{e}\,z > 0,$$

then

$$f(z) \equiv g(z).$$

An example of a family of functions $\{f_\alpha\}$ is provided by the Jacobi polynomials:

$$(2.17) \qquad R_n^{(\alpha,\beta)}(1-2x) = {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; x).$$

For them the following relation holds:

$$(2.18)\qquad \frac{d}{dx}\,\frac{x^{\alpha}}{\Gamma(\alpha+1)}\,R_n^{(\alpha,\beta)}(1-2x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,R_n^{(\alpha-1,\beta+1)}(1-2x),$$

and indeed

$$(2.19)\qquad I^{\mu}\,\frac{x^{\alpha}}{\Gamma(\alpha+1)}\,R_n^{(\alpha,\beta)}(1-2x) = \frac{x^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}\,R_n^{(\alpha+\mu,\beta-\mu)}(1-2x),$$

$\mathbb{R}\mathrm{e}\,\alpha > -1$, $\mathbb{R}\mathrm{e}\,\mu > 0$. See ASKEY [1, (3.6)] and ERDÉLYI et al. [4, 2.8 (22)].

Formula (2.19) with $\mu > 0$ contains an expression for Jacobi polynomials of order (α,β) with $\alpha > \beta$ as a convolution of a positive function and a Gegenbauer polynomial (= Jacobi polynomials with equal parameters). Hence a number of properties for the Jacobi polynomials such as

$$(2.20)\qquad |R_n^{(\alpha,\beta)}(x)| \le R_n^{(\alpha,\beta)}(1) = 1, \qquad -1 \le x \le 1,$$
$$\alpha \ge \beta \wedge \alpha+\beta \ge -1 \wedge \beta > -1,$$

can be derived from the corresponding properties for the Gegenbauer polynomials.

3. FORMULATION OF THE PROBLEM

The motivation for writing the present paper and the earlier paper [16] came from the following problem. In KOORNWINDER & SPRINKHUIZEN-KUYPER [9] we studied a class $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ of orthogonal polynomials in two variables. Let us call them Koornwinder polynomials, following K. RINGHOFER [12]. In [9, (6.20)] we obtained the differentiation formula

$$(3.1)\qquad D_-^{\gamma}\,\eta^{\alpha} R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \frac{1}{4}\,\alpha(\alpha+\gamma+\tfrac{1}{2})\,\eta^{\alpha-1} R_{n,k}^{\alpha-1,\beta+1,\gamma}(\xi,\eta),$$

where, after a change of variables

$$(3.2)\qquad (\xi,\eta) \to (x,y), \quad \xi = x, \quad \eta = \tfrac{1}{4}(x^2-y^2),$$

the partial differential operator D_-^{γ} equals

$$(3.3)\qquad D_-^{\gamma} = \frac{1}{4}\,D_{2\gamma+1}\ ,$$

with D_ν given by (1.1). It is important to have a formula corresponding to (3.1) in a similar way as (2.19) corresponds to (2.18), because such a formula may yield an analogue for the Koornwinder polynomials of inequality (2.20). Formula (3.1) can be written as

(3.4) $$D_\nu f_{\alpha+1}(x,y) = f_\alpha(x,y), \qquad x > y \geq 0,$$

(3.5) $$f_\alpha(x,x) = 0 \qquad \text{if } \mathrm{Re}\,\alpha > 0.$$

Comparison of (3.4) and (3.5) with (2.14) and (2.15) leads to the problem of deriving a fractional integral operator I_ν^μ which acts on suitable f such that

(3.6) $$I_\nu^{\mu_1} I_\nu^{\mu_2} f = I_\nu^{\mu_1+\mu_2} f,$$

(3.7) $$D_\nu I_\nu^{\mu+2} f = I_\nu^{\mu+2} D_\nu f = I_\nu^\mu f,$$

(3.8) $$I^0 f = f.$$

4. SOLUTION OF THE PROBLEM FOR ν IN $\mathbb{N}$

Let $\nu \in \mathbb{N}$. We will solve the problem stated in Section 3 by considering functions $\tilde{f}$ on $\mathbb{R}^{\nu+2}$ which are invariant under rotations around the x_0-axis. These functions $\tilde{f}$ correspond to functions f on

(4.1) $$\Omega := \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}:$$

(4.2) $$\tilde{f}(x_0, x_1, \ldots, x_{\nu+1}) = f(x,y),$$

where $x = x_0$ and $y = \sqrt{x_1^2 + x_2^2 + \ldots + x_{\nu+1}^2}$. The wave operator $\Box$ in $\mathbb{R}^{\nu+2}$ equals

(4.3) $$\Box := \partial_{x_0 x_0} - \partial_{x_1 x_1} - \ldots - \partial_{x_{\nu+1} x_{\nu+1}}.$$

It is easily checked that for functions f and $\tilde{f}$ corresponding to each other as in (4.2) the wave operator $\Box$ corresponds to D_ν:

$$\square\tilde{f}(x_0,x_1,\ldots,x_{\nu+1}) = D_\nu f(x,y). \tag{4.4}$$

Thus the initial value problem

$$D_\nu^k f(x,y) = g(x,y) \qquad \text{in } \Omega, \tag{4.5}$$

with initial conditions for f on a curve S in Ω, corresponds to

$$\square^k \tilde{f} = \tilde{g}, \qquad \text{in } \mathbb{R}^{\nu+2}, \tag{4.6}$$

with initial conditions for $\tilde{f}$ on the hypersurface $\tilde{S}$ corresponding to S.

The initial value problem (4.6) was solved by M. RIESZ [13] in terms of a fractional integral operator. The action of the fractional integral operator on distributions can be found in L. SCHWARTZ [15] or DE JAGER [8]. See also [16] for a related problem. In the following we give a sketch of the theory. Let P be a point in $\mathbb{R}^{\nu+2}$ with coordinates $(x_0^P,\ldots,x_{\nu+1}^P)$. The forward light cone $\tilde{L}_+(P)$ with its vertex in P is:

$$\tilde{L}_+(P) := \{(x_0,\ldots,x_{\nu+1}) \in \mathbb{R}^{\nu+1} \mid x_0 \geq x_0^P \wedge (x_0-x_0^P)^2-(x_1-x_1^P)^2-\ldots-(x_{\nu+1}-x_{\nu+1}^P)^2 \geq 0\}, \tag{4.7}$$

and similarly the backward light cone is given by:

$$\tilde{L}_-(P) := \{(x_0,\ldots,x_{\nu+1}) \in \mathbb{R}^{\nu+1} \mid x_0 \leq x_0^P \wedge (x_0-x_0^P)^2-(x_1-x_1^P)^2-\ldots-(x_{\nu+1}-x_{\nu+1}^P)^2 \geq 0\}. \tag{4.8}$$

We will use the notation $\tilde{L}_+$ and $\tilde{L}_-$ for a light cone with its vertex in the origin. Let f be a continuous function on $\mathbb{R}^{\nu+1}$. A hypersurface

$$\tilde{S} = \{(x_0,x_1,\ldots,x_{\nu+1}) \in \mathbb{R}^{\nu+2} \mid x_0 = f(x_1,\ldots,x_{\nu+1})\}$$

is called *admissible* if for any point P in $\tilde{S}$

$$\tilde{L}_+(P) \cap \tilde{S} = \tilde{L}_-(P) \cap \tilde{S} = \{P\}.$$

For an admissible hypersurface $\tilde{S}$ we define

$$(4.9) \qquad \tilde{S}_+ := \bigcup_{P \in \tilde{S}} \tilde{L}_+(P),$$

$$(4.10) \qquad \tilde{S}_- := \bigcup_{P \in \tilde{S}} \tilde{L}_-(P).$$

Hence $\tilde{S}_+ \cap \tilde{S}_- = \tilde{S}$. For each admissible hypersurface $\tilde{S}$ there exists an open neighbourhood $\tilde{K}$ of $\tilde{S}$ in $\mathbb{R}^{\nu+2}$ such that

(i) if $P \in \tilde{K}$ then $\tilde{S}_+ \cap \tilde{L}_-(P)$ is bounded;

(ii) if $P \in \tilde{K}$ then $\tilde{L}_-(P) \subset \tilde{K}$;

(iii) $\tilde{S}_- \subset \tilde{K}$.

If $\tilde{S}$ is a smooth admissible hypersurface, and if its normal everywhere makes an angle less than $\frac{1}{4}\pi - \delta$ (for some $\delta > 0$) with the x_0-axis, then we can choose $\tilde{K} := \mathbb{R}^{\nu+2}$. However, if this angle tends to $\frac{1}{4}\pi$, it may be necessary to take $\tilde{K}$ as a proper subset of $\mathbb{R}^{\nu+2}$. Let

$$\mathcal{D}(\tilde{K}) = C_c^\infty(\tilde{K})$$

with the usual topology ($\mathcal{D}(\tilde{K})$) is the inductive limit of spaces $\mathcal{D}_M(\tilde{K})$, M a compact subset of $\tilde{K}$, cf. [16, §7] and $\mathcal{D}'(\tilde{K})$ its dual. We will use the following spaces of distributions:

$$(4.11) \qquad \mathcal{D}'_+(\tilde{K}) := \{\tilde{f} \in \mathcal{D}'(\tilde{K}) \mid \operatorname{supp}(\tilde{f}) \subset \tilde{S}_+\},$$

$$(4.12) \qquad \mathcal{F}'_+ := \{\tilde{f} \in \mathcal{D}'(\mathbb{R}^{\nu+2}) \mid \operatorname{supp}(\tilde{f}) \subset \tilde{L}_+\}.$$

If $\tilde{f}$ and $\tilde{g}$ are both in $\mathcal{F}'_+$ then the convolution product $\tilde{f} * \tilde{g}$ exists as a distribution in $\mathcal{F}'_+$. If $\tilde{f}$ is in $\mathcal{F}'_+$ and $\tilde{g}$ in $\mathcal{D}'_+(\tilde{K})$ then $\tilde{f} * \tilde{g}$ exists in $\mathcal{D}'_+(\tilde{K})$. Both convolution products exist, since

$$\operatorname{supp}(\tilde{f}) \cap \operatorname{supp}((x_0,\ldots,x_{\nu+1}) \to \tilde{g}(y_0-x_0,\ldots,y_{\nu+1}-x_{\nu+1}))$$

is bounded for $(y_0,\ldots,y_{\nu+1}) \in \mathbb{R}^{\nu+2}$ or $\tilde{K}$, respectively. The convolution product of an arbitrary number of distributions in $\mathcal{F}'_+$ and exactly one distribution in $\mathcal{D}'_+(\tilde{K})$ is a distribution in $\mathcal{D}'_+(\tilde{K})$. Furthermore this convolution product is commutative and associative. Locally integrable functions on $\mathbb{R}^{\nu+2}$ or $\tilde{K}$ are identified with distributions:

(4.13) $$(\tilde{f},\tilde{\phi}) = \int \tilde{f}(x_0,\ldots,x_{\nu+1})\tilde{\phi}(x_0,\ldots,x_{\nu+1})\,dx_0\ldots dx_{\nu+1},$$

where the $\tilde{f}$ on the left hand side is a distribution and the $\tilde{f}$ on the right hand side is a locally integrable function. In both cases $\tilde{\phi}$ is a test function in $\mathcal{D}(\mathbb{R}^{\nu+2})$ or $\mathcal{D}(\tilde{K})$. Note that $\mathcal{D}(\tilde{K}) \subset \mathcal{D}(\mathbb{R}^{\nu+2})$.

