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O. DIEKMANN & D. HILHORST

HOW MANY JUMPS? VARIATIONAL CHARACTERIZATION OF THE LIMIT SOLUTION OF A SINGULAR PERTURBATION PROBLEM

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How many jumps? Variational characterization of the limit solution of a singular perturbation problem*)

by

O. Diekmann & D. Hilhorst

ABSTRACT

Using two alternative methods we describe the limiting behaviour, as $\varepsilon \downarrow 0$, of the solution y_ε of the nonlinear two-point boundary value problem $\varepsilon y'' + (g-y)y' = 0$, $y(0) = 0$, $y(1) = 1$, where g is a given function. The first method is based on the theory of maximal monotone operators, whereas the second one uses duality theory.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem, maximal monotone operator, convex analysis, duality theory*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Consider the two-point boundary value problem

$$\text{BVP} \qquad \begin{aligned} &\varepsilon y'' + (g-y)y' = 0, \\ &y(0) = 0, \quad y(1) = 1, \end{aligned}$$

where $g \in L_2 = L_2(0,1)$ is a given function and $y \in H^2$ is unknown. As we shall show, there exists for each $\varepsilon > 0$ a unique solution y_ε, which is increasing. We are interested in the limiting behaviour of y_ε as $\varepsilon \downarrow 0$.

Motivated by a physical application we previously studied a similar problem in a joint paper with L.A. Peletier [2]. Using the maximum principle as our main tool we were able to establish the existence of a unique limit solution y_0 under certain, physically reasonable, assumptions on the function g. In some cases we could characterize y_0 completely, in others, however, some ambiguity remained.

Here, inspired by the work of Grasman & Matkowsky [4], we shall resolve this ambiguity by using a variational formulation of the problem. In fact we shall present two different methods of analysis. The first one is based on the theory of maximal monotone operators, whereas the second one uses duality theory.

During our investigation of BVP we experienced that it could serve as a fairly simple, yet nontrivial, illustration of concepts and methods from abstract functional analysis. In order to demonstrate this aspect of the problem we shall spell out our arguments in some more detail than is strictly necessary.

The organization of the paper is as follows. In Section 2 we present the first method. We prove, by means of Schauder's fixed point theorem, that BVP has a solution y_ε for each $\varepsilon > 0$. Moreover, we show that BVP is equivalent to an abstract equation AE, involving a maximal monotone operator A, and to a variational problem VP, involving a convex, lower semi-continuous functional W. Subsequently we exploit these formulations in the investigation of the limiting behaviour of y_ε as $\varepsilon \downarrow 0$. (The idea of using the theory of maximal monotone operators was suggested to us by Ph. Clément.) It turns out that y_ε converges in L_2 to a limit y_0. Moreover, y_0 is abstractly characterized as the projection (in L_2) of g on $\overline{\mathcal{D}(A)}$. We conclude this section with some results about uniform convergence under restrictive assumptions.

In Section 3 we study a minimization problem P related to VP. We begin by proving that P has a unique solution. Next, we present a dual problem P^* and we deduce from the extremality relations between primal and dual problems that P and BVP are equivalent. Putting $\varepsilon = 0$ in P^* we obtain a formal limit problem P_0^*. Subsequently we associate with P_0^* a dual problem P_0^{**} and we show that the solution of P tends, as $\varepsilon \downarrow 0$, to the solution of P_0^{**}. A rewriting of P_0^{**} reveals the relation with the result of Section 2. This treatment of the problem has grown out of conversations with R. Témam who, in particular, indicated to one of us the appropriate functional analytic setting for studying the minimization problem.

In Section 4 we give concrete form to the characterization of y_0. In particular we present sufficient conditions for a function to be y_0 and we show, by means of examples, how these criteria can be used in concrete cases. The first part of the title originated from Example 4.

In Section 5 we make various remarks about generalizations and limitations of our approach.

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2. THE FIRST METHOD

2.1. THREE EQUIVALENT FORMULATIONS

In order to demonstrate the existence of a solution of BVP, let us first look at the auxiliary problem

$$u'' + (g-w)u' = 0,$$

$$u(0) = 0, \quad u(1) = 1,$$

where $w \in L_2$ is a given function. The solution of this linear problem is given explicitly by

$$u(x) = C(w) \int_0^x \exp\left(\int_0^\zeta (w(\xi) - g(\xi))d\xi\right)d\zeta$$

with

$$C(w) = \left(\int_0^1 \exp\left(\int_0^\zeta (w(\xi) - g(\xi))d\xi\right)d\zeta\right)^{-1}.$$

From this expression it can be concluded that $u' > 0$ and $0 \le u \le 1$. So if we write $u = Tw$, then T is a compact map of the closed convex set $\{w \in L_2 \mid 0 \le w \le 1\}$ into itself and hence, by Schauder's theorem, T must have a fixed point. Clearly this fixed point corresponds to a solution of BVP. Thus we have proved

PROPOSITION 2.1. *For each* $\varepsilon > 0$ *there exists a solution* $y_\varepsilon \in H^2$ *of* BVP. *Moreover, any solution* $y \in H^2$ *satisfies* (i) $y' > 0$ *and* (ii) $0 \le y \le 1$.

The a priori knowledge that y' is positive allows us to divide the equation by y'. In this manner we are able to reformulate the boundary value problem as an equivalent abstract equation

$$\text{AE} \qquad (I + \varepsilon A)y = g$$

where the (unbounded, nonlinear) operator $A: \mathcal{D}(A) \to L_2$ is defined by

$$Au = -\frac{u''}{u'} = -(\ell n\, u')' \tag{2.1}$$

with

$$\mathcal{D}(A) = \{u \in L_2 \mid u \in H^2,\ u' > 0,\ u(0) = 0,\ u(1) = 1\}. \tag{2.2}$$

PROPOSITION 2.2. *The operator* A *is monotone. Hence the solution of* AE *(and* BVP*) is unique.*

PROOF. Let $u_i \in \mathcal{D}(A)$ for $i = 1,2$ then

$$(Au_1 - Au_2, u_1 - u_2) = -\int ((\ell n\, u_1')' - (\ell n\, u_2')')(u_1 - u_2)$$

$$= \int (\ell n\, u_1' - \ell n\, u_2')(u_1' - u_2') \ge 0$$

(because $z \mapsto \ell n\, z$ is monotone on $(0,\infty)$; note that here and in the following we write $\int\phi$ to denote $\int_0^1 \phi(x)dx$.) Next, suppose $\varepsilon Ay_i = g - y_i$, $i = 1,2$, then $0 \le \varepsilon(Ay_1 - Ay_2, y_1 - y_2) = (g - y_1 - g + y_2, y_1 - y_2) = -\|y_1 - y_2\|^2$ and hence $y_1 = y_2$. $\square$

