stichting
mathematisch
centrum

AFDELING TOEGEPASTE WISKUNDE
(DEPARTMENT OF APPLIED MATHEMATICS)
TW 198/80 FEBRUARI

E.J.M. VELING

TRAVELLING WAVES IN AN INITIAL-BOUNDARY VALUE PROBLEM

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

1980 Mathematic Subject Classification: 35K60, 35B40, 92A10

Travelling waves in an initial-boundary value problem*)

by

E.J.M. Veling

ABSTRACT

In this paper we consider the initial-boundary value problem for a function u(x,t) satisfying a one-dimensional semilinear diffusion equation on the half-bounded interval $x \geq 0$. For a wide class of initial and boundary values a uniformly valid asymptotic expression will be given to which the solution converges exponentially. This expression is composed by a travelling wave and a solution of the stationary problem.

KEY WORDS & PHRASES: *semilinear diffusion, initial-boundary value problem, travelling waves, Lyapunov functional, asymptotic behaviour, exponential stability*

---

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we are interested in the equation

$$(1.1) \qquad u_t = u_{xx} + f(u), \quad (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+)$$

where f satisfies

(Hf1) $\quad f \in C^1([0,1]),\ f(0) = f(1) = 0,\ f'(0) < 0,\ f'(1) < 0,$
$\exists a,\quad 0 < a < 1 \quad .\ni. \quad f(u) < 0$ on $(0,a)$ and $f(u) > 0$ on $(a,1)$

and further

$$(\text{Hf2}) \qquad \int_0^1 f(u)\,du > 0.$$

The function u represents the density of an allele in a disploid population. Condition (Hf1) implies the heterozygote case, see ARONSON & WEINBERGER [1]. Besides the equation stands for a model for the signal propagation along transmission lines and is a degenerate case of the FitzHugh-Nagumo equation for the propagation of nerve pulses, see FIFE & McLEOD [4]. An interesting feature of this equation is the existence of *travelling waves*, i.e. solutions of the form $u(x,t) = U(z)$, $z = x-c_0t$, where U satisfies the ordinary differential equation

$$(1.2) \qquad \begin{cases} \dfrac{d^2}{dz^2} U + c_0 \dfrac{d}{dz} U + f(U) = 0, & z \in \mathbb{R}, \\ \lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to\infty} U(z) = 0. \end{cases}$$

It can be proved that U is monotone decreasing and that the value of c_0 is unique, because of (Hf1) and $c_0 > 0$, because of (Hf2), see [4] and HADELER & ROTHE [6]. We fix U through the requirement $U(0) = \frac{1}{2}$, so we have lost the freedom of translation along the z-axis.

If we look at the pure initial value problem - (1.1) together with the condition $u(x,0) = g(x)$ -, and we require

$$(1.3)\qquad \begin{cases} 0 \le g(x) \le 1, & x \in \mathbb{R}, \\ \liminf_{x\to-\infty} g(x) > a, & \limsup_{x\to\infty} g(x) < a \end{cases}$$

then the solution u(x,t) will converge to some translate of U(z) in an exponentially way, i.e. there exists constants z_0, K, ω, $K > 0$, $\omega > 0$, such that

$$(1.4)\qquad |u(x,t) - U(x-c_0t-z_0)| < Ke^{-\omega t}, \quad \text{uniformly } x \in \mathbb{R}.$$

See the paper of FIFE & McLEOD [4] for this result. Here we treat an initial-boundary value problem, so we have to specify besides the initial condition also the boundary function

$$(P)\qquad \begin{cases} u_t = u_{xx} + f(u), & (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+), \\ u(x,0) = g(x), & x \in \overline{\mathbb{R}^+}, \\ u(0,t) = h(t), & t \in \overline{\mathbb{R}^+}. \end{cases}$$

As we are interested in densities we add the conditions

(Hg1) $\quad 0 \le g(x) \le 1, \quad x \in \overline{\mathbb{R}^+}$,

(Hh1) $\quad 0 \le h(t) \le 1, \quad t \in \overline{\mathbb{R}^+}$.

To be sure that the solution satisfies some smoothness properties we require

(Hg2) $\quad g \in C^{2,\alpha}(\overline{\mathbb{R}^+})$

(Hh2) $\quad h \in C^{1,\alpha/2}(\overline{\mathbb{R}^+})$

for some α, $0 < \alpha < 1$.

For the notion $C^{1,\alpha/2}$, $C^{2,\alpha}(\overline{\mathbb{R}^+})$ see section 2. We require the following consistency conditions

$$(Hgh3)\qquad \begin{cases} h(0) = g(0), \\ \frac{d}{dt} h(0) = \frac{d^2}{dx^2} g(0) + f(g(0)). \end{cases}$$

The conditions (Hgh1), (Hgh2), (Hgh3) are sufficient to ensure the existence and uniqueness of a classical solution of (P), see Theorem 1.

The purpose of this paper is to prove that even in this case of a half-bounded x-interval the notion of convergence to a travelling wave is possible. Of course the boundary condition now plays an important role and if the limit value h(t) for t tending to infinity is not equal to the limit value of U(z) for z tending to minus infinity ($U(-\infty) = 1$) we need in the formulation of our result the function V_θ which is the solution of

$$(1.5) \qquad \begin{cases} \dfrac{d^2}{dx^2} V_\Theta + f(V_\theta) = 0, & x \in \mathbb{R}^+, \\ V_\theta(0) = \theta, \quad \lim_{x\to\infty} V_\theta(x) = 1, & 0 \le \theta \le 1. \end{cases}$$

Under the following conditions

(Hg4) $\qquad \limsup_{x \to \infty} g(x) < a,$

(Hh4) $\qquad \exists\theta,\ 0 \le \theta \le 1,\ \exists\gamma,\ \gamma > 0 .\ni. \theta - h(t) = O(e^{-\gamma t}),\ t \to \infty$

and a technical condition (Hh5), to be specified later it is possible to prove that the solution of (P) subject to conditions (Hf1-2), (Hg1-4), (Hh1-5) converges exponentially to an *asymptotic state*, i.e. there exists a z_0, K, ω, K > 0, ω > 0 such that

$$(1.6) \qquad |u(x,t) - U(x-c_0t-z_0) - V_\theta(x) + 1| < Ke^{-\omega t}, \text{ uniformly } x \in \overline{\mathbb{R}^+}.$$

See Theorem 2. We note that for θ = 1 $V_\theta(x) \equiv 1$. If we examine (1.6) we find immediately in view of the uniformity in x the following limit values for u(x,t)

i) $\quad x = x_0$, x_0 arbitrary, $\lim_{t\to\infty} u(x_0,t) = V_\theta(x_0)$,

ii) $\quad x = x_0 + c_1t$, $c_1 < c_0$, $\lim_{t\to\infty} u(x_0+c_1t_1,t) = 1$,

iii) $\quad x = x_0 + c_0t$, $\lim_{t\to\infty} u(x_0+c_0t,t) = U(x_0-z_0)$,

iv) $\quad x = x_0 + c_2t$, $c_2 > c_0$, $\lim_{t\to\infty} u(x_0+c_2t,t) = 0$.