The Riesz distribution Z_μ is a distribution in $\mathcal{F}'_+$ and is defined by

(4.14) $$Z_\mu(x_0,\ldots,x_{\nu+1}) = [\pi^{\frac{1}{2}\nu}2^{\mu-1}\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-\nu))]^{-1}\tilde{\rho}^{\mu-\nu-2},$$

where

(4.15) $$\tilde{\rho} = \begin{cases}(x_0^2 - x_1^2 - \ldots - x_{\nu+1}^2)^{\frac{1}{2}}, & \text{if } x_0 > (x_1^2 + \ldots + x_{\nu+1}^2)^{\frac{1}{2}},\\ 0, & \text{elsewhere,}\end{cases}$$

for $\mathbb{R}\mathrm{e}\,\mu > \nu$, and by

(4.16) $$(Z_\mu,\tilde{\phi}) = (Z_{\mu+2k},\Box^k\tilde{\phi}),$$

for $\mathbb{R}\mathrm{e}\,\mu > \nu - 2k$, $k \in \mathbb{N}$, $\phi \in \mathcal{D}(\mathbb{R}^{\nu+2})$.

The Riesz distributions have the following properties:

(4.17) $$Z_{\mu_1} * Z_{\mu_2} = Z_{\mu_1+\mu_2},$$

(4.18) $$\Box\, Z_{\mu+2} = Z_\mu,$$

(4.19) $$Z_0 = \delta,$$

$(Z_\mu,\tilde{\phi})$ is an entire function of μ for any $\tilde{\phi}$ in $\mathcal{D}(\mathbb{R}^{\nu+2})$ and Z_μ is a regular distribution (locally integrable function) for $\mathbb{R}\mathrm{e}\,\mu > \nu$. Within $\mathcal{D}'_+(\tilde{K})$ the differential equation (4.6) has the unique solution:

(4.20) $$\tilde{f} = \tilde{g} * Z_{2k} ,$$

where Z_{2k} is a Riesz distribution. The analogue for the wave operator of the fractional integral of Riemann and Liouville is defined for $\tilde{f}$ in $\mathcal{D}'_+(\tilde{K})$ by:

(4.21) $$\tilde{I}^{\mu}_{\nu}\tilde{f} = \tilde{f} * Z_{\mu}.$$

The operator $\tilde{I}^{\mu}_{\nu}$ has the properties:

(4.22) $$\tilde{I}^{\mu_1}_{\nu}\tilde{I}^{\mu_2}_{\nu}\tilde{f} = \tilde{I}^{\mu_1+\mu_2}_{\nu}\tilde{f},$$

(4.23) $$\Box\, \tilde{I}^{\mu+2}_{\nu}\tilde{f} = \tilde{I}^{\mu+2}_{\nu}\, \Box\, \tilde{f} = \tilde{I}^{\mu}_{\nu}\tilde{f},$$

(4.24) $$\tilde{I}^{0}_{\nu}\tilde{f} = \tilde{f}.$$

They follow immediately from (4.17), (4.18), (4.19), (4.21) and the associativity of the convolution product.

Until now $\tilde{S}$ was allowed to be an arbitrary admissible hypersurface and $\tilde{K}$ a corresponding neighbourhood as given after (4.10). In order to return to our original problem (4.5) we will suppose $\tilde{S}$ to be rotation invariant and then $\tilde{K}$ can also be chosen to be rotation invariant. If $\tilde{f}$ is a rotation invariant distribution in $\mathcal{D}'_+(\tilde{K})$ then $\tilde{I}^{\mu}_{\nu}\tilde{f}$ will have the same property, and (4.21) can be expressed in terms of the coordinates on Ω.

The rotation invariant (r.i.) test functions in $\mathcal{D}(\mathbb{R}^{\nu+2})$ form a closed subspace $\mathcal{D}_0(\mathbb{R}^{\nu+2})$. Similarly, let $\mathcal{D}_0(\mathbb{R}^2)$ consist of all functions in $\mathcal{D}(\mathbb{R}^2)$ which are even in the second variable. The spaces $\mathcal{D}_0(\mathbb{R}^{\nu+2})$ and $\mathcal{D}_0(\mathbb{R}^2)$ are isomorphic. The dual of $\mathcal{D}_0(\mathbb{R}^{\nu+2})$ is $\mathcal{D}'_0(\mathbb{R}^{\nu+2})$ and it consists of all r.i. distributions in $\mathcal{D}'(\mathbb{R}^{\nu+2})$. The set $\mathcal{D}'_0(\mathbb{R}^2)$ of even distributions in $\mathcal{D}'(\mathbb{R}^2)$ is the dual of $\mathcal{D}_0(\mathbb{R}^2)$. The spaces of distributions $\mathcal{D}'_0(\mathbb{R}^{\nu+2})$ and $\mathcal{D}'_0(\mathbb{R}^2)$ are also isomorphic. Note that a test function ϕ in $\mathcal{D}_0(\mathbb{R}^2)$ and a regular distribution ψ in $\mathcal{D}'_0(\mathbb{R}^2)$ are already determined by their restrictions to Ω:

(4.25) $$(\psi,\phi) := \frac{1}{2}\int_{\mathbb{R}^2} \psi(x,y)\phi(x,y)\,dxdy = \int_{\Omega} \psi(x,y)\phi(x,y)\,dxdy,$$

$\psi \in \mathcal{D}'_0(\mathbb{R}^2)$, regular, and $\phi \in \mathcal{D}_0(\mathbb{R}^2)$.

Let $\tilde{f}$ be a regular distribution in $\mathcal{D}'_0(\mathbb{R}^{\nu+2})$. At the one hand $\tilde{f}$ can be identified with a function f in $L^1_{loc}(\Omega, y^\nu dxdy)$ by (4.2). At the other hand $\tilde{f}$ corresponds to a regular distribution $\Psi(f)$ in $\mathcal{D}'_0(\mathbb{R}^2)$ by

$$(4.26) \qquad (\Psi(f),\phi) = (\tilde{f},\tilde{\phi}),$$

$\phi \in \mathcal{D}_0(\mathbb{R}^2)$. (In (4.26) $\tilde{f}$ need not to be regular. If $\tilde{f}$ is not regular, the distribution $\Psi(f)$ still exists but there is no function f corresponding to this distribution.) Formula (4.26) leads to

$$(4.27) \qquad (\tilde{f},\tilde{\phi}) = \int_{\mathbb{R}^{\nu+2}} \tilde{f}(x_0,\ldots,x_{\nu+1})\tilde{\phi}(x_0,\ldots,x_{\nu+1})dx_0 \ldots dx_{\nu+1}$$

$$= \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))} \int_\Omega f(x,y)\phi(x,y)y^\nu dxdy.$$

Together with (4.25) and (4.26) this implies

$$(4.28) \qquad \Psi(f) = \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))} f(x,y)y^\nu,$$

when $f \in L^1_{loc}(\Omega, y^\nu dxdy)$.

If confusion is possible we will write $\Psi_\nu(f)$ for $\Psi(f)$. Define the adjoint D^*_ν of D_ν by:

$$(4.29) \qquad (D^*_\nu\psi,\phi) := (\psi, D_\nu\phi), \qquad \psi \in \mathcal{D}'_0(\mathbb{R}^2), \ \phi \in \mathcal{D}_0(\mathbb{R}^2),$$

then for $f \in L^1_{loc}(\Omega, y^\nu dxdy) \cap C^2(\Omega)$

$$(4.30) \qquad D^*_\nu\Psi(f) = \Psi(D_\nu f).$$

Let K be the projection of $\tilde{K}$ on Ω and let S_+ be the projection of $\tilde{S}_+$ on Ω. The space $\mathcal{D}[K]$ consists of all ϕ in $\mathcal{D}_0(\mathbb{R}^2)$ for which $\mathrm{supp}(\phi) \cap \Omega \subset K$. Let $\mathcal{D}'[K]$ be its dual (consisting of even distributions). Then $\mathcal{D}'_+[K] := \{\psi \in \mathcal{D}'[K] \mid \mathrm{supp}(\psi) \cap \Omega \subset S_+\}$.

Let us define the fractional integral operator J_ν^μ by:

(4.31) $$(J_\nu^\mu \Psi(f),\phi) := (\tilde{I}_\nu^\mu \tilde{f},\tilde{\phi}),$$

for $\Psi(f) \in \mathcal{D}'_+[K]$, corresponding to $\tilde{f} \in \mathcal{D}'_+(\tilde{K})$. The fractional integral operator I_ν^μ acting on a function f in $L^1_{loc}(K,y^\nu dxdy)$ with $supp(f) \subset S_+$, is given by:

(4.32) $$(\Psi(I_\nu^\mu f),\phi) := (J_\nu^\mu \Psi(f),\phi).$$

If $\mathbb{R}e\,\mu - \nu \leq 0$, the left hand side of (4.32) contains a distribution which in general is not regular, but if $\mathbb{R}e\,(\mu-\nu) > 0$, then (4.32) results in:

(4.33) $$I_\nu^\mu f(x,y) = c(\mu,\nu) \int_{\xi,\zeta>0} f(x-\xi,\sqrt{(y-\eta)^2-\zeta^2})\cdot$$

$$\cdot\ (\xi^2-\eta^2-\zeta^2)_+^{\frac{1}{2}(\mu-\nu)-1}\zeta^{\nu-1}d\xi d\eta d\zeta$$

$$= c(\mu,\nu) \int_{\substack{\xi,\eta>0\\ 0<\phi<\pi}} f(x-\xi,\sqrt{y^2+\eta^2-2y\eta\cos\phi})$$

$$\cdot\ (\xi^2-\eta^2)_+^{\frac{1}{2}(\mu-\nu)-1}\eta^\nu(\sin\phi)^{\nu-1}\,d\xi d\eta d\phi,$$

with

(4.34) $$[c(\mu,\nu)]^{-1} = 2^{\mu-2}\Gamma(\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-\nu)),$$

$$f \in L^1_{loc}(K,y^\nu dxdy),\ supp(f) \subset S_+,\ \mathbb{R}e(\mu-\nu) > 0.$$

As a corollary of the theory of $\tilde{I}_\nu^\mu$ and (4.31) we obtain:

<u>LEMMA 4.1</u>. *Let* $\psi,\chi \in \mathcal{D}'_+[K]$, $\nu \in \mathbb{N}$, *then*

$$J_\nu^{\mu_1}J_\nu^{\mu_2}\psi = J_\nu^{\mu_1+\mu_2}\psi,$$

$$D_\nu^* J_\nu^{\mu+2} \psi = J_\nu^{\mu+2} D_\nu^* \psi = J_\nu^\mu \psi,$$

$$J_\nu^0 \psi = \psi,$$

$$(D_\nu^*)^k \psi = \chi \iff \psi = J_\nu^{2k} \chi.$$

COROLLARY 4.2. *Let* $\nu \in \mathbb{N}$, $\mathrm{Re}\,(\mu-\nu) > 0$, $\mathrm{Re}\,(\mu_i-\nu) > 0$, $i = 1,2$, $f \in L^1_{loc}(K,y^\nu dxdy)$, $\mathrm{supp}(f) \subset S_+$, *then*

$$I_\nu^{\mu_1} I_\nu^{\mu_2} f = I_\nu^{\mu_1+\mu_2} f,$$

$$D_\nu I_\nu^{\mu+2} f = I_\nu^{\mu+2} D_\nu f = I_\nu^\mu f \qquad \textit{in weak sense}$$

Here all functions are in $L^1_{loc}(K,y^\nu dxdy)$ *and* I_ν^μ *is given by* (4.33). *If in addition* $D_\nu^\ell f \in L^1_{loc}(K,y^\nu dxdy)$ *for* $\ell = [\frac{\nu+1}{2}]$, *then*

(4.35) $\qquad I_\nu^0 f = f, \quad$ *if* $\quad I_\nu^0 := I_\nu^{2\ell} D_\nu^\ell f.$

If $k \geq [\frac{\nu+1}{2}]$, $f,g \in L^1_{loc}(K,y^\nu dxdy)$, $\mathrm{supp}(f)$ *and* $\mathrm{supp}(g) \subset S_+$, *then the unique solution of*

(4.36) $\qquad D_\nu^k f = g$

is given by

(4.37) $\qquad f = I_\nu^{2k} g.$

If $0 \leq k < [\frac{1}{2}(\nu+1)]$ *then the unique solution of* (4.36) *is*

(4.38) $\qquad f = I_\nu^{2[\frac{1}{2}(\nu+1)]} D_\nu^\ell g, \qquad \ell = [\frac{1}{2}(\nu+1)] - k,$

provided that $D_\nu^\ell g \in L^1_{loc}(K,y^\nu dxdy)$. *Both* (4.37) *and* (4.38) *are weak solutions. They are classical solutions if* g *is* $2[\frac{1}{2}(\nu+1)]$*-times continuously differentiable and* $D_\nu^{[\frac{1}{2}(\nu+1)]} g \in L^1_{loc}(K,y^\nu dxdy)$.