We recall that a monotone operator A defined on a Hilbert space H is called *maximal monotone* if it admits no proper monotone extension (i.e., it is maximal in the sense of inclusion of graphs). It is well known that A is maximal monotone if and only if $\mathcal{R}(I + \varepsilon A) = H$ for each $\varepsilon > 0$ (see Brézis [1]). In our case, with $H = L_2$ and A defined in (2.1), this is just a reformulation of the existence result Proposition 2.1. Consequently we know

PROPOSITION 2.3. A *is maximal monotone.*

In search for yet another formulation let us write the equation in the form

$$-\varepsilon(\ell n\, y')' + y - g = 0$$

Hence, for any $\phi \in H_0^1$,

$$\varepsilon \int \phi'(\ell n\, y' + 1) + \int \phi(y-g) = 0.$$

Motivated by this calculation we define a functional $W\colon L_2 \to \overline{\mathbb{R}}$ by

$$W(u) = \varepsilon\Psi(u) + \frac{1}{2}\|u-g\|^2 \tag{2.3}$$

where

$$\Psi(u) = \begin{cases} \int u' \,\ell n\, u' & \text{if } u \in \mathcal{D}(\Psi), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\mathcal{D}(\Psi) = \{u \in L_2 \mid u \text{ is AC, } u' \geq 0,\ u'\,\ell n\, u' \in L_1,\ u(0) = 0,\ u(1) = 1\} \tag{2.5}$$

(here AC means absolutely continuous). Also we define a variational problem

VP $\qquad \operatorname{Inf}_{L_2} W.$

We note that the mappings $z \mapsto z\,\ell n\, z$ and $z \mapsto z^2$ are (strictly) convex (on $[0,\infty)$ and $(-\infty,\infty)$ respectively) and that W inherits this property because $\mathcal{D}(\Psi)$ is convex as well. Hence VP has at most one solution. For future use we observe that the convexity of $z \mapsto z\,\ell n\, z$ implies, for $z \geq 0$ and $\zeta > 0$, the inequality

$$z\,\ell n\, z - \zeta\,\ell n\,\zeta \geq (1+\ell n\,\zeta)(z-\zeta).$$

PROPOSITION 2.4. y_ε *solves* VP.

PROOF. Firstly we note that $y_\varepsilon \in \mathcal{D}(\Psi)$. So for any $u \in \mathcal{D}(\Psi)$

$$\begin{aligned} W(u) - W(y_\varepsilon) &= \varepsilon \int (u'\,\ell n\, u' - y_\varepsilon'\,\ell n\, y_\varepsilon') + \frac{1}{2}\|u-g\|^2 - \frac{1}{2}\|y_\varepsilon - g\|^2 \\ &\geq \varepsilon \int (1+\ell n\, y_\varepsilon')(u'-y_\varepsilon') + \int (y_\varepsilon - g)(u-y_\varepsilon) \\ &= \int \left(-\varepsilon \frac{y_\varepsilon''}{y_\varepsilon'} + y_\varepsilon - g\right)(u-y_\varepsilon) = 0. \qquad \square \end{aligned}$$

We recall that the *subgradient* $\partial\Psi$ of the convex functional Ψ is defined by

$$\partial\Psi(u) = \{\zeta \in L_2 \mid \Psi(v) - \Psi(u) \geq (\zeta, v-u),\ \forall v \in \mathcal{D}(\Psi)\}.$$

A calculation like the one above shows that, for $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(\Psi)$,

$$\Psi(v) - \Psi(u) \geq (Au, v-u).$$

Hence $A \subset \partial\Psi$, but, since $\partial\Psi$ is monotone and A is *maximal* monotone, we must have $A = \partial\Psi$. Likewise it follows that $\partial W = \varepsilon A + I - g$. These observations should clarify the relation between VP and AE.

One can show that Ψ (and hence W as well) is lower semicontinuous and subsequently one can use this knowledge to give a direct variational proof of the existence of a solution of VP. We refer to Theorem 3.2. below for a detailed proof of this result.

We summarize the main results of this subsection in the following theorem.

THEOREM 2.5. *The problems* BVP, AE *and* VP *are equivalent. In fact, for each* $\varepsilon > 0$, *there exists* $y_\varepsilon \in \mathcal{D}(A)$ *which solves each problem and no problem admits any other solution.*

2.2. LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

The fact that y_ε solves AE can be expressed as

$$y_\varepsilon = (I + \varepsilon A)^{-1} g.$$

Subsequently, the observation that A is maximal monotone provides a key to describing the limiting behaviour. For, it is known from the general theory of such operators (see Brézis [1, Section II.4, in particular Th. 2.2]) that

$$\lim_{\varepsilon \downarrow 0} (I + \varepsilon A)^{-1} g = \mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g,$$

where the expression at the right-hand side denotes the projection (in the sense of the underlying Hilbert space, hence L_2 in this case) of g on the closed convex set $\overline{\mathcal{D}(A)}$, or, in other words,

$$\mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g = y_0$$

where y_0 denotes the unique solution of the variational problem

$$\mathrm{Min}_{\overline{\mathcal{D}(A)}}\ W_0$$

with

$$W_0(u) = \|u-g\|^2.$$

Below we shall give a proof of this result for this special case, using techniques as in Brézis' book, but exploiting the fact that A is the subdifferential of the functional Ψ.

THEOREM 2.6.

$$\lim_{\varepsilon\downarrow 0} \|y_\varepsilon - y_0\| = 0.$$

PROOF. First of all we note that $\|y_\varepsilon\| \le 1$. We shall split the proof into three steps.

Step 1. Take any $z \in \mathcal{D}(A)$ then from

$$\Psi(y_\varepsilon) - \Psi(z) \ge (Az,\ y_\varepsilon - z)$$

it follows that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) \ge 0.$$

Step 2. By definition,

$$0 \ge W(y_\varepsilon) - W(z) = \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) + \tfrac{1}{2}\|g-y_\varepsilon\|^2 - \tfrac{1}{2}\|g-z\|^2.$$

Hence

$$\limsup_{\varepsilon\downarrow 0} \|g-y_\varepsilon\|^2 \le \|g-z\|^2, \qquad \forall z \in \mathcal{D}(A).$$

But then, in fact, the same must hold for all $z \in \overline{\mathcal{D}(A)}$.