In iii) we follow the wave while in ii) and iv) we travel respectively too slow and too fast to the right.

In section 2 we list some notations and properties of the functions which we consider. In section 3 we mention the main ingredients for the proof of Theorem 2 and we prove Theorem 1. The proof of Theorem 2 is based on the same techniques as were used in the paper of FIFE & McLEOD [4], i.e. the maximum principle and the use of a Lyapunov functional.

2. NOTATION AND STATEMENT OF RESULTS

We introduce the following notations and definitions

$$\mathbb{R}^+ = (0,\infty), \quad \overline{\mathbb{R}^+} = [0,\infty),$$
$$\Omega = (a,b) \subset \mathbb{R}, \quad -\infty \le a < b \le \infty,$$
$$Q_T = \{(x,t) \,|\, x \in \Omega,\ t \in (0,T)\},$$
$$S_T = \{(x,t) \,|\, x \in \partial\Omega,\ t \in (0,T)\},$$
$$B = \{(x,t) \,|\, x \in \Omega,\ t = 0\},$$
$$B_T = \{(x,t) \,|\, x \in \Omega,\ t = T\}.$$

If we drop the lower index T of Q_T and S_T we extend the upper bound of the t-interval to infinity. It is clear that $\partial Q_T = B \cup \overline{S_T} \cup B_T$. We define some classes of functions depending on a scalar variable ($u = u(x)$, and $u^{(i)}$ is the i-th derivative),

$C^m(\Omega) = \{u = u(x) \,|\, u$ m-times continuously differentiable, $x \in \Omega\}$,

$\bar{\Omega}$ compact : $C^m(\bar{\Omega}) = \{u = u(x) \,|\, u$ m-times continuously differentiable, $u, u^{(1)}, \ldots, u^{(m)}$ bounded in Ω and $u, u^{(1)}, \ldots, u^{(m)}$ can be extended to continuous functions on $\bar{\Omega}\}$,

$\bar{\Omega}$ not compact : $C^m(\bar{\Omega}) = \{u = u(x) \,|\, u$ m-times continuously differentiable and $u, u^{(1)}, \ldots, u^{(m)}$ bounded and uniformly continuous in $\Omega\}$.

$$|u|_0^\Omega = \sup_{x\in\Omega} |u(x)|, \quad |u|_m^\Omega = \sum_{i=0}^{m} |u^{(i)}|_0^\Omega,$$

$$H(u;\alpha;\Omega) = \sup_{x_1,x_2\in\Omega,x_1\neq x_2} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}, \quad 0 < \alpha \le 1,$$

$$C^{m,\alpha}(\bar{\Omega}) = \{u = u(x)\,|\,u \in C^m(\bar{\Omega}),\ H(u^{(m)};\alpha;\Omega) < \infty\},$$

$$|u|^{\Omega}_{m,\alpha} = |u|^{\Omega}_{m} + \sum_{i=0}^{m} H(u^{(i)};\alpha;\Omega).$$

We note that $C^m(\bar{\Omega})(C^{m,\alpha}(\bar{\Omega}))$ is a Banach space under the norm $|\cdot|^{\Omega}_{m}$ $(|\cdot|^{\Omega}_{m,\alpha})$. We denote by $C^{0,\alpha}(\bar{\Omega})$ the class of Hölder ($\alpha < 1$) or Lipschitz ($\alpha = 1$) continuous functions on $\bar{\Omega}$. Next we define some classes of functions depending on a two-dimensional argument, namely $(x,t) \in D \subseteq Q$, D open.

$$C^0(D) = \{u = u(x,t)\,|\,u \text{ continuous},\ (x,t) \in D\},$$

$$C^1(D) = \{u = u(x,t)\,|\,u,u_x \text{ continuous},\ (x,t) \in D\},$$

$$C^2(D) = \{u = u(x,t)\,|\,u,u_x,u_{xx},u_t \text{ continuous},\ (x,t) \in D\},$$ and $C^0(\bar{D})$, $C^1(\bar{D})$, $C^2(\bar{D})$ analogously as for a scalar variable,

$$d(P_1,P_2) = \{(x_1-x_2)^2 + |t_1-t_2|\}^{\frac{1}{2}}, \text{ with } P_i = (x_i,t_i),\ i = 1,2,$$

$$u(P) = u(x,t) \text{ for } P = (x,t),$$

$$|u|^D_0 = \sup_{P\in D} |u(P)|,$$

$$H(u;\alpha;D) = \sup_{P_1,P_2\in D,P_1\neq P_2} \frac{|u(P_1)-u(P_2)|}{d(P_1,P_2)^\alpha}, \quad 0 < \alpha \le 1,$$

$$C^{2,\alpha}(\bar{D}) = \{u = u(x,t)\,|\,u \in C^2(\bar{D}),\ H(u_{xx};\alpha;D) < \infty,\ H(u_t;\alpha;D) < \infty\},$$

$$|u|^D_\alpha = |u|^D_0 + H(u;\alpha;D),$$

$$|u|^D_{2,\alpha} = |u|^D_\alpha + |u_x|^D_\alpha + |u_{xx}|^D_\alpha + |u_t|^D_\alpha.$$

Now we study the stationary equation $u_{xx} + f(u) = 0$, where f satisfies (Hf1) and (Hf2). As a consequence of the model we are only interested in values of u with $0 \le u \le 1$. The reverse sign in (Hf2) can be treated by the change $u' = 1-u$. Equality in (Hf2) does not lead to travelling wave solutions.

The solutions of $u_{xx} + f(u) = 0$ satisfy the expression

(2.1) $\frac{1}{2}u_x^2(x) + F(u(x)) = k,$

where

(2.2) $F(u) = \int_0^u f(w)dw,$

and k is a constant; (2.1) defines a curve in the (u,u_x) plane. In this phase plane the singular points (0,0) and (1,0) are saddle points, while (a,0) is a center point. From (Hf1) and (Hf2) it follows that $\min_{0\le u\le 1} F(u) = F(a) < 0$ and that there exists a number κ, $a < \kappa < 1$ such that $F(\kappa) = 0$. For the choice $k = F(1)$ in (2.1) we find the solution $u \equiv 1$ or u corresponds with the stable $(u = V)$ or unstable manifold of $(u,u_x) = (1,0)$. We label V by θ such that $V_\theta(0) = \theta$, see (1.5). The following asymptotic behaviour holds

(2.3) $1 - V_\theta(x) = O(e^{-\nu x}), \; x \to \infty, \; \nu = \sqrt{-f'(1)}.$