5. A GENERALIZED CONVOLUTION

5.1. Distributions and test functions on Ω

Again Ω is given by (4.1). Some of the definitions below are already contained in Section 4. We repeat them here in order to have all definitions together.

(5.1) $$L_+(x,y) := \{(\xi,\eta) \in \Omega \mid \xi - x \geq |y-\eta|\}$$

is the forward light cone with vertex $(x,y) \in \Omega$, and

(5.2) $$L_-(x,y) := \{(\xi,\eta) \in \Omega \mid x - \xi \geq |y-\eta|\}$$

is the backward light cone with vertex $(x,y) \in \Omega$. The symbols L_+ and L_- will be used for the light cones with their vertex in the origin:

$$L_+ = \{(x,y) \in \Omega \mid x \geq y \ (\geq 0)\},$$

and

$$L_- = \{(x,y) \in \Omega \mid x \leq -y \ (\leq 0)\}.$$

A curve S of the form $\{(f(y),y) \mid y \geq 0\}$ (with f continuous on $[0,\infty)$) is *admissible* if for any point (x,y) in S:

$$L_+(x,y) \cap S = L_-(x,y) \cap S = \{(x,y)\}.$$

For an admissible curve S:

(5.3) $$S_+ = \bigcup_{(x,y)\in S} L_+(x,y),$$

(5.4) $$S_- = \bigcup_{(x,y)\in S} L_-(x,y).$$

Let K be an open neighbourhood of S in Ω such that

(i) if $(x,y) \in K$ then $S_+ \cap L_-(x,y)$ is bounded;

(ii) if $(x,y) \in K$ then $L_-(x,y) \subset K$;

(iii) $S_- \subset K$.

Let

$$\tilde{K} := \{(x,y) \in \mathbb{R}^2 \mid (x,|y|) \in K\}.$$

The space $\mathcal{D}_0(\mathbb{R}^2)$ is the closed subspace of $\mathcal{D}(\mathbb{R}^2)$ which consists of C_c^∞-functions which are even in the second variable. The space $\mathcal{D}_0(\tilde{K})$ consists of test functions ϕ in $\mathcal{D}_0(\mathbb{R}^2)$ with $\text{supp}(\phi) \cap \Omega \subset K$. As topological vector spaces $\mathcal{D}_0(\mathbb{R}^2)$ inherits the topology of $\mathcal{D}(\mathbb{R}^2)$ and $\mathcal{D}_0(\tilde{K})$ inherits the topology of $\mathcal{D}(\tilde{K})$. There are the following sets of distributions: The set $\mathcal{D}_0'(\mathbb{R}^2)$ consisting of all distributions in $\mathcal{D}'(\mathbb{R}^2)$ which are even in the second variable and $\mathcal{D}_0'(\tilde{K})$ consisting of all even distributions in $\mathcal{D}'(\tilde{K})$. Note that $\mathcal{D}_0'(\mathbb{R}^2)$ is the dual of $\mathcal{D}_0(\mathbb{R}^2)$ and $\mathcal{D}_0'(\tilde{K})$ is the dual of $\mathcal{D}_0(\tilde{K})$. Furthermore

(5.5) $$\mathcal{D}_{0+}'(\tilde{K}) := \{\psi \in \mathcal{D}_0'(K) \mid \text{supp}(\psi) \cap \Omega \subset S_+\},$$

and

(5.6) $$\mathcal{F}'_+ := \{\psi \in \mathcal{D}_0'(\mathbb{R}^2) \mid \text{supp}(\psi) \cap \Omega \subset L_+\}.$$

Again a regular distribution ψ in $\mathcal{D}_0'(\mathbb{R}^2)$ acts on a test function ϕ as given in (4.25).

5.2. A convolution structure on $\mathcal{D}'[\Omega]$

In this subsection ν will be a fixed complex number with

(5.7) $$\operatorname{Re} \nu > 0.$$

If $\nu \in \mathbb{N}$ a convolution structure is inherited from $\mathbb{R}^{\nu+2}$ by considering rotation invariant functions. The generalization of this convolution structure corresponds to the ordinary convolution for the first variable and the convolution for the Bessel functions of order $\frac{1}{2}(\nu-1)$ for the second variable.

A. SCHWARTZ [14] uses the same convolution structure but his function spaces are different. For ν fixed we again (cf. (4.28) identify a function f in $L^1_{loc}(\Omega, y^\nu dxdy)$ with a regular distribution $\Psi_\nu(f)$ in $L^1_{loc}(\Omega, dxdy)$:

$$(5.8) \qquad \Psi_\nu(f) = \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))} f(x,y)y^\nu .$$

DEFINITION 5.1. Let $\mathrm{Re}\,\nu > 0$. Let $f \in L^1_{loc}(K, y^\nu dxdy)$, $g,h \in L^1_{loc}(\Omega, y^\nu dxdy)$ be such that $\mathrm{supp}(f) \cap \Omega \subset S_+$, $\mathrm{supp}(g) \cap \Omega$ and $\mathrm{supp}(h) \cap \Omega \subset L_+$, then

$$(5.9) \qquad k \circledast g(x,y) := \frac{2\pi^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu)} \int_{\zeta>0} k(x-\xi, \sqrt{(y-\eta)^2+\zeta^2})g(\xi,\sqrt{\eta^2+\zeta^2})\zeta^{\nu-1} d\xi d\eta d\zeta$$

$$= \frac{2\pi^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu)} \int_{\substack{\eta>0 \\ 0<\phi<\pi}} k(x-\xi, \sqrt{y^2+\eta^2-2y\eta\cos\phi})g(\xi,\eta)\eta^\nu(\sin\phi)^{\nu-1} d\xi d\eta d\phi,$$

where k can be both f and h.

Note that the integral is over a bounded part of $\mathbb{R}^3$, so it converges for $\mathrm{Re}\,\nu > 0$. In this case the commutativity is clear from the first expression and the transformation of variables $(\xi,\eta) \to (x-\xi, y-\eta)$. In the following we will use the second part of (5.9). Then (5.9) is equivalent to:

$$(5.10) \qquad (k \circledast g)(x,y) = \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))} \int_{\eta>0} T^\nu_{x,y} \check{k}(\xi,\eta) \cdot g(\xi,\eta)\eta^\nu d\xi d\eta,$$

where

$$(5.11) \qquad \check{f}(x,y) := f(-x,y),$$

and the generalized translation operator $T^\nu_{x,y}$ is defined by:

$$(5.12)\qquad T^{\nu}_{x,y}f(\xi,\eta) := \frac{\Gamma(\frac{1}{2}(\nu+1))}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\nu)} \int_0^{\pi} f(\xi-x,\sqrt{y^2+\eta^2-2y\eta\cos\phi})(\sin\phi)^{\nu-1}\,d\phi.$$

The operator $T^{\nu}_{x,y}$ has the properties:

(i) $T^{\nu}_{x,y}f(\xi,\eta) = T^{\nu}_{\xi,\eta}\check{f}(x,y)$;

(ii) if f has its support in some forward light cone, then $T^{\nu}_{x,y}f(\xi,\eta)$ considered as a function of ξ and η (x and y fixed) has its support in a forward light cone, while considered as a function of x and y its support is contained in a backward light cone. Similar results hold for functions f with their supports in S_+, being the union of forward light cones with their vertices on the admissible line S in Ω.

Because of the commutativity of the convolution product (5.9)

$$(5.13)\qquad \int_{\eta>0} T^{\nu}_{x,y}\check{f}(\xi,\eta)\cdot g(\xi,\eta)\eta^{\nu}d\xi d\eta =$$

$$= \int_{\eta>0} f(\xi,\eta)T^{\nu}_{x,y}\check{g}(\xi,\eta)\eta^{\nu}d\xi d\eta,$$

for all functions f and g for which these integrals converge.

Note that

$$(5.14)\qquad (\Psi_{\nu}(f),\phi) = \frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))} \int_{\Omega} f(x,y)\phi(x,y)y^{\nu}dxdy$$

for $f \in L^1_{loc}(\Omega,y^{\nu}dxdy)$. Hence (5.10) yields:

$$(5.15)\qquad (\Psi_{\nu}(k\circledast g),\phi) = \left\{\frac{2\pi^{\frac{1}{2}(\nu+1)}}{\Gamma(\frac{1}{2}(\nu+1))}\right\}^2 \int_{\eta,y>0} T^{\nu}_{x,y}\check{k}(\xi,\eta)\cdot$$

$$\cdot\, g(\xi,\eta)\phi(x,y)\eta^{\nu}y^{\nu}d\xi d\eta dxdy$$

$$= (\Psi_{\nu}(k)(x,y)\Psi_{\nu}(g)(\xi,\eta), T^{\nu}_{-\xi,\eta}\phi(x,y)).$$

Here we used (5.8), (5.10), (5.13) and the properties of $T^{\nu}_{x,y}$. The function Φ: $(x,y,\xi,\eta) \to T^{\nu}_{-\xi,\eta}\phi(x,y)$ is in $C^{\infty}(\mathbb{R}^4)$, even in y and even in η. Its support is not compact, but the intersection M of supp(Ψ(k)) $\times$ supp(Ψ(g)) and supp(Φ) is compact in $\mathbb{R}^4$ (or in $\tilde{K} \times \mathbb{R}^2$ if supp(k) $\subset S_+$ and $\phi \in \mathcal{D}_0(\tilde{K})$).

Formula (5.15) suggests the definition of $\Psi(k) \circledast \Psi(g)$ for distributions. In that case $\Psi(k)(x,y)\Psi(g)(\xi,\eta)$ has to be interpreted as a tensor product of distributions. (Remember that this is defined as a distribution on $\mathcal{D}(\mathbb{R}^4)$ (or on $\mathcal{D}(\tilde{K}\times\mathbb{R}^2)$ by

$$(\Psi(k)(x,y)\Psi(g)(\xi,\eta), X(x,y,\xi,\eta)) = (\Psi(k)(x,y),(\Psi(g)(\xi,\eta), X(x,y,\xi,\eta))),$$

$X \in \mathcal{D}(\mathbb{R}^4)$ or $\mathcal{D}(\tilde{K}\times\mathbb{R}^2)$. It is equal to $(\Psi(g)(\xi,\eta),(\Psi(k)(x,y),X(x,y,\xi,\eta)))$, see TRÈVES [20, Th. 40.3].) Although the support of Φ is not compact, the right hand side of (5.15) is uniquely defined when we replace Φ by $\rho\Phi$, where ρ is a cut-off function which is 1 on the compact set M and 0 outside of a suitable chosen open neighbourhood V of M with $\bar{V}$ compact. Note that ρ can be chosen such that it is even in y and η. Furthermore, we have to investigate if the linear functional on $\mathcal{D}_0(\mathbb{R}^2)$ defined by the left hand side of (5.15) is continuous. Consider a sequence $\{\phi_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \phi_n = 0$ in $\mathcal{D}_0(\mathbb{R}^2)$ (or $\mathcal{D}_0(\tilde{K})$), then also $\lim_{n\to\infty} \rho\Phi_n = 0$ in $\mathcal{D}(\mathbb{R}^4)$ or $\mathcal{D}(\tilde{K}\times\mathbb{R}^2)$), where

$$\rho\Phi_n : (x,y,\xi,\eta) \to \rho(x,y,\xi,\eta)T^{\nu}_{-\xi,\eta}\phi_n(x,y).$$