Step 3. Since $\|y_\varepsilon\| \le 1$, $\{y_\varepsilon\}$ is weakly precompact in L_2. Take any $\{\varepsilon_n\}$ and $\tilde{y}$ such that $y_{\varepsilon_n} \rightharpoonup \tilde{y}$ in L_2, then

$$(*) \qquad \|g-\tilde{y}\|^2 \le \liminf_{n\to\infty} \|g-y_{\varepsilon_n}\|^2 \le \limsup_{n\to\infty} \|g-y_{\varepsilon_n}\|^2 \le \|g-z\|^2, \qquad \forall z \in \overline{\mathcal{D}(A)}.$$

Consequently $\tilde{y} = y_0$, which shows that the limit does not depend on the subsequence under consideration. Hence $y_\varepsilon \rightharpoonup y_0$. Finally, by taking $z = y_0$ in (*) it follows that in fact $y_\varepsilon \to y_0$. □

We note that

$$\overline{\mathcal{D}(A)} = \{u \in L_2 \mid u \text{ is nondecreasing, } 0 \le u \le 1\}.$$

So in general y_0 need not be continuous (nor does it need to satisfy the boundary conditions). However it is possible, as our next result shows, to establish uniform convergence to a continuous limit at the price of some conditions on g.

THEOREM 2.7. *Suppose* $g \in C^1$, $g(0) < 0$ *and* $g(1) > 1$. *Then* $y_0 \in C$ *and*

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le 1} |y_\varepsilon(x) - y_0(x)| = 0.$$

PROOF. The idea is to derive a uniform bound for y_ε'. We know already that $y_\varepsilon' > 0$ and we are going to show that $y_\varepsilon' \le \sup g'$. To this end we first observe that $g(0)-y_\varepsilon(0)<0$, and $g(1)-y_\varepsilon(1) > 0$, which, combined with the differential equation, shows that $y_\varepsilon''(0) > 0$ and $y_\varepsilon''(1) < 0$. Hence y_ε' assumes its maximum in an interior point, say $\bar{x}$. Next, differentiation of the differential equation followed by substitution of $y_\varepsilon''(\bar{x}) = 0$, $y_\varepsilon'''(\bar{x}) \le 0$, leads to the conclusion that $y_\varepsilon'(\bar{x}) \le g'(\bar{x})$. The uniform bound for y_ε' implies, by virtue of the Arzela-Ascoli theorem, that the limit set of $\{y_\varepsilon\}$ in the space of continuous functions is nonempty. Combination of this result with Theorem 2.6 leads to the desired conclusion. $\square$

In Section 4 we shall show that y_0 can be calculated in many concrete examples. Quite often it will turn out that y_0 is continuous (or piece-wise continuous). This motivates our next result.

THEOREM 2.8. *Suppose* y_0 *is continuous. Then* y_ε *converges to* y_0 *uniformly on compact subsets of* $(0,1)$.

PROOF. Let $I \subset (0,1)$ be a compact set. Put $\beta(\varepsilon) = \max\{y_\varepsilon(x)-y_0(x) \mid x \in I\}$ and let $\bar{x}(\varepsilon) \in I$ be such that $y_\varepsilon(\bar{x}(\varepsilon)) - y_0(\bar{x}(\varepsilon)) = \beta(\varepsilon)$. Suppose $\limsup_{\varepsilon\downarrow 0} \beta(\varepsilon) = \beta > 0$ and let $\{\varepsilon_n\}$ be such that $\beta(\varepsilon_n) \to \beta$ as $n \to \infty$. Choose $\delta \in (0,\delta_1)$, where δ_1 denotes the distance of 1 to I, such that $|y_0(x) - y_0(\xi)| \le \frac{1}{4}\beta$ if $|x-\xi| \le \delta$. Also, choose n_0 such that $\beta(\varepsilon_n) \ge \frac{3}{4}\beta$ for $n \ge n_0$. Then for $x \in [\bar{x}(\varepsilon_n), \bar{x}(\varepsilon_n)+\delta]$ and $n \ge n_0$ the following inequality holds:

$$\begin{aligned} y_{\varepsilon_n}(x) - y_0(x) &\ge y_{\varepsilon_n}(\bar{x}(\varepsilon_n)) - y_0(\bar{x}(\varepsilon_n)) + y_0(\bar{x}(\varepsilon_n)) - y_0(x) \\ &\ge \frac{3}{4}\beta - \frac{1}{4}\beta = \frac{1}{2}\beta. \end{aligned}$$

However, this leads to

$$\|y_{\varepsilon_n} - y_0\|^2 \ge \frac{1}{4}\delta\beta^2$$

which is in contradiction with Theorem 2.6. Hence our assumption $\beta > 0$ must be false and we arrive at the conclusion that $\limsup_{\varepsilon\downarrow 0} \max\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \le 0$. Essentially the same argument yields that $\liminf_{\varepsilon\downarrow 0} \min\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \ge 0$. Taking both statements together yields the result. □

It should be clear that appropriate analogous results can be proved if y_0 is piece-wise continuous. In Theorem 2.8 the sense of convergence is sharpened "a posteriori", that is, once the continuity of y_0 is established by other means. Note that our proof exploits the uniform one-sided bound $y_\varepsilon' > 0$.

3. THE SECOND METHOD

3.1. A VARIATIONAL EXISTENCE PROOF

In this section we study in some detail a minimization problem P which is a variant of VP. We shall use methods from convex analysis. In fact, our presentation follows closely Ekeland & Témam [3, Chapter III, Section 4] and in order to bring this out clearly we begin by introducing some notation in accordance with this reference. We define

$$V = AC = \{v \in L_2 \mid v' \in L_1\} \tag{3.1}$$

and we consider V as a Banach space provided with the norm

$$\|v\|_V = \|v\|_{L_2} + \|v'\|_{L_1}. \tag{3.2}$$

We denote by V^* the dual space of V. Next, we introduce $Y = L_1 \times L_2$ and a bounded linear mapping $\Lambda: V \to Y$ defined by

$$\Lambda v = (\Lambda_1 v, \Lambda_2 v) = (v', v). \tag{3.3}$$

Moreover, we introduce functionals G_1, G_2 and F defined on L_1, L_2 and V, respectively, as follows

$$G_1(w) = \begin{cases} \varepsilon \int w \,\ell n\, w + \frac{\varepsilon}{e} & \text{if } w \geq 0 \text{ and } w \,\ell n\, w \in L_1, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

$$G_2(w) = \frac{1}{2} \int (g - w)^2, \tag{3.5}$$

$$F(w) = \begin{cases} 0 & \text{if } w(0) = 0 \text{ and } w(1) = 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Finally, we call P the minimization problem

$$P \qquad \operatorname{Inf}_V J \tag{3.7}$$

where by definition

$$J(v) = G_1(\Lambda_1 v) + G_2(\Lambda_2 v) + F(v). \tag{3.8}$$

Clearly G_2 is (strictly) convex and lower semicontinuous (l.s.c.); consequently it is weakly lower semicontinuous (w.l.s.c.) as well (cf. [3, p. 10]). The next result shows that the same conclusion holds for G_1.