For the special choice $f = f_c = u(1-u)(u-a)$, with $0 < a < \frac{1}{2}$, we find explicitly

$$V_\theta(x) = 1 - \frac{2(1-a)}{\frac{2}{3}(2-a)+\frac{1}{3}\sqrt{2}\sqrt{1+a}\sqrt{1-2a}\,\sinh(\sqrt{1-a}\,x+B)},$$

$$B = \operatorname{arsinh}\left[\frac{\sqrt{2}(3-3a-(1-\theta)(2-a))}{(1-\theta)\sqrt{1+a}\sqrt{1-2a}}\right],$$

$$\nu = \sqrt{1-a}.$$

Next we study the travelling wave solutions, i.e. solutions depending only upon the variable $z = x-ct$. We write

(2.4) $u(x,t) = u(z+ct,t) \equiv v(z,t) = v(x-ct,t)$

where $v(z,t)$ satisfies

(2.5) $v_t = v_{zz} + cv_z + f(v).$

For travelling wave solutions we have $v_t = 0$. A consequence of our choice for f (Hf1-2) is the existence of a unique number $c_0 > 0$ and a function U(z) which satisfies

$$(2.6)\qquad \begin{cases} \dfrac{d^2}{dz^2} U + c_0 \dfrac{d}{dz} U + f(U) = 0, & z \in \mathbb{R}, \\ \lim\limits_{z\to-\infty} U(z) = 1, \ \lim\limits_{z\to\infty} U(z) = 0. \end{cases}$$

We fix U(z) by the condition $U(0) = \frac{1}{2}$. The uniqueness of the number c_0 will become clear by realising that in the phase plane the function U represents the unstable manifold of (1,0) which merges with the stable manifold of (0,0). Both points are saddles. See e.g. [4], [6] and McKEAN [7] for the corresponding phase portraits. The following asymptotic behaviour holds

$$(2.7)\qquad U(z) = O(e^{\beta_0 z}),\ z \to \infty,\ \beta_0 = -\tfrac{1}{2}[c_0 + \sqrt{c_0^2 - 4f'(0)}] < 0,$$

$$(2.8)\qquad 1 - U(z) = O(e^{\beta_1 z}),\ z \to -\infty,\ \beta_1 = -\tfrac{1}{2}[c_0 - \sqrt{c_0^2 - 4f'(1)}] > 0.$$

For the choice $f = f_c$ we find explicitly

$$U(z) = \frac{1}{1 + e^{z/\sqrt{2}}},$$

$$c_0 = \tfrac{1}{2}\sqrt{2}(1-2a),\ \beta_0 = -\tfrac{1}{2}\sqrt{2},\ \beta_1 = \tfrac{1}{2}\sqrt{2}\ .$$

We finally mention the results.

THEOREM 1. *Let the condition* (Hf1-2), (Hg1-3), (Hh1-3) *be satisfied, then problem* (P) *has a unique solution* $u \in C^{2,\alpha}(\bar{Q})$.

For the next theorem we need an additional property of u(x,t), namely

$$(2.9)\qquad \lim_{x\to\infty} \liminf_{t\to\infty} u(x,t) = 1.$$

We will formulate Theorem 2 under the hypothesis (Hh5), which is sufficient for (2.9). We note that the theorem remains true as long as (2.9) is valid.

Hypothesis (Hh5) can be interpreted by saying that the boundary function h exceeds some treshold value, for a sufficiently long time.

(Hh5) If $\theta \le \kappa$, then $h(t) \ge \eta > \kappa$ during some interval (t_1, t_1+T_η), where t_1 is arbitrary and T_η will be specified in Lemma 2.

THEOREM 2. *Let the conditions* (Hf1-2), (Hg1-4), (Hh1-5) *be satisfied, then there exists constants* z_0, K,ω, $K > 0$, $\omega > 0$ *such that the solution* $u(x,t)$ *of* (P) *satisfies*

1) $0 \le \theta < 1$

$|u(x,t) - U(x-c_0t-z_0) - V_\theta(x) + 1| < Ke^{-\omega t}$, *uniformly* $x \in \overline{\mathbb{R}^+}$,

2) $\theta = 1$

$|u(x,t) - U(x-c_0t-z_0)| < Ke^{-\omega t}$, *uniformly* $x \in \overline{\mathbb{R}^+}$.

In section 3 we give an outline of the proof of Theorem 1. In section 4 we prove a weaker form of Theorem 2, case 2 in the sense that the uniform convergence result only holds for $x \ge \delta > 0$, where δ is arbitrary. In section 5 we prove Theorem 2, case 1 using results of section 4 and an analysis which includes the local behaviour at $x = 0$. In section 6 we prove Theorem 2, case 2.

3. INGREDIENTS OF THE PROOF

For the study of the partial differential equation $u_t = u_{xx} + cu_x + f(u)$ ($c = 0$ and $c \ne 0$) we formulate the well known maximum principle and a consequence of it.

Strong Maximum Principle (see ARONSON & WEINBERGER [2]).
Let $u,v \in C^0(\bar{Q}_T) \cap C^2(Q_T \cup B_T)$, where the corresponding Ω is a possible unbounded interval in $\mathbb{R}$, and let

$$L[u] = u_{xx} + cu_x + f(u) - u_t, \tag{3.1}$$

where f is Lipschitz continuous on $[-K,K]$ for some $K > 0$. Suppose $|u|_0^{Q_T}$, $|v|_0^{Q_T} \le K$ and

$$L[u] \le L[v], \quad (x,t) \in Q_T \cup B_T,$$
$$u(x,0) \ge v(x,0), \quad x \in \Omega,$$

and when $S_T \neq \emptyset$

$$u(x,t) \ge v(x,t), \quad (x,t) \in \bar{S}_T,$$

then

$$u(x,t) \ge v(x,t), \quad (x,t) \in \bar{Q}_T.$$

If moreover

$$u(x,0) > v(x,0) \text{ in an open subset of } \Omega$$

then

$$u(x,t) > v(x,t), \quad (x,t) \in Q_T \cup B_T.$$

The requirement about the boundedness of u and v is much too strong. If $H(f;1;\mathbb{R}) < \infty$ (i.e. f is uniformly Lipschitz continuous) then we can relax the a priori boundedness condition; see the discussion in [2]. Also it is not necessary to require the uniform continuity of u and v if $\bar{Q}_T$ is not compact: continuity and the relaxed boundedness condition are enough.

DEFINITION 1. $u(x,t)$ is a *regular sub-(super-)solution* of the problem

$$(P') \qquad \begin{cases} u_t = u_{xx} + cu_x + f(u), & (x,t) \in Q_T \cup B_T, \\ u(x,0) = g(x), & x \in \Omega, \\ u(x,t) = h(t), & (x,t) \in \bar{S}_T, \end{cases}$$

if $u \in C^0(\bar{Q}_T) \cap C^2(Q_T \cup B_T)$, $L[u] \ge (\le)\ 0$, $(x,t) \in Q_T \cup B_T$, $u(x,0) \le (\ge)\ g(x)$, $x \in \Omega$, $u(x,t) \le (\ge)\ h(t)$, $(x,t) \in \bar{S}_T$.

DEFINITION 2. $u(x,t)$ is a *sub-(super-)solution* of the problem (P') if

$u(x,t) = \max_{1 \le i \le N} (\min)\ u_i(x,t)$ for a finite collection of regular sub-(super-) solutions $\{u_i\}_1^N$ for the problem (P').