Here ρ is chosen such that (supp(Ψ(k) $\times$ supp(Ψ(g))) $\cap$ supp Φ_n, n = 1,2,... is a subset of the compact set where $\rho = 1$. This choice is possible since $\phi_n \to 0$ in $\mathcal{D}_0(\mathbb{R}^2)$ (or $\mathcal{D}_0(\tilde{K})$) implies supp(ϕ_n) $\subset$ N, n = 1,2,..., where N is a compact subset of $\mathbb{R}^2$ or $\tilde{K}$. This proves the continuity of the linear functional in the left hand side of (5.15), since continuity is equivalent to sequential continuity on $\mathcal{D}_0(\mathbb{R}^2)$ and $\mathcal{D}_0(\tilde{K})$. Thus (5.15) leads to:

<u>DEFINITION 5.2</u>. Let $\psi,\zeta \in \mathcal{F}'_+$ and $\chi \in \mathcal{D}'_{0+}(\tilde{K})$, then

(5.16) $$(\psi\circledast\kappa,\phi) := (\psi(x,y)\kappa(\xi,\eta), T^{\nu}_{-\xi,\eta}\phi(x,y)),$$

where $\kappa = \chi$ or $\kappa = \zeta$, $\phi \in \mathcal{D}_0(\tilde{K})$ (if $\kappa = \chi$) or $\phi \in \mathcal{D}_0(\mathbb{R}^2)$ (if $\kappa = \zeta$). Hence

(5.17) $$\Psi(k \circledast g) = \Psi(k) \circledast \Psi(g).$$

LEMMA 5.3. *If ψ and $\kappa \in \mathcal{F}'_+$ then $\psi \circledast \kappa \in \mathcal{F}'_+$. If $\psi \in \mathcal{F}'_+$ and $\kappa \in \mathcal{D}'_{0+}(\tilde{K})$ then $\psi \circledast \kappa \in \mathcal{D}'_{0+}(\tilde{K})$.*

PROOF. The linearity of the functional $\psi \circledast \kappa$ is clear from the definition. The continuity of its action on $\mathcal{D}_0(\mathbb{R}^2)$ or $\mathcal{D}_0(\tilde{K})$ follows from the argument given before the definition. So $\psi \circledast \kappa$ is in $\mathcal{D}'_0(\mathbb{R}^2)$ or $\mathcal{D}'_0(\tilde{K})$. Consideration of the support of $\psi \circledast \kappa$ proves the lemma. □

Since $T^{\nu}_{-\xi,\eta}\phi(x,y) = T^{\nu}_{-x,y}\phi(\xi,\eta)$ it is clear that the convolution structure $\circledast$ is commutative. Suppose ψ, χ and ζ are in $\mathcal{F}'_+$ or two of them are in $\mathcal{F}'_+$ and the third one is in $\mathcal{D}'_{0+}(\tilde{K})$. Let ϕ be a test function in $\mathcal{D}_0(\mathbb{R}^2)$ (or in the smaller set $\mathcal{D}_0(\tilde{K})$ if one of the distributions is in $\mathcal{D}'_{0+}(\tilde{K})$). Consider $\psi \circledast \chi \circledast \zeta$:

$$(\psi \circledast (\chi \circledast \zeta),\phi) = (\psi(\xi,\eta)\chi(z,w)\zeta(x,y),T^{\nu}_{-z,w}T^{\nu}_{-\xi,\eta}\phi(x,y)),$$

while

$$((\psi \circledast \chi) \circledast \zeta,\phi) = (\psi(\xi,\eta)\chi(z,w)\zeta(x,y),T^{\nu}_{-\xi,\eta}T^{\nu}_{-z,w}\phi(x,y)).$$

Here we used the commutativity of the convolution. Hence the associativity follows from the relation

$$T^{\nu}_{-z,w}T^{\nu}_{-\xi,\eta}\phi(x,y) = T^{\nu}_{-\xi,\eta}T^{\nu}_{-z,w}\phi(x,y). \tag{5.18}$$

From (5.12) it is clear that (5.18) is equivalent to

$$T^{\nu}_{0,w}T^{\nu}_{0,\eta}\phi(x+z+\xi,y) = T^{\nu}_{0,\eta}T^{\nu}_{0,w}\phi(x+z+\xi,y). \tag{5.19}$$

This relation will be proved using the Hankel transform.

5.3. Hankel transformation

The "modified" Bessel function $\mathcal{J}_{\beta}(y)$ is defined by

(i) $\mathcal{J}_{\beta}(x)$ is a solution of

(5.20) $\Delta_{2\beta+1}u(y) + u(y) = 0, \qquad \beta \neq -1,-2,\ldots,$

where

(5.21) $\Delta_{2\beta+1} := \partial_{yy} + (2\beta+1)y^{-1}\partial_y ,$

(ii) $\mathcal{J}_\beta(0) = 1$, $\mathcal{J}_\beta'(0) = 0$.

Then

(5.22) $\mathcal{J}_\beta(y) = 2^\beta\Gamma(\beta+1)y^{-\beta}J_\beta(y),$

where $J_\beta(y)$ is a Bessel function.

It is checked by differentiation and integration by parts that for $\nu = 2\beta + 1$

(5.23) $T^{2\beta+1}_{0,y}\ \Delta_{2\beta+1}\ u(\eta) = \Delta_{2\beta+1}\ T^{2\beta+1}_{0,y}\ u(\eta), \qquad \mathrm{Re}\ \beta > -\frac{1}{2},$

where $T^\nu_{0,y}$ is defined in (5.12). I found this very useful remark in MUKHLISOV [10]. This commutation relation (5.23) and the definition of $\mathcal{J}_\beta$ result in the product formula:

(5.24) $\mathcal{J}_\beta(y)\mathcal{J}_\beta(\eta) = T^{2\beta+1}_{0,\eta}\ \mathcal{J}_\beta(y).$

This product formula is well-known (cf. WATSON [21, p. 367]). In terms of the function $\mathcal{J}_\beta$ the Hankel transform $u^\wedge$ of a function u is given by:

(5.25) $u^\wedge(\lambda) = (2^\beta\Gamma(\beta+1))^{-1}\int_0^\infty u(y)\mathcal{J}_\beta(\lambda y)y^{2\beta+1}dy, \qquad \beta \in \mathbb{R},\ \beta \geq -\frac{1}{2}.$

For functions u in $L^2([0,\infty),y^{2\beta+1}dy)$ the transform $u^\wedge$ exists in the same space and $(u^\wedge)^\wedge = u$ again. See TITCHMARSH [18, Ch. VIII].

5.4. The associativity

Let ϕ: $(x,y) \to \phi(x,y)$ in $C_c^\infty(\mathbb{R}^2)$, even in y. Consider the Hankel transform with $\beta = \frac{1}{2}(\nu-1)$, $\nu > 0$, of $T^\nu_{0,\eta}\phi(x,y)$ with respect to the second variable y. It is possible to take the Hankel transform because $T^{\nu}_{0,\eta}\phi(x,y)$ is in $C^{\infty}_{c}([0,\infty))$, and thus in $L^2([0,\infty),y^{\nu}dy)$, considered as a function of y.

$$(5.26)\qquad (T^{\nu}_{0,\eta}\phi(x,\cdot))^{\wedge}(\lambda) =$$

$$= (2^{\frac{1}{2}(\nu-1)}\Gamma(\tfrac{1}{2}(\nu+1)))^{-1}\int_0^{\infty} T^{\nu}_{0,\eta}\phi(x,y)\mathcal{J}_{\frac{1}{2}(\nu-1)}(\lambda y)y^{\nu}dy$$

$$= (2^{\frac{1}{2}(\nu-1)}\Gamma(\tfrac{1}{2}(\nu+1)))^{-1}\int_0^{\infty} \phi(x,y)T^{\nu}_{0,\eta}\mathcal{J}_{\frac{1}{2}(\nu-1)}(\lambda y)y^{\nu}dy$$

$$= (\phi(x,\cdot))^{\wedge}(\lambda)\mathcal{J}_{\frac{1}{2}(\nu-1)}(\lambda\eta),$$

with use of the product formula (5.24).

Formula (5.26) tells us that (5.19) is true after a Hankel transformation, but then (5.19) must hold for $\nu > 0$ since the Hankel transformation is invertible. For complex values of ν, $\mathrm{Re}\,\nu > 0$, (5.19) holds since it depends analytically on ν. Thus we proved the associativity:

$$(5.27)\qquad \psi \circledast (\chi \circledast \zeta) = (\psi \circledast \chi) \circledast \zeta.$$

Other important properties of the generalized translation operator $T^{\nu}_{x,y}$ as given in (5.12) are:

$$(5.28)\qquad \partial_x T^{\nu}_{\xi,\eta}\phi(x,y) = (T^{\nu}_{\xi,\eta}\partial_x\phi)(x,y),$$

$$(5.29)\qquad (\partial_{yy}+\nu y^{-1}\partial_y)T^{\nu}_{\xi,\eta}\phi(x,y) = (T^{\nu}_{\xi,\eta}(\partial_{yy}+\nu y^{-1}\partial_y)\phi)(x,y)\,.$$

The first one is clear from (5.12), while the second one is equivalent to (5.23). Application of (5.28) and (5.29) on the convolution product as given in Definitions 5.1 and 5.2 results in:

COROLLARY 5.4. *Let Q be one of the operators* ∂_x, $\partial_{yy}+\nu y^{-1}\partial_y$, D_ν, *then:*

$$(5.30)\qquad Q(f \circledast g) = (Qf) \circledast g = f \circledast (Qg),$$

for suitable functions f *and* g. *For distributions* ψ *and* χ *this yields:*

$$(5.31) \qquad Q^*(\psi \otimes \chi) = (Q^*\psi) \otimes \chi = \psi \otimes Q^*\chi,$$

where

$$(5.32) \qquad (Q^*\psi, \phi) := (\psi, Q\phi),$$

for any test function ϕ.

6. A FRACTIONAL INTEGRAL OPERATOR FOR D_ν

In this section it is supposed that $\mathrm{Re}\,\nu > 0$. The function z_μ is defined by:

$$(6.1) \qquad z_\mu := [\pi^{\frac{1}{2}\nu} 2^{\mu-1} \Gamma(\tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}(\mu-\nu))]^{-1} \rho^{\mu-\nu-2},$$

with

$$(6.2) \qquad \rho(x,y) := \begin{cases} (x^2-y^2)^{\frac{1}{2}}, & \text{if } x > y \geq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

cf. (4.14). If $\mathrm{Re}(\mu-\nu) > 0$ then z_μ is in $L^1_{loc}(\Omega, y^\nu dxdy)$ and it corresponds to a distribution $q_\mu = \Psi_\nu(z_\mu)$ in $\mathcal{F}'_+ \cap L^1_{loc}(\Omega, dxdy)$ by (5.8). Analytical continuation with respect to μ leads to:

<u>DEFINITION 6.1</u>. The distribution q_μ in $\mathcal{F}'_+$ is defined for $\mathrm{Re}\,\mu-\nu > 0$, $\mathrm{Re}\,\nu > 0$ by (4.25) and:

$$(6.3) \qquad q_\mu(x,y) := \Psi_\nu(z_\mu)(x,y) = h(\mu,\nu)\{\rho(x,y)\}^{\mu-\nu-2} y^\nu,$$

$$(6.4) \qquad [h(\mu,\nu)]^{-1} = \pi^{-\frac{1}{2}} 2^{\mu-2} \Gamma(\tfrac{1}{2}\mu) \Gamma(\tfrac{1}{2}(\mu-\nu)) \Gamma(\tfrac{1}{2}(\nu+1)).$$

For $\mathrm{Re}\,(\mu-\nu) > -2k$ and $\phi \in \mathcal{D}_0(\mathbb{R}^2)$:

$$(6.5) \qquad (q_\mu, \phi) := (q_{\mu+2k}, D_\nu^k \phi).$$

It is a corollary of (6.10) below that (6.5) is independent of the choice of k.