PROPOSITION 3.1. *G_1 is convex and w.l.s.c..*

PROOF. Let the function $k\colon \mathbb{R} \to \overline{\mathbb{R}}$ be defined by

$$k(y) = \begin{cases} \varepsilon y \,\ell n\, y + \frac{\varepsilon}{e} & \text{if } y \ge 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

Then k is Borel measurable, l.s.c. and positive. Hence, in other words, it is a normal positive integrand (cf. [3, p. 216]). Rewriting G_1 as

$$G_1(w) = \int k(w(\cdot)), \tag{3.10}$$

we observe that the l.s.c. of k and Fatou's lemma imply that G_1 is l.s.c.:

$$\int k(w(\cdot)) \le \int \liminf_{m\to\infty} k(w_m(\cdot)) \le \liminf_{m\to\infty} \int k(w_m(\cdot))$$

whenever $w_m \to w$ strongly in L_1. Since obviously G_1 is convex the result follows. □

THEOREM 3.2. *For each $\varepsilon > 0$, P has a unique solution.*

PROOF. First we note that the functional J is bounded from below on V. Let $\{u_m\}$ be a minimizing sequence. We intend to show that $\{u_m\}$ is bounded in L_2 and that $\{u_m'\}$ is bounded in L_1 and equi-integrable (cf. [3, p. 223]). Indeed, from

$$\varepsilon \int u_m' \,\ell n\, u_m' + \frac{\varepsilon}{e} + \frac{1}{2}\int (g-u_m)^2 \le C$$

we deduce that $u_m' \ge 0$, that

$$\int u_m^2 \le C_1, \quad \int u_m' \le C_2$$

and that

$$\int_{\Omega(M)} u_m' \le (\ell n\, M)^{-1} \int_{\Omega(M)} u_m' \,\ell n\, u_m' \le (\ell n\, M)^{-1} \frac{C}{\varepsilon}$$

where $\Omega(M) = \{x \mid u_m'(x) \ge M\}$ and $M > 1$. Thus, given any constant $\delta > 0$, we have that

$$\int_{\Omega(M)} u_m' \le \delta$$

provided $M > \exp \frac{C}{\varepsilon\delta}$.

We conclude that $\{u_m\}$ is weakly precompact in L_2 and that $\{u_m'\}$ is weakly precompact in L_1 (cf. [3, p. 223]). If $u_m \rightharpoonup u$ in L_2 and $u_m' \rightharpoonup w$ in L_1, then the usual manipulations with distributional derivatives show that $u' = w$, and consequently that $u \in V$. Moreover, from $u_m(x) = \int_0^x u_m'(\xi)d\xi$ we deduce that $u(x) = \int_0^x u'(\xi)d\xi$ and thus that $u(0) = 0$. Likewise it follows that $u(1) = 1$. So $F(u) = 0$. Since the functionals G_1 and G_2 are w.l.s.c. on L_1 and L_2 respectively, it follows that $u = u_\varepsilon$ is a solution of P. Since, furthermore, J is strictly convex the solution is unique. □

3.2. THE DUAL PROBLEM

In Subsection 2.1 we proved the equivalence of BVP and VP by showing that the solution of BVP (whose existence was proven first) also solves VP. Here we want to go the other way around, i.e., we want to show that the solution of P also solves BVP. In order to do so we shall first determine a dual problem and subsequently we shall utilize the extremality relations.

We embed P into a wider class of perturbed problems P(p) as follows:

$$(3.11)\qquad P(p)\qquad \operatorname{Inf}_V \Phi(\cdot,p)$$

where $p = (p_1,p_2) \in Y$ and where by definition

$$(3.12)\qquad \Phi(v,p) = G_1(\Lambda_1 v - p_1) + G_2(\Lambda_2 v - p_2) + F(v).$$

With respect to these perturbations the dual problem P^* is given by (cf. [3, Section III.4])

$$(3.13)\qquad P^*\qquad \operatorname{Sup}_{Y^*} - \Phi^*(0,\cdot),$$

where $Y^* = L_\infty \times L_2$ and Φ^* is the polar function of Φ, that is

$$(3.14)\qquad \Phi^*(v^*,p^*) = \sup\{\langle v^*,v\rangle_V + \langle p^*,p\rangle_Y - \Phi(v,p) \mid v \in V,\ p \in Y\}.$$

Hence,

$$(3.15)\qquad \Phi^*(0,p^*) = \sup\{\langle p^*,p\rangle_Y - \Phi(v,p) \mid v \in V,\ p \in Y\} =$$

$$= \sup_{v\in V}\sup_{p\in Y}\{\langle p^*,p\rangle_Y - G_1(\Lambda_1 v - p_1) - G_2(\Lambda_2 v - p_2) - F(v)\} =$$

$$= \sup_{v\in V}\sup_{q\in Y}\{\langle p^*,\Lambda v - q\rangle_Y - G_1(q_1) - G_2(q_2) - F(v)\} =$$

$$= \sup_{v\in V}\sup_{q\in Y}\{<-p_1^*,q_1>_{L_1} - G_1(q_1) + <-p_2^*,q_2>_{L_2} - G_2(q_2) + <\Lambda^*p^*,v>_V - F(v)\}$$

$$= G_1^*(-p_1^*) + G_2^*(-p_2^*) + F^*(\Lambda^*p^*),$$

where G_1^*, G_2^* and F^* denote the polar functions of G_1, G_2 and F, respectively, and $\Lambda^*: Y^* \to V^*$ denotes the adjoint of Λ. We shall determine the functionals G_1^*, G_2^* and F^* in order to arrive at an explicit expression for P^*.

Let us first consider G_1^*. We know that (cf. (3.10)) $G_1(w) = \int k(w(\cdot))$ and since k is a normal positive integrand we can interchange integration and taking the polar (cf. [3, Prop. 2.1, p. 251]):

$$G_1^*(p_1^*) = \int k^*(p_1^*(\cdot)),$$

where

$$k^*(z) = \sup\{yz - k(y) \mid y \geq 0\} = \varepsilon \exp(\frac{z}{\varepsilon} - 1) - \frac{\varepsilon}{e}.$$

In the same manner we find

$$G_2^*(p_2^*) = \int \tfrac{1}{2}(p_2^*)^2 + gp_2^*.$$

Next we calculate F^*:

$$F^*(\Lambda^*p^*) = \sup\{<p^*,\Lambda v>_Y \mid v \in V,\ v(0) = 0,\ v(1) = 1\}$$

$$= \int (ip_2^* + p_1^*) + \sup\{\int (p_1^*u' + p_2^*u) \mid u \in V,\ u(0) = u(1) = 0\}.$$