Comparison Theorem (see FIFE [3]).
Let $\underline{u}(x,t)$, $\bar{u}(x,t)$ *be respectively sub-, and super-solutions of the problem* (P'), *where* f *is uniformly Lipschitz continuous. Let*

$$\underline{u}(x,0) \le g(x) \le \bar{u}(x,0), \quad x \in \Omega,$$
$$\underline{u}(x,t) \le h(t) \le \bar{u}(x,t), \quad (x,t) \in \bar{S}_T,$$

then we can bound a solution $u(x,t)$ *of* (P') *by*

$$\underline{u}(x,t) \le u(x,t) \le \bar{u}(x,t), \quad (x,t) \in \bar{Q}_T$$

and further

1) $\underline{u}(x,t) < \bar{u}(x,t), \quad (x,t) \in Q_T$

or

2) $\underline{u}(x,t) \equiv \bar{u}(x,t), \quad (x,t) \in Q_T.$

In case 1) *either* $\underline{u}$ *or* $\bar{u}$ *may also be a solution; in case* 2) *both* $\underline{u}$ *and* $\bar{u}$ *have to be the solution.*

The proof of the existence of the solution $u \in C^{2,\alpha}(\bar{Q})$ (Theorem 1) requires a priori bounds for the derivatives of u. These estimates are developed by Schauder for the elliptic case and extended by Friedman to the parabolic case, see FRIEDMAN [5]. In the proof of Theorem 2 we use a corollary of these estimates for the semilinear case, which we formulate below.

A Priori Estimate Theorem (see [3], [4]).
Let $Q = (a,b) \times (t_0,t_1)$, $t_0 \ge 0$, $-a$, b, t_1 *possible infinite. Let* $Q_\delta = (a+\delta,b-\delta) \times (t_0+\delta,t_1)$, $0 < \delta < \min((b-a)/2, t_1-t_0)$. *Let* $u \in C^2(Q)$ *and let* u *satisfy* $u_t = u_{xx} + cu_x + f(u)$, $(x,t) \in Q$, *with* $|u|_0^Q \le K$ *and* $f \in C^{0,1}([-K,K])$. *Then the following estimates hold for some* α, $0 < \alpha < 1$, *where* C *is a constant, depending only on* δ *and* α

$$(3.2)\qquad |u|_0^{Q_\delta} + |u_x|_0^{Q_\delta} \le C(|u|_0^Q + |f\circ u|_0^Q),$$

$$(3.3)\qquad |u|_0^{Q_\delta} + |u_x|_0^{Q_\delta} + |u_{xx}|_0^{Q_\delta} + |u_t|_0^{Q_\delta} \le C\{H(f\circ u;1;Q)\cdot (|f\circ u|_0^Q + |u|_0^Q) + |u|_0^Q\},$$

$$(3.4)\qquad H(u_{xx};\alpha;Q_\delta),\ H(u_t;\alpha;Q_\delta) \le C\{H(f\circ u;1;Q)\cdot(|f\circ u|_0^Q + |u|_0^Q) + |u|_0^Q\}.$$

The derivation of (3.2) is based on Friedman's $(1+\delta)$-estimate (see [5], Ch. 7. Thm. 4) and the derivation of (3.3) and (3.4) is based on the interior estimate (see [5], Ch. 3, Thm 5). Note that the constant C does not depend on Q, u and f but only on the distance between Q_δ and the parabolic boundary of Q.

<u>REMARK</u>. We note the difference in formulation of (3.3) with FIFE [3], formula (4.4b). This stems from the fact that in Fife's formula (4.2b) the term $|h|_1^Q$ should be changed into $|h|_0^Q + H(h;\alpha;Q)$ according the interior estimate. The continuity with respect to t enters then in the estimate. The corresponding term in Fife's formula (4.4b) becomes $|f'\circ u|_0^{Q_{\delta/2}}\cdot (|u|_0^{Q_{\delta/2}} + H(u;\alpha;Q_{\delta/2}))$. The second factor can be estimated in the same way as (3.2), because the term $H(u;\alpha;Q_{\delta/2})$ was present in the original $(1+\delta)$-estimate. The final result is (3.3).

<u>PROOF OF THEOREM 1</u>. By (Hg1), (Hh1) and the fact that $f(0) = f(1) = 0$ we can take $\underline{u} \equiv 0$, $\bar{u} \equiv 1$ as sub-, and supersolution. So we know by the Strong Maximum Principle that $0 \le u(x,t) \le 1$. For the proof of Theorem 1 we adapt the technique in OLEINIK & KRUHKZKOV [8] in an obvious way. They treat the corresponding initial value problem in theorem 14. Conditions (Hg2-3), (Hh2-3) give the required smoothness. We note that in stead of $f \in C^1([0,1])$ it is sufficient for this Theorem 1 to take $f \in C^{0,1}([0,1])$.

4. A LEMMA

In this section we prove a weaker form of Theorem 2 for the case $\theta = 1$, but the main ingredients of the proof are already present. In Theorem 2 we give a convergence result which is uniform on the entire half line $\overline{\mathbb{R}^+}$. Here, however, we shall exclude a neighbourhood of the origin.

LEMMA 1. *Let the conditions* (Hf1-2), (Hg1-4), (Hh1-3) *and* (Hh4) *with* $\theta = 1$ *be satisfied, then for arbitrary* $\delta > 0$ *there exists constants* z_0, K,ω, $K > 0$, $\omega > 0$, *such that the solution* $u(x,t)$ *of problem* (P) *satisfies*

$$|u(x,t) - U(x-c_0t-z_0)| < Ke^{-\omega t}, \text{ uniformly } x \geq \delta > 0.$$

The most complicated part of the proof of this lemma is the construction of sub- and supersolutions. Our intention is to bound the solution between translates of the function $U(z)$. In the same spirit as FIFE & McLEOD [4] we try as subsolution

$$\underline{u} = \max(0, U(x-c_0t+s(t)) - q(t)),$$

where we require at this stage that $q > 0$, and that q and s tend monotonically to a limit value for $t \to \infty$, with $q(\infty) = 0$. For an application of the Comparison Theorem we have to check the following conditions

i) $L[u] \geq 0$, $(x,t) \in Q$, see (3.1) for $L[u]$, with $c = 0$,
ii) $u(x,0) \leq g(x)$, $x \in \overline{\mathbb{R}^+}$,
iii) $u(0,t) \leq h(t)$, $t \in \overline{\mathbb{R}^+}$,

both for $u_1 = 0$ and for $u_2 = U(x-c_0t+s(t)) - q(t)$. The conditions on f,g and h imply trivially that u_1 is a regular subsolution. We evaluate $L[u_2]$.