LEMMA 6.2.

(6.6) $$z_{\mu_1} \circledast z_{\mu_2} = z_{\mu_1+\mu_2}, \qquad \mathrm{Re}\,(\mu_i-\nu) > 0,\ i = 1,2,$$

(6.7) $$q_{\mu_1} \circledast q_{\mu_2} = q_{\mu_1+\mu_2}, \qquad \mu_1,\mu_2 \in \mathbb{C}.$$

PROOF. If $\mathrm{Re}\,(\mu_i-\nu) > 0$, $i = 1,2$, then (5.9), (6.1) and (6.2) yield:

$$z_{\mu_1} \circledast z_{\mu_2} = \text{const.} \int_{\substack{0<\xi<x\\ \zeta>0}} [(x-\xi)^2 - (y-\eta)^2 - \zeta^2]_+^{\frac{1}{2}(\mu_1-\nu)-1} \cdot$$
$$\cdot\ (\xi^2-\eta^2-\zeta^2)_+^{\frac{1}{2}(\mu_2-\nu)-1}\ \zeta^{\nu-1}d\xi d\eta d\zeta.$$

After a Lorentz transformation on ξ and η this results in:

$$z_{\mu_1} \circledast z_{\mu_2} = \text{const.} \int_{\substack{0<\xi<\sqrt{x^2-y^2}\\ \zeta,x>0}} [(\sqrt{x^2-y^2} - \xi)^2 - \eta^2 - \zeta^2]_+^{\frac{1}{2}(\mu_1-\nu)-1} \cdot$$
$$\cdot\ (\xi^2 - \eta^2 - \zeta^2)^{\frac{1}{2}(\mu_2-\nu)-1}\ \zeta^{\nu} d\xi d\eta d\zeta$$

$$= \begin{cases} \text{const.}\ (x^2-y^2)_+^{\frac{1}{2}(\mu_1+\mu_2-\nu)-1}, & x > 0,\\ 0, & x \le 0, \end{cases}$$

which is seen from the tranformation $(\xi,\eta,\zeta) \to \sqrt{x^2-y^2}\,(\xi',\eta',\zeta')$. Thus

(6.8) $$z_{\mu_1} \circledast z_{\mu_2} = \text{const.}\ z_{\mu_1+\mu_2}.$$

The constant can be calculated by evaluation of the integral. Though, it is easier to consider (as did M. RIESZ [13]) the convolution of z_μ and the function e^x. Although the support of e^x extends to infinity, the convolution with z_μ, $\mu \in \mathbb{C}$ exists, and is associative since e^x is rapidly decreasing for $x \to -\infty$ and z_μ is of finite algebraic growth.

$$(6.9) \qquad z_\mu \circledast e^x = [2^{\mu-2}\Gamma(\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-\nu))]^{-1} \cdot$$

$$\cdot \int\limits_{\substack{\xi,\eta>0 \\ 0<\phi<\pi}} e^{x-\xi}(\xi^2-\eta^2)_+^{\frac{1}{2}(\mu-\nu)-1}\eta^\nu(\sin\phi)^{\nu-1}\,d\xi d\eta d\phi = e^x,$$

since

$$\int\limits_{\substack{\xi,\eta>0 \\ 0<\phi<\pi}} e^{-\xi}(\xi^2-\eta^2)_+^{\frac{1}{2}(\mu-\nu)-1}\eta^\nu(\sin\phi)^{\nu-1}\,d\xi d\eta d\phi =$$

$$= \int_0^\infty e^{-\xi}\xi^{\mu-1}d\xi \int_0^1 (1-\eta^2)^{\frac{1}{2}(\mu-\nu)-1}\eta^\nu d\eta \int_0^\pi (\sin\phi)^{\nu-1}\,d\phi$$

$$= 2^{\mu-2}\Gamma(\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-\nu)).$$

From (6.9) it is clear that the constant in (6.8) is equal to 1. This proves (6.6). If $\mathrm{Re}\,(\mu_i-\nu) > 0$, $i = 1,2$, then (6.7) is a corollary of (6.6), (6.3) and (5.17). For $\mu_i \in \mathbb{C}$, $i = 1,2$, (6.7) is proved by analytical continuation with (6.5). □

In (6.5) we already used that:

$$(6.10) \qquad D_\nu z_{\mu+2} = z_\mu,$$

which implies

$$(6.11) \qquad D_\nu^* q_{\mu+2} = q_\mu.$$

Here (6.10) is checked by computation.

<u>DEFINITION 6.3</u>. If $\psi \in \mathcal{F}'_+$ or $\mathcal{D}'_{0+}(\tilde{K})$, then

(6.12) $\quad J_\nu^\mu(\psi) := q_\mu \circledast \psi.$

If $f \in L^1_{loc}(\Omega, y^\nu dxdy)$ and $supp(f) \subset L_+$ or if $f \in L^1_{loc}(K, y^\nu dxdy)$ and $supp(f) \subset S_+$, then

(6.13) $\quad \Psi_\nu(I_\nu^\mu f) := J_\nu^\mu(\Psi_\nu(f)),$

if the right hand side of (6.13) is a function in $L^1_{loc}(\Omega, dxdy)$ or $L^1_{loc}(K, dxdy)$.

Hence, for $\mathbb{R}e\,(\mu-\nu) > 0$:

(6.14) $\quad I_\nu^\mu(f) = z_\mu \circledast f.$

Note that (6.14) is equal to (4.33).

THEOREM 6.4. *For any* $\psi \in \mathcal{F}'_+$ *or* $\mathcal{D}'_{0+}(\tilde{K})$:

(i) $\quad J_\nu^{\mu_1} J_\nu^{\mu_2}\psi = J_\nu^{\mu_1+\mu_2}\psi,$

(ii) $\quad D_\nu^* J_\nu^{\mu+2}\psi = J_\nu^{\mu+2} D_\nu^*\psi = J_\nu^\mu\psi,$

(iii) $J_\nu^0\psi = \psi.$

PROOF. The first and the second part of the theorem follow from (6.7), (6.11) (6.12) and the associativity of the convolution $\circledast$. The third part is a consequence of the other two, and is proved in the same way as in Lemma 2.2. Here, the adjoint operator $J_\nu^{\mu*}$ to J_ν^μ, defined by $(\psi, J_\nu^{\mu*}\phi) = (J_\nu^\mu\psi, \phi)$, $\phi \in \mathcal{D}_0(\mathbb{R}^2)$ or $\phi \in \mathcal{D}_0(\tilde{K})$, corresponds to taking the convolution with a positive function if $\mu, \nu \in \mathbb{R}$ and $(\mu-\nu) > 0$. □

COROLLARY 6.5. *Let* ψ, χ *be both in* $\mathcal{F}'_+$ *or both in* $\mathcal{D}'_{0+}(\tilde{K})$, *then the unique solution of*

$$(D_\nu^*)^k\psi = \chi$$

is given by

$$\psi = J_\nu^{2k}\chi.$$

LEMMA 6.6. *Let* Q *be one of the differential operators* ∂_x *or* $\partial_{yy} + \nu y^{-1}\partial_y$, *then*

$$Q^* J_\nu^\mu \psi = J_\nu^\mu Q^* \psi,$$

where $\psi \in F'_+$ *or* $\mathcal{D}'_{0+}(\widetilde{K})$ *and* Q^* *is defined by* (5.32).

COROLLARY 6.7. = *Corollary* 4.2 *with the restriction* $\nu \in \mathbb{N}$ *replaced by* $\mathbb{R}e\, \nu > 0$. *The function* f *is also allowed to be in* $L^1_{loc}(\Omega, y^\nu dxdy)$ *if* $supp(f) \subset L_+$.

COROLLARY 6.8. *Let* Q *be one of the differential operators* ∂_x *or* $\partial_{yy} + \nu y^{-1}\partial_y$, *then*

$$Q\, I_\nu^\mu f = I_\nu^\mu Q\, f,$$

if f *and* Q f *are in* $L^1_{loc}(\Omega, y^\nu dxdy)$ *and* $supp(f) \subset L_+$ *or if they are in* $L^1_{loc}(K, y^\nu dxdy)$ *and* $supp(f) \subset S_+$.

7. APPLICATION OF THE FRACTIONAL INTEGRAL OPERATOR

In this section we return to our original problem as stated in Section 3. Iteration of (3.1) in terms of $(\xi,\eta) = (x, \frac{1}{4}(x^2-y^2))$ yields:

(7.1) $$D^\mu_{2\gamma+1}\, (x^2-y^2)^{\alpha+\mu} R^{\alpha+\mu,\beta-\mu,\gamma}_{n,k}(x, \tfrac{1}{4}(x^2-y^2)) =$$

$$= 2^{2\mu}\, \frac{\Gamma(\mu+\alpha+1)\Gamma(\mu+\alpha+\gamma+\frac{3}{2})}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+\frac{3}{2})}\, (x^2-y^2)^\alpha R^{\alpha,\beta,\gamma}_{n,k}(x, \tfrac{1}{4}(x^2-y^2)),$$

$\mu \in \mathbb{N}$.

The function f_α: $(x,y) \to (x^2-y^2)^\alpha R^{\alpha,\beta,\gamma}_{n,k}(x, \frac{1}{4}(x^2-y^2))$, $x > y \geq 0$, $(x,y) \to 0$, elsewhere, is in $L^1_{loc}(\Omega, y^{2\gamma+1}dxdy)$ if $\mathbb{R}e\,(\alpha+1) > 0$, $\mathbb{R}e\,(\gamma+1) > 0$, $\mathbb{R}e\,(\alpha+\gamma+\frac{3}{2}) > 0$. Corollary 6.7 leads to:

$$(7.2)\qquad (x^2-y^2)^{\alpha+\mu} R_{n,k}^{\alpha+\mu,\beta-\mu,\gamma}(x,\tfrac{1}{4}(x^2-y^2)) =$$

$$= 2^{2\mu} \frac{\Gamma(\mu+\alpha+1)\Gamma(\mu+\alpha+\gamma+\frac{3}{2})}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+\frac{3}{2})} I_{2\gamma+1}^{2\mu} (x^2-y^2)^{\alpha} R_{n,k}^{\alpha,\beta,\gamma}(x,\tfrac{1}{4}(x^2-y^2)),$$

for $\mu \in \mathbb{N}$, $\mathrm{Re}\,\gamma > -\frac{1}{2}$, $\mathrm{Re}\,\alpha > -1$, $\mathrm{Re}\,(\mu-\gamma-\frac{1}{2}) > 0$ and I_ν^μ given by (4.33). The condition $\mathrm{Re}\,\gamma > -\frac{1}{2}$ corresponds to $\mathrm{Re}\,\nu > 0$, which is needed in (4.33), but which can be extended somewhat (see Section 9, Remark 1). Similarly the condition $\mathrm{Re}\,(\mu-\gamma-\frac{1}{2}) > 0$ is obtained from (4.33). The conditions $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,\gamma > -\frac{1}{2}$ together imply $\mathrm{Re}\,(\alpha+\gamma+\frac{3}{2}) > 0$.

LEMMA 7.1. *Suppose that the conditions given after* (7.2) *hold, then* (7.2) *is true for* $\mu \in \mathbb{C}$.

PROOF. We know that (7.2) holds for $\mu \in \mathbb{N}$. There are two ways to prove (7.2) for $\mu \in \mathbb{C}$, $\mathrm{Re}\,(\mu-\gamma-\frac{1}{2}) > 0$ (cf. the remark after (2.16)).

(i) Use Carlson's theorem (Theorem 2.4) and estimate $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$ as a function of α. This estimation can be obtained from the expansion (7.3) below.