Here we made the transformation $v = u + i$, where i denotes the function $i(x) = x$, in order to arrive at homogeneous boundary conditions. Since $\mathcal{D}$ is dense in the set $\{u \in V \mid u(0) = u(1) = 0\}$ we conclude that

$$F^*(\Lambda^*p^*) = \begin{cases} \int(ip_2^* + p_1^*) & \text{if } p_2^* = (p_1^*)' \text{ in the sense of distributions,} \\ +\infty & \text{otherwise.} \end{cases}$$

Collecting all results we arrive at the following explicit formulation:

$$(3.16)\quad P^* \quad \mathrm{Sup}\{\int (\frac{\varepsilon}{e} - \varepsilon e^{-\frac{p_1^*}{\varepsilon} - 1} - p_1^* + (g - i)p_2^* - \tfrac{1}{2}(p_2^*)^2) \mid p^* \in L_\infty \times L_2,\ p_2^* = (p_1^*)'\}$$

From known properties of P one can deduce that P^* has a solution. Indeed, since

(i) Φ is convex and inf P is finite,

(ii) the function $p \mapsto \Phi(i,p)$ is finite and continuous at the point $p = 0$, we are in a position to conclude from [3, Prop. 2.3, p. 51] that P^* has a solution and that $\inf P = \sup P^*$. Finally, we deduce from the strict convexity of the functional in (3.16) that the solution of P^* is unique.

3.3. THE EXTREMALITY RELATIONS

In virtue of [3, Prop. 2.4, p. 52] the following claims are equivalent:

(i) v is a solution of P, p^* is a solution of P^*

(ii) $v \in V$ and $p^* \in Y^*$ satisfy the extremality relation

$$\Phi(v,0) + \Phi^*(0,p^*) = 0. \tag{3.17}$$

In the present case (3.17) can be decoupled as follows:

$$\begin{aligned} 0 &= \Phi(v,0) + \Phi^*(0,p^*) \\ &= G_1(\Lambda_1 v) + G_1^*(-p_1^*) + G_2(\Lambda_2 v) + G_2^*(-p_2^*) + F(v) + F^*(\Lambda^* p^*) \\ &= \{G_1(\Lambda_1 v) + G_1^*(-p_1^*) - \langle -p_1^*, \Lambda_1 v\rangle_{L_1}\} + \\ &+ \{G_2(\Lambda_2 v) + G_2^*(-p_2^*) - \langle -p_2^*, \Lambda_2 v\rangle_{L_2}\} + \\ &+ \{F(v) + F^*(\Lambda^* p^*) - \langle \Lambda^* p^*, v\rangle_V\}. \end{aligned}$$

Since each of these expressions in brackets is nonnegative, actually each of them must be zero. Thus we find

$$\int (\varepsilon v' \ln v' + \varepsilon e^{-\frac{p_1^*}{\varepsilon} - 1} + v' p_1^*) = 0, \tag{3.18}$$

$$\int (\tfrac{1}{2}(g-v)^2 + \tfrac{1}{2}(p_2^*)^2 - g p_2^* + v p_2^*) = \tfrac{1}{2}\int (p_2^* - g + v)^2 = 0, \tag{3.19}$$

$$v(0) = 0, \quad v(1) = 1, \quad p_2^* = (p_1^*)'. \tag{3.20}$$

In order to draw further conclusions from (3.18), consider the function f defined by

$$f(x) = \varepsilon\lambda \ln \lambda + \varepsilon e^{-\frac{x}{\varepsilon} - 1} + \lambda x,$$

for fixed $\lambda \geq 0$. If $\lambda = 0$, then $f > 0$. If $\lambda > 0$, then the convex function f is nonnegative and it attains its minimum, zero, at the point $x = -\varepsilon(1 + \ln \lambda)$. Consequently

(3.18) implies that $v' > 0$ and that

$$(3.21) \qquad p_1^* = -\varepsilon(1 + \ell n\ v').$$

Likewise (3.19) implies that

$$(3.22) \qquad p_2^* = g - v.$$

Finally, combination of (3.20) - (3.22) leads to

$$(3.23) \qquad \begin{cases} -\varepsilon(\ell n\ v')' + v = g. \\ v(0) = 0, \quad v(1) = 1. \end{cases}$$

So if v is the solution of P then v satisfies (3.23). From the fact that $g \in L_2$ we deduce that $\ell n\ v' \in H^1$ and consequently that $v \in H^2$. Hence v satisfies BVP.

Conversely, let v be the solution of BVP. Define p_1^* and p_2^* by (3.21) and (3.22), respectively. Then v and $p^* = (p_1^*, p_2^*)$ satisfy the extremality relation (3.17) and consequently v solves P while p^* solves P^*.

3.4. LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

Formally we can associate with P^* the following limiting problem

$$(3.24) \qquad P_0^* \qquad -\mathrm{Inf}\{\int (q + (i - g)q' + \tfrac{1}{2}(q')^2) \mid q \in C\},$$

where by definition

$$(3.25) \qquad C = \{q \in H^1 \mid q \ge 0\}$$

(note that the condition $p_2^* = (p_1^*)'$ motivates the choice of the underlying space and that we choose $q \ge 0$ because otherwise $\varepsilon \exp(-\frac{q}{\varepsilon})$ tends to $-\infty$ as $\varepsilon \downarrow 0$).

P_0^* consists of minimizing a strictly convex, continuous and coercive functional on a closed convex subset of the reflexive space H^1. Hence there exists a unique solution of P_0^*, which we shall call q_0.

Defining functionals G_3 and G_4 on L_2 and H^1 as follows:

$$(3.26) \qquad G_3(w) = \int \frac{1}{2} w^2 + (i - g)w$$

$$(3.27) \qquad G_4(w) = \begin{cases} \int w & \text{if } w \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

we rewrite

$$P_0^* \qquad -\mathrm{Inf}\{G_3(\Lambda_1 q) + G_4(q) \mid q \in H^1\}$$

where now Λ_1, defined by $\Lambda_1 q = q'$, is considered as a bounded linear mapping of H^1 into L_2.