$$(4.1) \qquad L[u_2] = U_{xx} + f(U-q) + c_0U_x - \dot{s}U_x + \dot{q}$$

$$= U_{xx} + c_0U_x + f(U) + f(U-q) - f(U) - \dot{s}U_x + \dot{q}$$

$$= f(U-q) - f(U) - \dot{s}U_x + \dot{q}.$$

To give estimates for this expression we study the difference $f(U-q) - f(U)$ and the behaviour of U_x. For convenience we extend the domain of f as follows

$$\bar{f}(u) = \begin{cases} f'(0)u, & u < 0, \\ f(u), & 0 \le u \le 1, \\ f'(1)(u-1), & 1 < u, \end{cases}$$

then there exists a constant $K > 0$ such that

$$\begin{aligned} &\bar{f}(u-q) - f(u) \ge -Kq, \quad 0 \le q \le 1,\ 0 \le u \le 1, \\ &\bar{f}(u+q) - f(u) \le \ Kq, \quad 0 \le q \le 1,\ 0 \le u \le 1, \end{aligned} \tag{4.2}$$

where

$$K = \sup_{0\le u\le 1} f'(u), \tag{4.3}$$

and further, for any choice of q_1, $0 < q_1 < a$ and q_2, $0 < q_2 < 1-a$ there exists positive numbers μ_1, μ_2, δ_1 and δ_2 such that

$$\begin{cases} \bar{f}(u-q) - f(u) \ge \mu_1 q, & 0 \le q \le 1,\ 0 \le u \le \delta_1, \\ f(u+q) - f(u) \le -\mu_1 q, & 0 \le q \le q_1 < a,\ 0 \le u \le \delta_1, \\ f(u-q) - f(u) \ge \mu_2 q, & 0 \le q \le q_2 < 1-a,\ 1-\delta_2 \le u \le 1, \\ \bar{f}(u+q) - f(u) \le -\mu_2 q, & 0 \le q \le 1,\ 1-\delta_2 \le u \le 1, \end{cases} \tag{4.4}$$

see [4]. We note that in any case $\delta_1 + q_1 < a$ and $\delta_2 + q_2 < 1-a$.

The purpose of the function q is to satisfy ii), so we have to choose $q(0) > 0$. The other requirement on q (q tends to zero monotonically) implies $\dot{q} < 0$, so we have to balance this negative term in (4.1) by $f(U-q)-f(U)$ or by $(-\dot{s}U_x)$. We know that $U(z)$ is monotonically decreasing with $\lim_{|z|\to\infty} \frac{d}{dz}U(z)=0$. Hence for any choice of δ_1,δ_2 it follows that

$$\ell = \sup_{\delta_1\le U\le 1-\delta_2} \frac{d}{dz} U(z) \tag{4.5}$$

is bounded away from zero. If we choose $\dot{s} > 0$ then $-\dot{s}U_x$ is positive and bounded away from zero for values $0 < \delta_1 \le U \le 1-\delta_2 < 1$ and fixed t. For the remaining part of the range of U ($[0,\delta_1)$ and $(1-\delta_2,1]$) we use (4.4). We fix q_1,q_2 then we know $\delta_1,\delta_2,\mu_1,\mu_2$ and we define the functions

(4.6) $$q(t) = q(0)e^{-\beta t}, \quad s(t) = s(0) + \frac{K+\beta}{\beta(-\ell)} q(0)(1-e^{-\beta t}),$$

where

(4.7) $$\beta \le \mu = \min(\mu_1,\mu_2).$$

Later on we restrict β furthermore. We consider (4.1).

1. $0 \le U < \delta_1$, $L[u_2] \ge \mu_1 q - \beta q \ge 0$,
2. $\delta_1 \le U \le 1-\delta_2$, $L[u_2] \ge -Kq + (-\ell)((K+\beta)/(-\ell))q - \beta q = 0$,
3. $1-\delta_2 < U \le 1$, $L[u_2] \ge \mu_2 q - \beta q \ge 0$.

It is thus possible to fulfill condition i). Next we turn to condition ii). By (Hg4) it is always possible to choose $s(0)$ and $q(0)$ such that

(4.8) $$U(x+s(0)) - q(0) \le g(x), \qquad x \in \mathbb{R}^+, \; q(0) < 1-a.$$

From the monotonicity of $U(z)$ it follows that any larger value of $s(0)$ also suffices. Finally we examine condition iii). By (Hh4) we know that there exists constants C_1, T_1 such that

$$1 - h(t) \le C_1 e^{-\gamma t}, \qquad t > T_1,$$

thus for some number $T_2 > T_1$

$$\begin{aligned} u_2(0,t) - h(t) &= U(-c_0 t + s(t)) - q(t) - h(t) \\ &\le U(-c_0 t + s(0)) - 1 + 1 - h(t) - q(t) \\ &\le C_1 e^{-\gamma t} - q(0)e^{-\beta t} < 0, \qquad t > T_2, \end{aligned}$$

if we choose $\beta < \gamma$ and T_2 large enough. By enlarging $s(0)$ it is also possible to fulfill condition iii) for $0 \le t \le T_2$, while it does not disturb the estimate above. So by application of the Comparison Theorem we find

(4.9) $$\underline{u}(x,t) = \max(0,U(x-c_0t+s(t)) - q(t) < u(x,t), \qquad (x,t) \in Q.$$

In an analogous way it is possible to construct a supersolution

(4.10) $\bar{u}(x,t) = \min(1, U(x-c_0t-\bar{s}(t)) + \bar{q}(t)$

where

(4.11) $\bar{q}(t) = \bar{q}(0)e^{-\beta t}, \ \bar{s}(t) = \bar{s}(0) + \frac{K+\beta}{\beta(-\ell)}\bar{q}(0)(1-e^{-\beta t}).$

By examining the corresponding condition ii) we need $\bar{q}(0) \geq \limsup_{x\to\infty} g(x)$, while for application of (4.4) we need $\bar{q}(0) < a$, so we find (Hg4) in a natural way. We pay attention to the corresponding condition iii). From (2.8) we learn that there exists constants C_2, T_3 such that

$$1 - U(-c_0t-\bar{s}(0)) \leq C_3e^{-\beta_1 c_0 t}, \quad t > T_3,$$

thus for some number $T_4 > T_3$

$$\begin{aligned} &U(-c_0t - \bar{s}(t)) + \bar{q}(t) - h(t) \\ &\geq U(-c_0t - \bar{s}(0)) - 1 + 1 - h(t) + \bar{q}(t) \\ &\geq -C_3e^{-\beta_1 c_0 t} + \bar{q}(0)e^{-\beta t} > 0, \quad t > T_3 \end{aligned}$$

if we choose $\beta < \beta_1 c_0$ and T_3 large enough. So finally we set

(4.12) $\beta < \min(\beta_1 c_0, \gamma, \mu).$

With these estimates and the knowledge of the asymptotic behaviour of $U(z)$, $|z| > \infty$, namely (see (2.7), (2.8))