(ii) Use the generalized power series expansion

$$(7.3)\qquad R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta) = \sum_{\ell=0}^{k} \sum_{m=\ell}^{n} c_{n,k;m,\ell}^{\alpha,\beta,\gamma} Z_{m,\ell}^{\gamma}(\xi,\eta),$$

where $Z_{m,\ell}^{\gamma}$ is a zonal polynomial of James (see Subsection 8.4) and calculate

$$(7.4)\qquad I_{2\gamma+1}^{2\mu} (x^2-y^2)^{\alpha} Z_{m,\ell}^{\gamma}(x,\tfrac{1}{4}(x^2-y^2)).$$

Both methods work. The latter leads to some interesting results which can be found in Section 8. □

REMARK 7.2. If we are able to prove the conjecture:

$$(7.5)\qquad |R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)| \le 1, \qquad \alpha \ge \beta \ge -\tfrac{1}{2},\ \gamma \ge -\tfrac{1}{2},$$

(ξ,η) in the region of orthogonality, for some real values of the parameters $(\alpha,\beta,\gamma) = (\alpha_0,\beta_0,\gamma)$, $\gamma \ge -\frac{1}{2}$, then (7.2) can be used to prove (7.5) for those (α,β,γ) with $\alpha+\beta = \alpha_0+\beta_0$ and $\alpha > \alpha_0+\gamma+\frac{1}{2}$ (see KOORNWINDER & SPRINKHUIZEN-KUYPER [9, Corollary 8.5] and SPRINKHUIZEN-KUYPER [17, §7], for the conjecture and the values of (α,β,γ) for which it is proved). In order to prove (7.5) in this case we use the positivity of the operator $I^{2\mu}_{2\gamma+1}$ with $\mu = \alpha-\alpha_0$ and $R^{\alpha,\beta,\gamma}_{0,0}(\xi,\eta) \equiv 1$. The latter relation results in:

$$(7.6) \qquad (x^2-y^2)^{\alpha+\mu} = 2^{2\mu}\,\frac{\Gamma(\mu+\alpha+1)\,\Gamma(\mu+\alpha+\gamma+\frac{3}{2})}{\Gamma(\alpha+1)\,\Gamma(\alpha+\gamma+\frac{3}{2})}\,I^{2\mu}_{2\gamma+1}(x^2-y^2)^{\alpha}.$$

<u>REMARK 7.3</u>. In Section 9 the definition of $I^{2\mu}_{2\gamma+1}$ is extended to the case $\mathrm{Re}\,\gamma > -1$. See figure 1 in that section for those cases where $I^{2\mu}_{2\gamma+1}$ is an operator with a positive kernel.

8. AN INTEGRAL OPERATOR WHICH HAS JACOBI FUNCTIONS AS EIGENFUNCTIONS

8.1. The integral operator

For f in $L^1_{loc}(\Omega, y^\nu dxdy)$, $\mathrm{supp}(f) \subset L_+$ or f in $L^1_{loc}(K, y^\nu dxdy)$, $\mathrm{supp}(f) \subset S_+$ and $\mathrm{Re}\,(\mu-\nu) > 0$, we have (cf. Definition 6.3, (6.14) and (4.33)):

$$(8.1) \qquad I^\mu_\nu f(x,y) = c(\mu,\nu) \int\limits_{\substack{\xi<x \\ 0<\phi<\pi}} f(\xi,\eta)\{(x-\xi)^2-(y^2+\eta^2-2y\eta\cos\phi)\}_+^{\frac{1}{2}(\mu-\nu)-1} \cdot \eta^\nu(\sin\phi)^{\nu-1}\,d\xi\, d\eta\, d\phi,$$

where

$$(8.2) \qquad [c(\mu,\nu)]^{-1} = 2^{\mu-2}\,\Gamma(\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}\mu)\,\Gamma(\tfrac{1}{2}(\mu-\nu)).$$

Using hyperbolic coordinates in (8.1):

(8.3) $\quad x = r\,\mathrm{ch}\,t, \quad y = r\,\mathrm{sh}\,t, \quad \xi = \rho\,\mathrm{ch}\,\tau, \quad \eta = \rho\,\mathrm{sh}\,\tau,$

we get

(8.4)
$$I_\nu^\mu f(r\ \mathrm{ch}\ t, r\ \mathrm{sh}\ t) = c(\mu,\nu) \int\limits_{\substack{0<\phi<\pi \\ \tau>0 \\ r>\rho>0}} f(\rho\ \mathrm{ch}\ \tau, \rho\ \mathrm{sh}\ \tau) \cdot$$
$$\cdot\ [r^2 + \rho^2 - 2r\rho(\mathrm{ch}\ t\ \mathrm{ch}\ \tau - \mathrm{sh}\ t\ \mathrm{sh}\ \tau \cos\phi)]_+^{\frac{1}{2}(\mu-\nu)-1}$$
$$\cdot\ \rho^{\nu+1}(\mathrm{sh}\ \tau)^\nu (\sin\phi)^{\nu-1} d\rho d\tau d\phi.$$

Substitution of

(8.5) $\quad f(r\ \mathrm{ch}\ t, r\ \mathrm{sh}\ t) = r^\alpha F(t)$

in (8.4) and replacing ρ by $r\rho$ yields

(8.6)
$$r^{-\alpha-\mu} I_\nu^\mu r^\alpha F(t) = c(\mu,\nu) \int\limits_{\substack{0<\rho<1 \\ \tau>0 \\ 0<\phi<\pi}} F(\tau) \cdot$$
$$\cdot\ [1 + \rho^2 - 2\rho(\mathrm{ch}\ t\ \mathrm{ch}\ \tau - \mathrm{sh}\ t\ \mathrm{sh}\ \tau\ \cos\phi)]_+^{\frac{1}{2}(\mu-\nu)-1} \cdot$$
$$\cdot\ \rho^{\alpha+\nu+1}(\mathrm{sh}\ \tau)^\nu (\sin\phi)^{\nu-1} d\rho d\tau d\phi.$$

<u>DEFINITION 8.1.</u>

(8.7)
$$J_{\nu,\alpha}^\mu F(t) := c(\mu,\nu) \int\limits_{\substack{0<\rho<1 \\ \tau>0 \\ 0<\phi<\pi}} F(\tau) \cdot$$
$$\cdot\ [1 + \rho^2 - 2\rho(\mathrm{ch}\ t\ \mathrm{ch}\ \tau - \mathrm{sh}\ t\ \mathrm{sh}\ \tau\ \cos\phi)]_+^{\frac{1}{2}(\mu-\nu)-1}$$
$$\cdot\ \rho^{\alpha+\nu+1}(\mathrm{sh}\ \tau)^\nu (\sin\phi)^{\nu-1}\ d\rho d\tau d\phi,$$

$\mathrm{Re}\,\mu-\nu > 0$, $\mathrm{Re}\,\nu > 0$, F such that the integral converges, $c(\mu,\nu)$ given by (8.2).

8.2. The convolution structure for Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$.

In this section T_t and $*$ will be used for the generalized translation operator and the convolution, respectively, corresponding to the expansion in Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$. As a reference for this convolution structure and other facts about Jacobi functions we used FLENSTED-JENSEN & KOORNWINDER [5]. In [5] the more general case of Jacobi functions of order (α,β) is considered. Though, in all formulas it is possible to take the limit $\beta \downarrow -\frac{1}{2}$.

The Jacobi function ϕ_λ is given by

(8.8) $$\phi_\lambda(t) = \phi_\lambda^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}(t) := {}_2F_1(\tfrac{1}{2}(\tfrac{1}{2}\nu+i\lambda),\tfrac{1}{2}(\tfrac{1}{2}\nu-i\lambda);\tfrac{1}{2}(\nu+1);-\mathrm{sh}^2 t).$$

Other expressions for ϕ_λ are obtained by using the transformation formulas for the hypergeometric function ${}_2F_1$:

(8.9a) $$\phi_\lambda(t) = {}_2F_1(\tfrac{1}{2}\nu+i\lambda,\tfrac{1}{2}\nu-i\lambda;\tfrac{1}{2}(\nu+1);\tfrac{1}{2}(1-\mathrm{ch}\ t))$$

(8.9b) $$= (\mathrm{ch}\ t)^{-\frac{1}{2}\nu-i\lambda}\,{}_2F_1(\tfrac{1}{2}(\tfrac{1}{2}\nu+i\lambda),\tfrac{1}{2}(\tfrac{1}{2}\nu+1+i\lambda);\tfrac{1}{2}(\nu+1);\mathrm{th}^2 t).$$

A definition corresponding to the definition we used for the functions J_β (cf. Section 5.2) is:

The Jacobi function $\phi_\lambda(t)$ is the solution of the differential equation

(8.10) $$E_\nu u(t) = -(\tfrac{1}{4}\nu^2 + \lambda^2)u(t),$$

such that $u(0) = 1$, $u'(0) = 0$. Here

(8.11) $$E_\nu := \partial_{tt} + \nu\,\mathrm{cth}\,t\,\partial_t.$$

The generalized translation operator T_t:

$$T_t F(\tau) := \pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}(\nu+1))[\Gamma(\tfrac{1}{2}\nu)]^{-1} \int_0^\pi F(\operatorname{arch}[\operatorname{ch} t \operatorname{ch} \tau - \operatorname{sh} t \operatorname{sh} \tau \cos\psi]) \cdot$$

$$\cdot (\sin\psi)^{\nu-1} d\psi, \qquad \mathrm{Re}\,\nu > 0$$

has the property:

(8.12) $E_\nu T_t F(\tau) = T_t E_\nu F(\tau)$.

The proof of (8.12) is obtained from

(8.13) $\{\partial_{\tau\tau} + \nu \operatorname{cth} \tau\, \partial_\tau + (\operatorname{sh}\tau)^{-2}(\partial_{\psi\psi} + (\nu-1)\operatorname{ctg}\psi\, \partial_\psi)\}F(A) =$

$= F''(A) + \nu \operatorname{cth} A\, F'(A)$,

where

$$A = \operatorname{arch}[\operatorname{ch} t \operatorname{ch} \tau - \operatorname{sh} t \operatorname{sh} \tau \cos\psi],$$

and

$$\int_0^\pi (\partial_{\psi\psi} + (\nu-1)\operatorname{ctg}\psi\, \partial_\psi)F(A)(\sin\psi)^{\nu-1} d\psi = (\sin\psi)^{\nu-1}\partial_\psi F(A)\Big|_0^\pi = 0.$$

The commutation relation (8.12) results in the product formula for the Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$.

(8.14) $\phi_\lambda(t)\phi_\lambda(\tau) = T_t\phi_\lambda(\tau)$.

The Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$ form a continuous orthogonal set with respect to the weight

(8.15) $d\mu(t) = \pi^{-\frac{1}{2}} 2^{\nu-\frac{1}{2}} (\operatorname{sh} t)^\nu dt$.

The convolution product is defined by

(8.16) $F * G(t) := \int_{\tau>0} F(\tau) T_t G(\tau) d\mu(\tau)$

$$= \pi^{-\frac{1}{2}} 2^{\nu-\frac{1}{2}} \int_{\tau>0} F(\tau) T_t G(\tau) (\operatorname{sh}\tau)^\nu d\tau,$$

for suitable functions F and G. The convolution is both commutative and associative.

Consider the Jacobi transform:

$$(8.17)\qquad F^{\wedge}(\lambda) := \int_0^\infty F(t)\phi_\lambda(t)\,d\mu(t)\,.$$

The inverse transformation and more details can be found in FLENSTED-JENSEN & KOORNWINDER [5]. A corollary of (8.16), (8.17) and (8.14) is:

$$(8.18)\qquad F * \phi_\lambda(t) = F^{\wedge}(\lambda)\cdot\phi_\lambda(t),$$

for suitable functions F.