Next we construct the dual problem P_0^{**} of P_0^* relative to the perturbed functional

$$G_3(\Lambda_1 q - r) + G_4(q), \qquad r \in L_2.$$

We find

$$(3.28) \qquad P_0^{**} \qquad \mathrm{Inf}\{G_3^*(-v) + G_4^*(\Lambda_1^* v) \mid v \in L_2\},$$

where

$$(3.29) \qquad G_3^*(v) = \frac{1}{2}\int (v+g-i)^2$$

and

$$(3.30) \qquad G_4^*(v) = \sup\{\langle v,q\rangle_{H^1} - \int q \mid q \in \mathcal{C}\}$$

$$= \sup\{\langle v-1,q\rangle_{H^1} \mid q \in \mathcal{C}\}$$

$$= \begin{cases} 0 & \text{if } (1-v) \in \mathcal{C}^*, \\ +\infty & \text{otherwise,} \end{cases}$$

where we have put $\int p = \langle 1,p\rangle_{H^1}$ (according to the natural identification of L_2 with a subspace of $(H^1)^*$) and where

$$(3.31) \qquad \mathcal{C}^* = \{v \in (H^1)^* \mid \langle v,q\rangle_{H^1} \geq 0,\ \forall q \in \mathcal{C}\}.$$

Performing the change of function $u = v + i$ we get

$$P_0^{**} \qquad \mathrm{Inf}\{\frac{1}{2}\int (g-u)^2 \mid u \in \Omega\},$$

where by definition

$$(3.32) \qquad \Omega = \{u \in L_2 \mid 1 - \Lambda_1^*(u-i) \in \mathcal{C}^*\}.$$

<u>PROPOSITION 3.3</u>. $\Omega = \overline{\mathcal{D}(A)}$

<u>PROOF</u>. By definition

$$\Omega = \{u \in L_2 \mid \int (q + (i-u)q') \geq 0,\ \forall q \in \mathcal{C}\}.$$

Let $u \in \Omega$ and $q \in \mathcal{C} \cap H_0^1$ then, using $\int(q+iq') = 0$, we obtain $\int uq' \leq 0$, which shows that $u' \geq 0$ in the sense of distributions. Next, suppose that u assumes values larger than one on a set of positive measure. Since u is non-decreasing this set can be taken to be an interval with 1 as its right endpoint and, say, λ as its left endpoint. Take $q(x) = 0$ for $0 \leq x \leq \lambda$ and q strictly increasing for $x > \lambda$, then we arrive at the contradiction

$$\int (q + (i-u)q') < \int_\lambda^1 (q(x) + (x-1)q'(x))dx = 0$$

It follows that $u \leq 1$. Likewise one can show that $u \geq 0$, so we conclude that $u \in \overline{\mathcal{D}(A)}$ indeed.

Conversely, suppose $u \in \mathcal{D}(A)$ and $q \in \mathcal{C}$ then

$$\int (q + (i-u)q') = u(0)q(0) - (u(1)-1)q(1) + \int u'q \geq 0$$

and consequently $u \in \Omega$. Finally, if $u \in \overline{\mathcal{D}(A)}$ we arrive at the same conclusion by using an approximating sequence in $\mathcal{D}(A)$ and by noting that $\int(q+(i-u)q')$ depends continuously on u. $\square$

Thus we showed that P_0^{**} is precisely the reduced problem considered in Subsection 2.2. We recall that it has precisely one solution y_0. Expressing the extremality relation

$$G_3(\Lambda_1 q_0) + G_3^*(i-y_0) + \langle y_0 - i, \Lambda_1 q_0\rangle_{L_2} = 0$$

as

$$\int (q_0' + y_0 - g)^2 = 0$$

we obtain that

(3.33) $\qquad q_0' = g - y_0.$

The other extremality relation

$$G_4(q_0) + G_4^*(\Lambda_1^*(y_0 - i)) - \langle \Lambda_1^*(y_0 - i), q_0\rangle_{H^1} = 0$$

yields the relation

(3.34) $\qquad q_0(1) = \int y_0 q_0'.$

THEOREM 3.4. *For each* $\varepsilon > 0$ *let* y_ε *denote the solution of* P *and* $p_\varepsilon^* = (p_{\varepsilon 1}^*, p_{\varepsilon 2}^*)$ *the solution of* P^*. *Moreover, let* y_0 *denote the solution of* P_0^{**} *and* q_0 *the solution of* P_0^*. *Then*

(i) $$\lim_{\varepsilon \downarrow 0} \| p_{\varepsilon 1}^* - q_0 \|_{H^1} = 0$$

(ii) $$\lim_{\varepsilon \downarrow 0} \| y_\varepsilon - y_0 \|_{L_2} = 0$$

PROOF. First we want to show that $p_{\varepsilon 1}^*$ is bounded in H^1 uniformly in ε. Since $0 \le y_\varepsilon \le 1$, y_ε and $(p_{\varepsilon 1}^*)' = p_{\varepsilon 2}^* = g - y_\varepsilon$ are bounded in L_2 uniformly in ε. The definition of J implies

$$0 \le \operatorname{Inf} P \le \frac{1}{2} \int (g - i)^2 + \frac{\varepsilon}{e} .$$

Using $\operatorname{Sup} P^* = \operatorname{Inf} P$ we obtain

$$0 \le \int \left(\frac{\varepsilon}{e} - \varepsilon e^{-\frac{p_{\varepsilon 1}^*}{\varepsilon} - 1} - p_{\varepsilon 1}^* + (g - i) p_{\varepsilon 2}^* - \frac{1}{2}(p_{\varepsilon 2}^*)^2\right) \le \frac{1}{2} \int (g - i)^2 + \frac{\varepsilon}{e} .$$

From $y_\varepsilon' = \exp(-\frac{1}{\varepsilon} p_{\varepsilon 1}^* - 1)$ and $\int y_\varepsilon' = y_\varepsilon(1) - y_\varepsilon(0) = 1$ we deduce that

(3.35) $$\lim_{\varepsilon \downarrow 0} \varepsilon \int e^{-\frac{1}{\varepsilon} p_{\varepsilon 1}^* - 1} = 0$$

Combination of these results yields a uniform bound for $|\int p_{\varepsilon 1}^*|$. Hence there exists $\theta = \theta(\varepsilon) \in [0,1]$ such that $p_{\varepsilon 1}^*(\theta)$ is uniformly bounded and, finally, we obtain

$$\int (p_{\varepsilon 1}^*)^2 = \int_0^1 \left(p_{\varepsilon 1}^*(\theta) + \int_\theta^x (p_{\varepsilon 1}^*)'(\xi) d\xi\right) dx \le C.$$

So there exists a sequence $\{\varepsilon_n\}$ and a function $w \in H^1$ such that $p_{\varepsilon_n 1}^*$ converges to w weakly in H^1 and strongly in L_2. Next we intend to show that $w = q_0$.