$$\begin{aligned} U(z) &= O(e^{\beta_0 z}), \ z \to \infty, \ \beta_0 = -\frac{c_0}{2} - \sigma_0, \ \sigma_0 > \frac{c_0}{2}, \\ 1 - U(z) &= O(e^{\beta_1 z}), \ z \to -\infty, \ \beta_1 = -\frac{c_0}{2} + \sigma_1, \ \sigma_1 > \frac{c_0}{2}, \end{aligned}$$

we can give the estimates for the function $u(x,t)$

(4.13) $u(x,t) < C_4(e^{(-\frac{c_0}{2}-\sigma)(x-c_0t)} + e^{-\beta t}), \quad x-c_0t \geq 0,$

$$(4.14)\qquad 1-u(x,t) < C_4\left(e^{(-\frac{c_0}{2}+\sigma)(x-c_0t)} + e^{-\beta t}\right), \quad -c_0t \le x-c_0t \le 0,$$

where $\sigma = \min(\sigma_0,\sigma_1)$, so $\beta_0 \le -\frac{c_0}{2} - \sigma < 0$, $\beta_1 \ge -\frac{c_0}{2} + \sigma > 0$. As was noted in section 2 $u(x,t) = v(z,t)$, $z = x-c_0t$ and v satisfies (2.5). Thus the estimates (4.13), (4.14) can be translated directly into estimates for $v(z,t)$.

We want to apply the A Priori Estimate Theorem for the derivatives and give a pointwise bound for $v_z(z,t)$. By choosing $Q = (z-\frac{2}{3}\delta, z+\frac{2}{3}\delta) \times (t-\frac{\delta}{3c_0},\infty)$, and $Q_\delta = (z-\frac{1}{3}\delta, z+\frac{1}{3}\delta) \times (t,\infty)$ we find by (3.2)

$$|v_z(z,t)| \le |v_z|_0^{Q_\delta} < C(|v|_0^Q + |f\circ v|_0^Q).$$

There exists positive constants k,K such that $-ku \le f(u) \le Ku$, and $-k(1-u) \le f(u) \le K(1-u)$, where $k = -\inf_{0\le u\le 1} f'(u)$ and $K = \sup_{0\le u\le 1} f'(u)$. Now we can estimate the right hand side by means of (4.13) and (4.14). This yields

$$(4.15)\qquad |v_z(z,t)| < C_5\left(e^{(-\frac{c_0}{2}-\sigma)z} + e^{-\beta t}\right), \quad z \ge 0,\ t \ge \delta/c_0,$$

$$(4.16)\qquad |v_z(z,t)| < C_5\left(e^{(-\frac{c_0}{2}+\sigma)z} + e^{-\beta t}\right), \quad -c_0t+\delta \le z \le 0,\ t \ge \delta/c_0.$$

The need to take the supremum over the larger domain Q has been met by enlarging the constant C_4 by a factor depending on δ. For values of $z \le 0$ we study the function $w(z,t) = 1-v(z,t)$ and the corresponding equation. We treat the higher derivatives in the same way and we translate the results for the function $u(x,t)$. Then finally we arrive at the following estimates

$$(4.17)\qquad |u_x(x,t)|,\ |u_{xx}(x,t)|,\ |u_t(x,t)|,$$
$$H(u_{xx};\alpha;Q_1), H(u_t;\alpha;Q_1) < C_6\left(e^{(-\frac{1}{2}c_0-\sigma)(x-c_0t)} + e^{-\beta t}\right),$$
$$(x,t) \in Q_1 = \{(x,t) \mid x-c_0t \ge 0,\ t \ge \delta/c_0\},$$

$$(4.18)\qquad |u_x(x,t)|,\ |u_{xx}(x,t)|,\ |u_t(x,t)|,$$
$$H(u_{xx};\alpha;Q_2), H(u_t;\alpha;Q_2) < C_6\left(e^{(-\frac{1}{2}c_0+\sigma)(x-c_0t)} + e^{-\beta t}\right),$$
$$(x,t) \in Q_2 = \{(x,t) \mid -c_0t+\delta \le x-c_0t \le 0,\ t \ge \delta/c_0\}.$$

It will be convenient to extend the domain of the function u(x,t) from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R} \times \mathbb{R}^+$. This will be done in the following way

$$(4.19) \qquad \tilde{u}(x,t) = \begin{cases} 1, & x \le \delta,\ t \ge 0, \\ \psi(x,t), & \delta \le x \le 2\delta,\ t \ge 0, \\ u(x,t), & 2\delta \le x, \quad t \ge 0, \end{cases}$$

where $\psi(x,t)$ represents a smooth connection between the functions u and 1

$$(4.20) \qquad \psi(x,t) = 1 + \chi\left(\frac{2x-3\delta}{\delta}\right)(u(x,t)-1),$$

$$(4.21) \qquad \chi(y) = \frac{\int_{-1}^{y} e^{1-\frac{1}{1-s^2}}\,ds}{\int_{-1}^{1} e^{1-\frac{1}{1-s^2}}\,ds}, \qquad -1 \le y \le 1.$$

We shall use the notation $\tilde{v}(z,t) = \tilde{u}(x,t)$. Now the same type of estimates are valid for $\tilde{u}$ with possibly a larger constant (say C_7), but without the restriction $x \ge \delta$, because firstly the constant part of $\tilde{u}$ does not give any problem and secondly the derivatives of the connection ψ are all bounded in view of the boundedness of u and χ and their corresponding derivatives.

Thus we have found the same estimates as Fife & McLeod did in their treatment for the pure initial value problem. The rest of the proof of this lemma follows the lines of [4]. For the sake of completeness we sketch this remaining part.

1. $\{\tilde{v}(\cdot,t) \mid t \ge t_0 = \delta/c_0\}$ is relatively compact in $C^2(\overline{\mathbb{R}})$.

For a sequence $\{t_n\}_1^\infty$, $t_n \to \infty$, $n \to \infty$, we apply Arzela-Ascoli's Theorem on the interval [-K,K] for every K. Then there exists a subsequence $\{t_{n,K}\}$ such that $\tilde{v}(\cdot,t_{n,K})$ converges in $C^2([-K,K])$, and there exists a subsequence $\{t_{n,K+1}\} \subset \{t_{n,K}\}$ such that $\tilde{v}(z,t_{n,K+1})$ converges to $\bar{v}(z)$ on [-K-1,K+1], and $\bar{v}(z)$ satisfies also the a priori estimates for the limit case $t = \infty$, so $\frac{d^2}{dz^2}\bar{v}(z)$ is uniformly continuous on $\mathbb{R}$. Choose now numbers K and T such that

$$(4.22)\qquad \left|\left(\frac{\partial}{\partial z}\right)^i(\tilde{v}(z,t) - \bar{v}(z))\right| < \varepsilon, \quad |z| < K,\ t > T,\ i = 0,1,2.$$

Choose N such that $t_{N,K} > T$ then

$$(4.23)\qquad \left|\left(\frac{\partial}{\partial z}\right)^i(\tilde{v}(z,t) - \bar{v}(z))\right| < \varepsilon, \quad |z| \le K,\ n \ge N,\ i = 0,1,2.$$

(4.22) and (4.23) together imply that there exists a subsequence $\{t_n'\} \subset \{t_n\}$, $t_n \to \infty$ such that $\lim_{n\to\infty} v(z,t_n') = \bar{v}(z)$ in $C^2(\bar{\mathbb{R}})$.