8.3. The action of $J^{\mu}_{\nu,\alpha}$ on Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$

Using the notations of Subsection 8.2 (thus $*$ is the convolution product for the Jacobi functions of order $(\frac{1}{2}(\nu-1),-\frac{1}{2})$, we can write the integral operator $J^{\mu}_{\nu,\alpha}$ (see (8.7)) in the form:

$$(8.19)\qquad J^{\mu}_{\nu,\alpha}F(t) = F * k(t),$$

where

$$(8.20)\qquad k(t) = c(\mu,\nu)\pi 2^{-\nu+\frac{1}{2}}\Gamma(\tfrac{1}{2}\nu)[\Gamma(\tfrac{1}{2}(\nu+1))]^{-1}\int_0^1 (1+\rho^2-2\rho\ \mathrm{ch}\ t)_+^{\frac{1}{2}(\mu-\nu)-1}\cdot$$

$$\cdot\ \rho^{\alpha+\nu+1}d\rho$$

$$= \frac{2^{-(\mu+2\nu+\alpha-\frac{1}{2})}\pi\Gamma(\alpha+\nu+2)}{\Gamma(\frac{1}{2}\mu)\Gamma(\frac{1}{2}(\nu+1))\Gamma(\frac{1}{2}(\mu+\nu)+\alpha+2)}(\mathrm{ch}\ t)^{-(\alpha+\nu+2)}\cdot$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}(\alpha+\nu+2),\tfrac{1}{2}(\alpha+\nu+3);\tfrac{1}{2}(\mu+\nu)+\alpha+2;(\mathrm{ch}\ t)^{-2}).$$

The second expression for k(t) follows from evaluation of the integral in the first expression:

$$\int_0^1 (1+\rho^2-2\rho \text{ ch } t)^{\frac{1}{2}(\mu-\nu)-1}\rho^{\alpha+\nu+1}d\rho$$

$$= (e^{-t})^{\alpha+\nu+2}\int_0^1 (1-e^{-2t}s)^{\frac{1}{2}(\mu-\nu)-1}(1-s)^{\frac{1}{2}(\mu-\nu)-1}s^{\alpha+\nu+1}ds,$$

and this leads to the given formula when we use:

$${}_2F_1(a,b;c;z) = \Gamma(c)[\Gamma(b)\Gamma(c-b)]^{-1}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$

and

$${}_2F_1(a,b;a-b+1;z) = (1+z)^{-a}{}_2F_1(\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2};a-b+1;4z(1+z)^{-2}),$$

see ERDÉLYI et al. [4].

Combination of (8.18) and (8.19) results in:

THEOREM 8.2.

(8.21) $$J^{\mu}_{\nu,\alpha}\phi_{\lambda}^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}(t) = d(\mu,\nu,\alpha,\lambda)\phi_{\lambda}^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}(t),$$

where

$$d(\mu,\nu,\alpha,\lambda) = k^{\wedge}(\lambda) = 2^{-\mu}\frac{\Gamma(\frac{1}{2}\alpha+1+\frac{1}{2}(\frac{1}{2}\nu+i\lambda))\Gamma(\frac{1}{2}\alpha+1+\frac{1}{2}(\frac{1}{2}\nu-i\lambda))}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\mu+1+\frac{1}{2}(\frac{1}{2}\nu+i\lambda))\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\mu+1+\frac{1}{2}(\frac{1}{2}\nu-i\lambda))},$$

$$\mathbb{R}\text{e }(\mu-\nu) > 0,\ \mathbb{R}\text{e }\nu > 0,\ |\text{Im}\lambda| < \mathbb{R}\text{e }(\alpha+2+\tfrac{1}{2}\nu).$$

PROOF. In order to calculate $k^{\wedge}(\lambda)$, we use the second expression for $k(t)$, $\phi_\lambda(t)$ as given in (8.9b), the power series expansion of ${}_2F_1$, and the following formulas

$$\int_0^\infty (\text{ch } t)^{-2x-2y+1}(\text{sh } t)^{2y-1}dt = \tfrac{1}{2}B(x,y), \quad \mathbb{R}\text{e } x,y > 0,$$

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathbb{R}\text{e }(c-a-b) > 0,$$

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

If we use the expression of k(t) in terms of the hypergeometric function instead of the integral representation as the definition for k(t), then the conditions on the parameters can be weakened to: $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,\nu > -1$, $|\mathrm{Im}\lambda| < \mathrm{Re}\,(\alpha+2+\frac{1}{2}\nu)$. □

8.4. The action of I_ν^μ on the zonal polynomials of James

The zonal polynomial of James $Z^\gamma_{m,\ell}(\xi,\eta)$ is given by the formula:

$$Z^\gamma_{m,\ell}(\xi,\eta) := \frac{(2\gamma+1)_{m-\ell}}{(\gamma+\frac{1}{2})_{m-\ell}}\,\eta^{\frac{1}{2}(m-\ell)}R^{(\gamma,\gamma)}_{m-\ell}(\tfrac{1}{2}\eta^{-\frac{1}{2}}\xi), \tag{8.22}$$

see KOORNWINDER & SPRINKHUIZEN-KUYPER [9]). Here $R^{(\gamma,\gamma)}_{m-\ell}(x)$ is a Jacobi polynomial (see (2.17)). In terms of the coordinates $\xi = x = r\,\mathrm{ch}\,t$ and $\eta = \frac{1}{4}(x^2-y^2) = \frac{1}{4}r^2$, $y = r\,\mathrm{sh}\,t$ and with $\gamma = \frac{1}{2}(\nu-1)$, we obtain

$$Z^\gamma_{m,\ell}(r\,\mathrm{ch}\,t,\tfrac{1}{4}r^2) = \frac{(\nu)_{m-\ell}}{(\frac{1}{2}\nu)_{m-\ell}2^{m+\ell}}\,r^{m+\ell}\,{}_2F_1(-m+\ell,m-\ell+\nu;\tfrac{1}{2}(\nu+1);\tfrac{1}{2}(1-\mathrm{ch}t)) \tag{.23}$$

$$= \frac{(\nu)_{m-\ell}}{(\frac{1}{2}\nu)_{m-\ell}2^{m+\ell}}\,r^{m+\ell}\phi^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}_{i(m-\ell+\frac{1}{2}\nu)}(t),$$

with $i^2 = -1$.

THEOREM 8.3.

$$I^{2\mu}_{2\gamma+1}(x^2-y^2)^\alpha Z^\gamma_{m,\ell}(x,\tfrac{1}{4}(x^2-y^2)) = \tag{8.24}$$

$$= 2^{-2\mu}\,\frac{\Gamma(\ell+\alpha+1)\Gamma(m+\alpha+\gamma+\frac{3}{2})}{\Gamma(\ell+\alpha+\mu+1)\Gamma(m+\alpha+\mu+\gamma+\frac{3}{2})}\,(x^2-y^2)^{\alpha+\mu}Z^\gamma_{m,\ell}(x,\tfrac{1}{4}(x^2-y^2)),$$

$$\mathrm{Re}\,(\mu-\gamma-\tfrac{1}{2}) > 0,\quad \mathrm{Re}\,\gamma > -\tfrac{1}{2},\quad \mathrm{Re}\,\alpha+\ell+1 > 0.$$

PROOF. With $x = r\,\mathrm{ch}\,t$, $y = r\,\mathrm{sh}\,t$, $\gamma = \frac{1}{2}(\nu-1)$ the right hand side of (8.24) is equal to:

$$I_{\nu}^{2\mu} \frac{(\nu)_{m-\ell}}{(\frac{1}{2}\nu)_{m-\ell}2^{m+\ell}} r^{2\alpha+m+\ell} \phi_{i(m-\ell+\frac{1}{2}\nu)}^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}(t) =$$

$$= \frac{(\nu)_{m-\ell}}{(\frac{1}{2}\nu)_{m-\ell}2^{m+\ell}} r^{2\alpha+m+\ell+2\mu} J^{2\mu}_{\nu,2\alpha+m+\ell} \phi_{i(m-\ell+\frac{1}{2}\nu)}^{(\frac{1}{2}(\nu-1),-\frac{1}{2})}(t)$$

$$= d(2\mu,\nu,2\alpha+m+\ell,i(m-\ell+\tfrac{1}{2}\nu))(x^2-y^2)^{2\alpha+2\mu} Z^{\gamma}_{m,\ell}(x,\tfrac{1}{4}(x^2-y^2)).$$

Here we used (8.6), (8.7) and Theorem 8.2. The conditions on the parameters also follow Theorem 8.2. □

Substitution of (8.24) in (7.3) results in (7.2) and so (7.2) holds for $\mathrm{Re}\,\gamma > -\frac{1}{2}$, $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,(\mu-\gamma-\frac{1}{2}) > 0$.

9. SOME REMARKS

REMARK 9.1. Because of the commutativity of the convolution in (6.14) (see also (8.1)), the action of I_{ν}^{μ} ($\mathrm{Re}\,\mu > \mathrm{Re}\,\nu > 0$) on a function f which satisfies the conditions in Definition 6.3 can be written as:

(9.1) $$I_{\nu}^{\mu} f(x,y) = c(\mu,\nu) \int\limits_{\substack{\xi<x \\ 0<\phi<\pi}} f(\xi,\eta) \cdot$$

$$\cdot [(x-\xi)^2 - y^2 - \eta^2 - 2y\eta\cos\phi]_+^{\frac{1}{2}(\mu-\nu)-1} \cdot$$

$$\cdot \eta^{\nu}(\sin\phi)^{\nu-1}\, d\xi d\eta d\phi,$$

$$[c(\mu,\nu)]^{-1} = 2^{\mu-2}\Gamma(\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-\nu)).$$

Integration over ϕ leads to:

(9.2a) $$I_{\nu}^{\mu} f(x,y) = \int f(\xi,\eta) K_{\nu}^{\mu}(x,y;\xi,\eta)\, d\xi d\eta,$$

where

(9.2b) $$K_\nu^\mu(x,y;\xi,\eta) = \frac{2^{1-\mu}}{\{\Gamma(\frac{1}{2}\mu)\}^2}[(x-\xi)^2-(y-\eta)^2]^{\frac{1}{2}\mu-1}y^{-\frac{1}{2}\nu}\eta^{\frac{1}{2}\nu}\cdot$$

$$\cdot\ {}_2F_1(1-\tfrac{1}{2}\nu,\tfrac{1}{2}\nu;\tfrac{1}{2}\mu;[(x-\xi)^2-(y-\eta)^2](4y\eta)^{-1}),$$

$$\text{if } \xi-\eta < x-y < \xi+\eta \wedge x+y > \xi+\eta, \quad y,\eta > 0,$$

(9.2c) $$K_\nu^\mu(x,y;\xi,\eta) = \frac{2^{2-\mu}\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}\mu)\Gamma(\frac{1}{2}(\nu+1))\Gamma(\frac{1}{2}(\mu-\nu))}[(x-\xi)^2-(y-\eta)^2]^{\frac{1}{2}(\mu-\nu)-1}$$

$$\cdot\ \eta^\nu {}_2F_1(1-\tfrac{1}{2}(\mu-\nu),\tfrac{1}{2}\nu;\nu;4y\eta[(x-\xi)^2-(y-\eta)^2]^{-1}),$$

$$\text{if } x-y > \xi+\eta, \quad y,\eta > 0,$$

(9.2d) $$K_\nu^\mu(x,y;\xi,\eta) = 0, \qquad \text{elsewhere.}$$

If f is a continuous function with its support in L_+ or S_+ then the integral in (9.2a) converges if

(9.3) $$\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\mu+\nu > 0 \quad \text{and} \quad \mathrm{Re}\,\mu-\nu > 0.$$

Because K_ν^μ depends analytically on ν and is continuous in ξ and η outside a set of measure zero, formula (9.2a) leads to an extension of the operator I_ν^μ to the region $\mathrm{Re}\,\nu > -1$. It is defined for functions f which satisfy the conditions in Definition 6.3, for $\mathrm{Re}\,\nu > -1$.