From the fact that p_ε^* solves P^* we deduce that

$$\int \left(\frac{\varepsilon}{e} - \varepsilon e^{-\frac{1}{\varepsilon} p_{\varepsilon 1}^* - 1} - p_{\varepsilon 1}^* + (g - i)(p_{\varepsilon 1}^*)' - \frac{1}{2}((p_{\varepsilon 1}^*)')^2\right)$$

$$\ge \int \left(\frac{\varepsilon}{e} - \varepsilon e^{-\frac{1}{\varepsilon} q_0 - 1} - q_0 + (g - i) q_0' - \frac{1}{2}(q_0')^2\right).$$

Furthermore, the functional $q \mapsto \int (q + (i - g) q' + \frac{1}{2}(q')^2)$ is convex and continuous on H^1 and thus w.l.s.c. Hence

$$(*)\qquad \int (w + (i-g)w' + \tfrac{1}{2}(w')^2)$$

$$\le \liminf_{n\to\infty} \int (p^*_{\varepsilon_n 1} + (i-g)(p^*_{\varepsilon_n 1})' + \tfrac{1}{2}((p^*_{\varepsilon_n 1})')^2)$$

$$\le \limsup_{n\to\infty} \int (p^*_{\varepsilon_n 1} + (i-g)(p^*_{\varepsilon_n 1})' + \tfrac{1}{2}((p^*_{\varepsilon_n 1})')^2)$$

$$\le \int (q_0 + (i-g)q_0' + \tfrac{1}{2}(q_0')^2).$$

We observe that $w \ge 0$ (else (3.35) could not be true). Since q_0 is the unique solution of P_0^*, necessarily $w = q_0$. Inserting this into (*) we obtain that in fact $p^*_{\varepsilon_n 1}$ converges to q_0 strongly in H^1. Moreover, since the limit does not depend on the sequence under consideration (i) follows. Finally, we arrive at (ii) by noting that $y_\varepsilon = g - (p^*_{\varepsilon 1})'$ and $y_0 = g - q_0'$. □

4. CALCULATION OF y_0

We recall that y_0 is the unique solution of the variational problem $\mathrm{Min}_{\overline{\mathcal{D}(A)}}\, W_0$, where $W_0(u) = \|u - g\|^2$. It is well known (for instance, see Ekeland-Témam [3, II, 2.1]) that one can equivalently characterize y_0 as the unique solution of the *variational inequality*:

$$(4.1)\qquad \text{find } y \in \overline{\mathcal{D}(A)} \text{ such that } (y-g, v-y) \ge 0,\ \forall v \in \overline{\mathcal{D}(A)}.$$

Already from the reduced differential equation $(g-y)y' = 0$, it can be guessed that y_0 is possibly composed out of pieces where it equals g and pieces where it equals a constant. Of course, if $y_0 = g$ in some open interval, g has to be nondecreasing in that interval. The characterization of y_0 by (4.1) can be used to find conditions on the "allowed" constants.

<u>THEOREM 4.1</u>. *Suppose* $y \in \overline{\mathcal{D}(A)}$ *has the following property: there exists a partition* $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ *of* $[0,1]$ *and a subset* L *of* $\{0,1,\ldots,n-1\}$ *such that*:
(i) if $i \notin L$ then $y(x) = g(x)$ for $x \in [x_i, x_{i+1}]$,
(ii) If $i \in L$ then $y(x) = C_i$ for $x \in [x_i, x_{i+1}]$ and

$$\int_x^{x_{i+1}} (C_i - g(\xi))\,d\xi \ge 0, \qquad \forall x \in [x_i, x_{i+1}],\ \text{if } C_i \in [0,1),$$

$$\int_{x_i}^{x} (C_i - g(\xi))\,d\xi \le 0, \qquad \forall x \in [x_i, x_{i+1}],\ \text{if } C_i \in (0,1],$$

(so in particular, if $C_i \in (0,1)$, $\int_{x_i}^{x_{i+1}} (C_i - g(\xi))\,d\xi = 0$).

Then $y = y_0$.

PROOF. According to (4.1) it is sufficient to check that

$$I(v) = \int (y-g)(v-y) \geq 0, \qquad \forall v \in \overline{\mathcal{D}(A)}\ .$$

In fact it is sufficient to check this for all $v \in \overline{\mathcal{D}(A)} \cap H^1$ (since this set is dense in $\overline{\mathcal{D}(A)}$ and I is continuous). We note that $I(v) = \Sigma_{i\in L}\, I_i(v)$, where

$$I_i(v) = \int_{x_i}^{x_{i+1}} (C_i - g(\xi))(v(\xi) - C_i)d\xi.$$

If $C_i = 0$ then

$$I_i(v) = -v(x_i) \int_{x_i}^{x_{i+1}} g(\xi)d\xi - \int_{x_i}^{x_{i+1}} v'(\xi) \int_{\xi}^{x_{i+1}} g(x)dxd\xi \geq 0.$$

If $C_i \in (0,1)$ then

$$I_i(v) = \int_{x_i}^{x_{i+1}} v'(\xi) \int_{\xi}^{x_{i+1}} (C_i - g(x))dxd\xi \geq 0.$$

If $C_i = 1$ then

$$I_i(v) = (v(x_{i+1}) - 1) \int_{x_i}^{x_{i+1}} (C_i - g(\xi))d\xi - \int_{x_i}^{x_{i+1}} v'(\xi) \int_{x_i}^{\xi} (C_i - g(x))dxd\xi \geq 0$$

Hence indeed $I(v) \geq 0$, $\forall v \in \overline{\mathcal{D}(A)} \cap H^1$. □

The sufficient conditions of the theorem can be used as a kind of algorithm to compute y_0 in concrete cases. We shall illustrate this idea by means of a number of examples (some of which are almost literally taken from [2]).

EXAMPLE 1. Suppose g is nondecreasing, then

$$y_0(x) = \begin{cases} 0 & \text{if } g(x) \leq 0, \\ g(x) & \text{if } 0 \leq g(x) \leq 1, \\ 1 & \text{if } g(x) \geq 1. \end{cases}$$

EXAMPLE 2. Suppose g is nonincreasing, then $y_0(x) = C$ with

$$C = \begin{cases} 0 & \text{if } \int g \le 0, \\ \int g & \text{if } 0 \le \int g \le 1, \\ 1 & \text{if } \int g \ge 1. \end{cases}$$

EXAMPLE 3. Suppose that $g \in C^1$ is such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0 < b < c < 1$ and $0 < g(c) < g(b) < 1$. Let g_1^{-1} denote the inverse of g on [0,b] and g_2^{-1} the inverse of g on [c,1]. Define two points a and d by

$$a = g_1^{-1}(g(c)), \qquad d = g_2^{-1}(g(b)).$$

Then $g([a,b]) = g([c,d])$. (See Figure 1).