2. Estimates for $\bar{v}(z)$.

For the function $\bar{v}(z) = \bar{v}(x-c_0t)$ we can give the estimate

$$(4.24)\qquad U(x-c_0t+s(\infty)) \le \bar{v}(x-c_0t) \le U(x-c_0t-\bar{s}(\infty)).$$

We remark that both $s(\infty)$, $\bar{s}(\infty)$ are finite numbers, so the limit function is bounded by two translated travelling waves. It is possible to prove that the limit function itself is a travelling wave, i.e. there exists a z_0 such that $\bar{v}(x-c_0t) = U(x-c_0t-z_0)$. The mathematical tool for proving this purpose is a Lyapunov functional V. To avoid difficulties with convergence we transform $\tilde{v}(z,t)$ to $w(z,t)$, where w lies in the domain of the functional V

$$(4.25)\qquad w(z,t) = \begin{cases} 1, & z \le \varepsilon t - 1, \\ \sigma_-(z,t), & -\varepsilon t - 1 \le z \le -\varepsilon t, \\ \tilde{v}(z,t), & -\varepsilon t \le z \le \varepsilon t, \\ \sigma_+(z,t), & \varepsilon t \le z \le \varepsilon t + 1, \\ 0, & \varepsilon t + 1 \le z, \end{cases}$$

where we have chosen

$$(4.26)\qquad \varepsilon < 2\beta/c_0,$$

and

$$(4.27)\qquad \begin{aligned} \sigma_-(z,t) &= 1 + \chi(2z+2\varepsilon t+1)(\tilde{v}(z,t)-1), \\ \sigma_+(z,t) &= (1 - \chi(2z-2\varepsilon t-1))(\tilde{v}(z,t)). \end{aligned}$$

3. The Lyapunov functional V.

Define the following functional

$$V[w] = \int_{-\infty}^{\infty} e^{c_0 z}\{\tfrac{1}{2}w_z^2 - F(w) + H(-z)F(1)\}dz, \tag{4.28}$$

where $H(z) = 0$, $z < 0$, $H(z) = 1$, $z > 0$; see (2.2) for $F(w)$. We can make the following sequence of statements as a result of the definition of w and (4.17), (4.18) and (4.26)

i) $V[w(\cdot,t)]$ is bounded, independently of $t \geq \delta/c_0$.

ii) Let $V(t) \equiv V[w(\cdot,t)]$, then $\frac{d}{dt}V(t)$ exists and

$$\frac{d}{dt}V(t) = \dot{V}(t) = -\int_{-\infty}^{\infty} e^{c_0 z}\{w_{zz} + c_0 w_z + f(w)\}w_t dz.$$

iii) Let

$$Q[w] = \int_{-\infty}^{\infty} e^{c_0 z}\{w_{zz} + c_0 w_z + f(w)\}^2 dz,$$

then $Q[w] \geq 0$ and

$$\lim_{t\to\infty} |\dot{V}(t) + Q[w](t)| = 0.$$

iv) $\limsup_{t\to\infty} \dot{V}(t) \leq 0$ and even $\limsup_{t\to\infty} \dot{V}(t) = 0$, because otherwise $V[w(\cdot,t)]$ tends to minus infinity, which contradicts i). Thus there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} \dot{V}(t_n) = 0$.

v) $\lim_{n\to\infty} Q[w](t_n) = 0$.

vi) There exist a subsequence $\{t_n'\} \subset \{t_n\}$ such that

$$\lim_{n\to\infty} Q[w](t_n') = 0 \quad \text{and} \quad w(z,t_n') \to \bar{v}(z) \text{ in } C^2(\bar{\mathbb{R}}).$$

vii) For any bounded interval I vi) implies

$$\lim_{n\to\infty} \int_I e^{c_0 z}\{w_{zz} + c_0 w + f(w)\}^2\big|_{t=t_n'} dz = 0,$$

and the same limit is also equal to

$$\int_I e^{c_0 z}\{\bar{v}_{zz} + c_0\bar{v}_z + f(\bar{v})\}^2 dz.$$

So we have $\bar{v}_{zz} + c_0\bar{v}_z + f(\bar{v}) = 0$ for $z \in I$. We know $\bar{v}(-\infty) = 1$, $\bar{v}(\infty) = 0$, thus from the uniqueness in the phase plane, modulo translation, there exists a z_0, such that $\bar{v}(z) = U(z-z_0)$.

viii) We know that $w(z,t_n') \to U(z-z_0)$, $n \to \infty$ in $C^2(\bar{\mathbb{R}})$ and thus also $\tilde{v}(z,t_n') \to U(z-z_0)$, $n \to \infty$. We prove that z_0 does not depend on the choice of the sequence $\{t_n\}$. If for some n_0 $|\tilde{v}(z,t'_{n_0})-U(z-z_0)| < \varepsilon$ we can construct sub- and supersolutions in the same way as at the beginning of the proof of this lemma, such that for all $t \geq t'_{n_0}$ $|\tilde{v}(z,t)-U(z-z_0)| < \varepsilon$ (and also for the corresponding derivatives) by choosing $q(0)$, $\bar{q}(0) = O(\varepsilon)$ and $s(0)$, $\bar{s}(0) = z_0 + O(\varepsilon)$.

4. The rate of convergence is exponential.

For the proof of this statement we refer directly to the paper of Fife & McLeod ([4], section 5). In our notation they find

$$|\tilde{v}(z,t) - U(z-z_0)| < Ke^{-\omega t}, \text{ uniformly } z \in \mathbb{R}.$$

We remark that it follows from their proof that $\omega < \beta$, so also $\omega < \gamma$. See also the discussion in section 5. Keeping in mind that $\tilde{v}(z,t) = v(z,t) = u(x,t)$ for $x \geq 2\delta$ it follows

$$|u(x,t) - U(z-z_0)| < Ke^{-\omega t}, \text{ uniformly } x \geq 2\delta.$$

Thus we have proved the statement of Lemma 1 (δ was arbitrary). At this point we need a better analysis for extending the domain of uniformity up to $x = 0$. This will be done in the next section for the general case ($0 \leq \theta < 1$). In section 6 we apply the same technique for the particular case $\theta = 1$.

5. THE GENERAL CASE

In the preceding section it was possible to prove a uniform convergence result for $x \in [\delta,\infty)$, $\delta > 0$ arbitrary, by the special choice of boundary function $h(t)$, because $\lim_{t\to\infty} h(t) = 1$ and $\lim_{t\to\infty} U(x-c_0t-z_0) = 1$ for a fixed number x. In the general case the limit value of $h(t)$ is θ, $0 \le \theta < 1$, so it will be impossible to prove a result like lemma 1. The influence of the boundary function will play an important role in this section. As in ARONSON & WEINBERGER [1] we encounter in this initial-boundary value problem a threshold effect (see [1], Theorems 5.3 and 5.4).