REMARK 9.2. From the original definition of I_ν^μ (cf. (8.1)) it is clear that I_ν^μ is an operator with a positive kernel if $\mu > \nu > 0$. The sign of K_ν^μ in (9.2) for $\nu > -1$ depends on the sign of the hypergeometric functions in (9.2b) and (9.2c). Using similar techniques as GASPER [6] used in order to find the sign of the convolution kernel for the Jacobi polynomials, we find:

(9.4) $$K_\nu^\mu(x,y;\xi,\eta) \geq 0 \qquad \text{if} \quad 0 > \nu > -1 \wedge \mu+\nu \geq 2$$
$$\text{or if } \mu > \nu \geq 0.$$

and $K^{\mu}_{\nu}(x,y;\xi,\eta)$ is both positive and negative if $-1 < \nu < 0$ and $0 < \mu+\nu < 2$.

REMARK 9.3. A corollary of Remark 9.1 is that (7.2) holds as a classical integral for

(9.5) $\quad \mathbb{R}e\,\gamma > -1, \quad \mathbb{R}e\,(\alpha+1) > 0, \quad \mathbb{R}e\,(\mu-\gamma-\frac{1}{2}) > 0, \quad \mathbb{R}e\,(\alpha+\gamma+\frac{3}{2}) > 0.$

From Remark 9.2 we see that $I^{2\mu}_{2\gamma+1}$ is an operator with a positive kernel if

(9.6a) $\quad \mu+\gamma-\frac{1}{2} \geq 0 \quad$ and $\quad -1 < \gamma < -\frac{1}{2},$

or if

(9.6b) $\quad \mu > \gamma+\frac{1}{2} \geq 0,$

see figure 1.

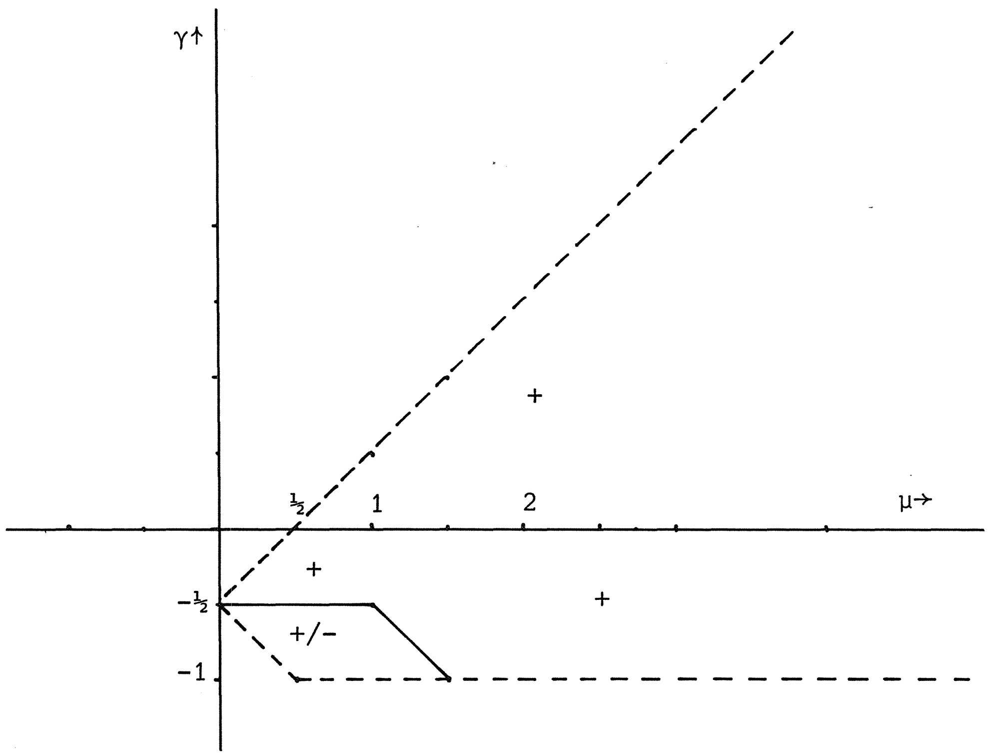

Figure 1. The sign of $I^{2\mu}_{2\gamma+1}$.

REMARK 9.4. Corollaries 6.7 and 6.8 give the action of the operators ∂_x, $\partial_{yy} + \nu y^{-1}\partial_y$ and D_ν on $I_\nu^\mu f$. The following relation can also be useful:

$$y^{-1}\partial_y I_\nu^\mu f(x,y) = I_{\nu+2}^\mu y^{-1}\partial_y f(x,y), \tag{9.7}$$

for suitable functions f.

For μ an integer, relation (9.7) is a corollary of

$$y^{-1}\partial_y D_\nu = D_{\nu+2} y^{-1}\partial_y.$$

For other values of μ (9.7) can be proved by differentiation of (9.2).

REMARK 9.5. If $\nu \downarrow 0$ in (9.2) then we obtain:

$$I_0^\mu f(x,y) = \int f(\xi,\eta) K_0^\mu(x,y;\xi,\eta)\,d\xi d\eta, \tag{9.8a}$$

where

$$K_0^\mu(x,y;\xi,\eta) = 2^{1-\mu}[\Gamma(\tfrac{1}{2}\mu)]^{-2}[(x-\xi)^2 - (y-\eta)^2]^{\frac{1}{2}\mu-1}, \tag{9.8b}$$

$$\text{if } \xi-\eta < x-y < \xi+\eta \wedge x+y > \xi+\eta, \quad y,\eta > 0,$$

$$K_0^\mu(x,y;\xi,\eta) = 2^{1-\mu}[\Gamma(\tfrac{1}{2}\mu)]^{-2}\{[(x-\xi)^2 - (y-\eta)^2]^{\frac{1}{2}\mu-1} + \tag{9.8c}$$

$$+ [(x-\xi)^2 - (y+\eta)^2]^{\frac{1}{2}\mu-1}\}, \quad \text{if } x-y > \xi+\eta, \quad y,\eta > 0,$$

$$K_0(x,y;\xi,\eta) = 0, \qquad \text{elsewhere.} \tag{9.8d}$$

Formula (9.8b) is the Riesz kernel for $n = 2$. Formula (9.8c) is obtained from (9.8b) by adding the kernel with η replaced by $-\eta$. Formally this relation also holds for $\nu \neq 0$.

REMARK 9.6. R.M. DAVIS [2] solved the initial value problem for the operator

$$-L = \sum_{i=1}^{n-1} \partial_{x_i x_i} - \partial_{yy} - \nu y^{-1}\partial_y$$

with initial conditions for $y = y_0 > 0$. The Riesz kernel which she found ([2, p. 211]) is:

$$V^{\mu}(x_i,y;\xi_i,\eta) = 2^{1-\mu}\pi^{1-\frac{1}{2}n}[\Gamma(\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}(\mu-n)+1)]^{-1}\eta^{\frac{1}{2}\nu}y^{-\frac{1}{2}\nu}\cdot$$

$$\cdot\ [(y-\eta)^2-(x_1-\xi_1)^2 - \ldots - (x_{n-1}-\xi_{n-1})^2]^{\frac{1}{2}(\mu-n)}\cdot$$

$$\cdot\ {}_2F_1(1-\tfrac{1}{2}\nu,\tfrac{1}{2}\nu;\tfrac{1}{2}(\mu-n)-1;[(x_1-\xi_1)^2+\ldots+(x_{n-1}-\xi_{n-1})^2-(y-\eta)^2](4y\eta)^{-1}).$$

If $n = 2$ this kernel is equal to the kernel in (9.2b), besides a complex factor since she considers a different region of the plane.

REMARK 9.7. Suppose that $f(x,y)$ only depends on the Lorentz distance $r = \sqrt{x^2-y^2}$ and that the admissible line S is the sheet $x < 0$ of the hyperbola $x^2 - y^2 = 1$. The corresponding neighbourhood K can be chosen as the interior of L_-. In that case the fractional integral operator for $D_\nu = \partial_{xx} - \partial_{yy} - \nu y^{-1}\partial_y$ corresponds to the fractional integral operator for $d^2/dr^2 + (\nu+1)r^{-1}\, d/dr$, which is given in [16].

ACKNOWLEDGEMENT.

I am grateful to Tom Koornwinder since his suggestions led to many improvements in this paper.

REFERENCES

[1] ASKEY, A., *Orthogonal Polynomials and Special Functions,* Reg. Conf. Ser. Appl. Math. 21, Soc. Ind. Appl. Math., Philadelphia, Pennsylvania, 1975.

[2] DAVIS, R.M., *On a regular Cauchy problem for the Euler-Poisson-Darboux equation,* Ann. Mat. Pura Appl. IV, 42 (1956) 205-226.

[3] ERDÉLYI, A., *Fractional integrals of generalized functions,* in: "Fractional Calculus and its Applications" (B. Ross, ed.), Lecture Notes in Math. 457, Springer-Verlag, Berlin, 1975.

[4] ERDÉLYI, A., W. MAGNUS, F. OBERHETTINGER & F.G. TRICOMI, *Higher Transcendental Functions,* Vol. I, McGraw-Hill, New York, 1953.

[5] FLENSTED-JENSEN, M. & T.H. KOORNWINDER, *The convolution structure for Jacobi function expansions,* Ark. Mat. 11 (1973) 245-262.

[6] GASPER, G., *Banach algebras for Jacobi series and positivity of a kernel,* Ann. Math. 95 (1972) 261-280.

[7] GELFAND, I.M. & G.E. SHILOV, *Generalized Functions,* Vol. I, Academic Press, New York, 1964.

[8] JAGER, E.M. de, *Theory of distributions,* in: "*Mathematics Applied to Physics*", (É. Roubine, ed.), Springer-Verlag, Berlin, 1970.

[9] KOORNWINDER, T.H. & I.G. SPRINKHUIZEN-KUYPER, *Generalized power series expansions for a class of orthogonal polynomials in two variables,* SIAM J. Math. Anal. 9 (1978) 457-483.

[10] MUKHLISOV, F.G., *Existence and uniqueness of a solution for certain partial differential equations with the Bessel differential operator,* II, Izv. VUZ. Mat. 22 (1978) 58-63.

[11] OLDHAM, K.B. & J. SPANIER, *The Fractional Calculus,* Academic Press, New York, 1974.

[12] RINGHOFER, K., *Koornwinder's polynomials and representations of the conformal group,* J. Math. Phys., to appear.

[13] RIESZ, M., *L'intégrale de Riemann-Liouville et le problème de Cauchy,* Acta. Math. 81 (1949) 1-223.

[14] SCHWARTZ, A., *The structure of the algebra of Hankel transforms and the algebra of Hankel-Stieltjes transforms,* Can. J. Math. 23 (1971) 236-246.

[15] SCHWARTZ, L., *Théorie des Distributions,* Hermann, Paris, 1966.

[16] SPRINKHUIZEN-KUYPER, I.G., *A fractional integral operator corresponding to negative powers of a certain second order differential operator,* Rep. TW 182, Math. Centrum, Amsterdam, 1978, to appear in: J. Math. Anal. Appl.

[17] SPRINKHUIZEN-KUYPER, I.G., *Orthogonal polynomials in two variables. A further analysis of the polynomials orthogonal over a region bounded by two lines and a parabola,* SIAM J. Math. Anal. 7 (1976) 501-518.

[18] TITCHMARSH, E.C. , *Introduction to the Theory of Fourier Integrals,* Oxford University Press, Oxford, 1937.

[19] TITCHMARSH, E.C., *The Theory of Functions,* Oxford University Press, Oxford, 1939.

[20] TRÈVES, F., *Topological Vector Spaces, Distributions and Kernels,* Academic Press, New York, 1967.

[21] WATSON, G.N., *A Treatise on the Theory of Bessel Functions,* Cambridge University Press, Cambridge, 1944.