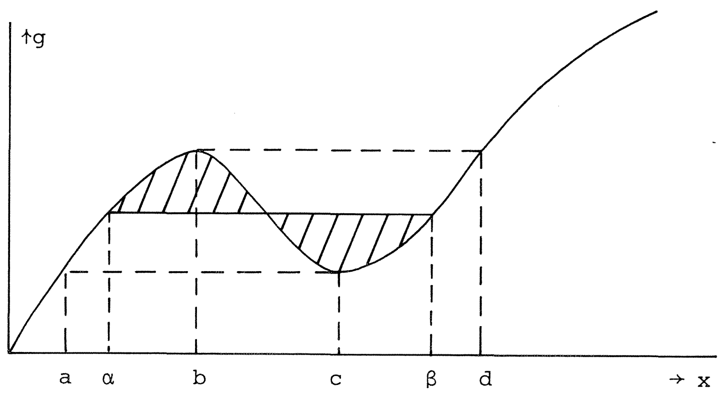

Figure 1

On [a,b] we define a mapping G by

$$G(x) = \int_x^{g_2^{-1}(g(x))} (g(x) - g(\xi))\,d\xi.$$

Then $G(a) < 0$, $G(b) > 0$ and on (a,b)

$$G'(x) = g'(x) \int_x^{g_2^{-1}(g(x))} d\xi > 0.$$

Consequently G has a unique zero on [a,b], say for $x = \alpha$. The function y_0 has the tendency to follow g as much as possible. However, it also has to be nondecreasing. So the inverse function of y_0 must "jump" from a point on [a,b] to a point on [c,d]. In view of Theorem 4.1 this jump can only take place between α and $\beta = g_2^{-1}(\alpha)$. We leave it to the reader to verify (by checking all requirements of Theorem 4.1) that

$$y_0(x) = \begin{cases} 0 & \text{if } x \le \alpha \text{ and } g(x) \le 0, \\ g(x) & \text{if } x \le \alpha \text{ and } g(x) \ge 0, \\ g(\alpha) & \text{if } \alpha \le x \le \beta, \\ g(x) & \text{if } x \ge \beta \text{ and } g(x) \le 1, \\ 1 & \text{if } x \ge \beta \text{ and } g(x) \ge 1. \end{cases}$$

It should be clear that the differentiability of g is not strictly necessary for our arguments to apply. In fact the monotonicity of G follows from straightforward geometrical considerations and the condition $G(\alpha) = 0$ has a corresponding interpretation (see Figure 1).

EXAMPLE 4. If g has more maxima and minima the construction of candidates for y_0 can be based on essentially the same idea as outlined in Example 3. However, it becomes more complicated since the number of possibilities becomes larger (see [2] for some more details). For instance, if g has a graph as shown in Figure 2, looking at zero's of functions like G above leaves us with two possible candidates: one with two "jumps" (a-b,c-d) and one with a "two-in-one jump" ($\alpha - \beta$).

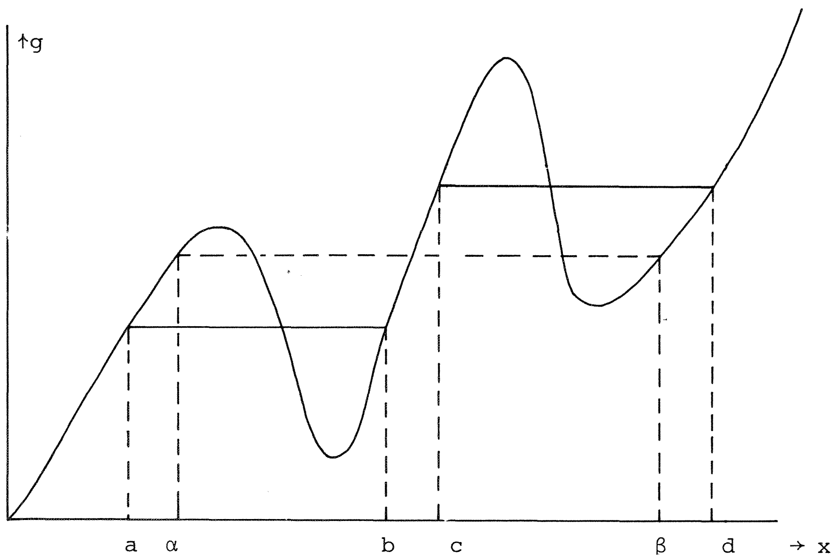

Figure 2

In [2] we were unable to decide in such a situation which was the actual limit. But now it can be read off from the picture that only the one with two "jumps" satisfies the requirements of Theorem 4.1, and hence this one must actually be y_0. (The other one corresponds to a saddle point of the functional W_0 restricted to $\overline{\mathcal{D}(A)}$.) It is in this sense that y_0 must have as many "jumps" as possible.

5. CONCLUDING REMARKS

(i) In all our examples y_0 satisfies the reduced equation $(g - y)y' = 0$. However, this equation is by no means sufficient to characterize y_0 completely. Our analysis clearly shows that the reduced variational problem $\mathrm{Min}_{\overline{\mathcal{D}(A)}}\ W_0$ contains much more information than the reduced differential equation.

(ii) In [2] we were actually interested in a boundary value problem of the type

(5.1) $$\varepsilon x y'' + (g - y)y' = 0, \qquad 0 < x < 1,$$

(5.2) $$y(0) = 0, \qquad y(1) = 1,$$

which arises from the assumption of radial symmetry in a two-dimensional geometry. This problem can be analysed in completely the same way as we did with BVP in this paper, by choosing as the underlying Hilbert space the weighted L_2-space corresponding to the measure $d\mu(x) = x^{-1}dx$. For instance, the operator $\tilde{A}$ defined by

$$(\tilde{A}u)(x) = -x\,\frac{u''(x)}{u'(x)}$$

with

$$\mathcal{D}(\tilde{A}) = \{u \in L^2(d\mu) \mid u' \in C(0,1],\ u' > 0,\ u(1) = 1,\ i\ \frac{u''}{u'} \in L^2(d\mu)\}$$

is clearly monotone in this space. The surjectivity of $I + \varepsilon\tilde{A}$ can be proved with the aid of an auxiliary problem and Schauder's fixed point theorem. (Note that some care is needed in checking that the functions which occur belong to the right space and that the solution operator is compact. This turns out to be all right. We refer to Martini's thesis [5] where related problems are treated in full detail.) Hence $\tilde{A}$ is maximal monotone. Subsequently it follows that, for given $g \in L_2(d\mu)$, the solution y_ε tends, as $\varepsilon \downarrow 0$, to a limit y_0 in $L_2(d\mu)$ and that y_0 is the projection in $L_2(d\mu)$ of g onto the closed convex set

$$\overline{\mathcal{D}(\tilde{A})} = \{u \in L_2(d\mu) \mid u \text{ is nondecreasing, } 0 \le u \le 1\}.$$

The second method carries over to this situation as well.

(iii) In [2] we were also interested in the situation where the differential equation (5.1), assumed to hold for $0 < x < \infty$, is supplemented by the condition

(5.3) $$\lim_{x\to\infty} y(x) = 1.$$

Intuitively one believes that similar results should be true in this situation. However, the present approach does not carry over directly and, in fact, the noncompactness of the domain presents serious mathematical difficulties.

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