If the boundary function $h(t)$ fulfills some condition (Hh5), which can be interpretated by saying that this function exceeds some value over a long enough period then $\lim_{x\to\infty} \liminf_{t\to\infty} u(x,t) = 1$ ([1], Theorem 5.4) while there exists another condition which is sufficient to ensure that $\lim_{x\to\infty} \limsup_{t\to\infty} u(x,t) = 0$ ([1], Theorem 5.3). We shall only consider the first case and prove the convergence to a travelling wave in a certain sense. We shall reformulate Theorem 5.4 of [1] but first we introduce the function $Q_\eta(x)$ as solution of

$$(5.1)\qquad \begin{cases} \dfrac{d^2}{dx^2} Q_\eta + f(Q_\eta) = 0, & x \in \mathbb{R}, \\ Q_\eta(0) = \eta,\ \dfrac{d}{dx} Q_\eta(0) = 0, & \kappa < \eta < 1. \end{cases}$$

We find $Q_\eta(x)$ by choosing $k = F(\eta)$ in (2.1). Q_η satisfies the bounds $0 \le Q_\eta \le 1$ only for $x \in [-\ell_\eta, \ell_\eta]$ where

$$\ell_\eta = \int_0^\eta \frac{du}{\sqrt{2(k-F(u))}},$$

and

$$Q_\eta(\pm\ell_\eta) = 0, \qquad Q_{\eta_x}(\pm\ell_\eta) = \mp\sqrt{2k}\ .$$

<u>LEMMA 2</u> (see [1], Theorem 5.4) *Let the conditions* (Hf1-2), (Hg1-3), (Hh1-4) *with* $g \equiv 0$ *be satisfied. There exists for any* $\eta \in (\kappa,1)$ *a positive number* T_η *such that if*

$$h(t) \geq \eta,\ t \in (t_1, t_1+T_\eta) \quad \textit{for some } t_1 > 0$$

then the solution u *of problem* (P) *satisfies*

(5.2) $$u(x, t_1+T_\eta) \geq Q_\eta(x-1-\ell_\eta),\ x \in (1, 1+2\ell_\eta),$$

and

$$\lim_{x\to\infty} \liminf_{t\to\infty} u(x,t) = 1.$$

Lemma 2 proves the existence of the number T_η in condition (Hh5).

Next we shall prove the existence of positive constants c_1, C, γ^*, T_0 such that

(5.3) $$1 - u(c_1 t, t) < Ce^{-\gamma^* t}, \quad t > T_0.$$

Consider therefore the following problem

(P') $$\begin{cases} \underline{u}_t = \underline{u}_{xx} + f(\underline{u}), & x > 0,\ t > t_1+T_\eta, \\ \underline{u}(x, t_1+T_\eta) = \begin{cases} Q_\eta(x-1-\ell_\eta), & x \in (1, 1+2\ell_\eta), \\ 0 & , x \in [0,1] \cup [1+2\ell_\eta, \infty), \end{cases} \\ \underline{u}(0,t) = 0, & t > t_1+T_\eta. \end{cases}$$

The solution of (P') is a subsolution for the solution of (P) so that $\underline{u}(x,t) \leq u(x,t)$, $x \geq 0$, $t \geq t_1+T_\eta$. Now we apply the Proposition 5.1 in [1] to show that $\underline{u}(x,t)$ is nondecreasing in t and $\lim_{t\to\infty} \underline{u}(x,t) = \phi(x)$ uniformly on bounded x-intervals, where $\phi(x)$ is the smallest nonnegative solution of the equation $\phi_{xx} + f(\phi) = 0$, $x > 0$ such that

i) $\phi(0) = \lim_{t\to\infty} \underline{u}(0,t) = 0$,

ii) $\phi(x) \geq Q_\eta(x-1-\ell_\eta)$, $x \in (1, 1+2\ell_\eta)$.

We shall show that these conditions imply that $\phi(x) = V_0(x)$ (see section 2 for $V_0(x)$). Clearly $\phi(0) = V_0(0)$. Next we shall prove that $V_0(x)$ satisfies ii). Choose any number p, $0 \leq p \leq \eta$, and define x_1 and x_2 by $V_0(x_1) = p$,

$Q_\eta(x_2-1-\ell_\eta) = p$, $x_2 \le 1 + \ell_\eta$. By the monotonicity of $V_0(x)$ ii) is equivalent with the inequality $x_1 \le x_2$. From (2.1) we learn

$$x_1 = \int_0^p \frac{du}{\sqrt{2(F(1)-F(u))}} < \int_0^p \frac{du}{\sqrt{2(F(\eta)-F(u))}} = x_2-1,$$

because $F(1) > F(\eta)$, so the statement follows. From the convergence of $\underline{u}(x,t)$ to $V_0(x)$ uniformly on bounded x-intervals we learn that for any given x-interval $[0,X]$ and for any given $\varepsilon > 0$, there exists a time $T_0 = T_0(X,\varepsilon)$ such that

(5.4) $$0 < V_0(x) - \underline{u}(x,T_0) < \varepsilon, \quad x \in (0,X].$$

<u>DEFINITION 3</u>. Define the function $m(x,t) = \underline{u}(x,t+T_0)$, $x \ge 0$, $t \ge 0$. Then $m(x,t)$ is the solution of problem (P")

(P") $$\begin{cases} m_t = m_{xx} + f(m), & (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+), \\ m(x,0) = \underline{u}(x,T_0), & x \in \overline{\mathbb{R}^+}, \\ m(0,t) = 0, & t \in \overline{\mathbb{R}^+}. \end{cases}$$

For the solution of problem (P") we shall construct a subsolution $\underline{m}(x,t)$. The construction of this subsolution is complicated and we do the calculations in an Appendix (section 7). From that construction we learn that there exists positive constants c_1, $\tilde{C}$, γ^* such that $c_1 < c_0$ and

(5.5) $$1 - \underline{m}(c_1t+c_1T_0,t) < \tilde{C}e^{-\gamma^* t}, \quad t > 0.$$

See formula (7.59) in the Appendix. Because of the inequalities $\underline{m}(x,t) \le m(x,t) = \underline{u}(x,t+T_0) \le u(x,t+T_0)$ we learn from (5.5) that also

(5.6) $$1 - u(c_1,t,t) \le 1 - \underline{m}(c_1t,t-T_0) < \tilde{C}e^{-\gamma^*(t-T_0)}, \quad t > T_0,$$

which is equivalent with (5.3).

<u>DEFINITION 4</u>. Define the following subdomain of Q

(5.7) $$Q^* = \{(x,t) \mid x > \bar{x}(t) = c_1 t, \quad t > 0\}.$$

In this domain Q^* we apply Lemma 1. We formulate the result as Lemma 3.

LEMMA 3. *Let the conditions* (Hf1-2), (Hg1-4), (Hh1-5) *be satisfied, then for arbitrary* $\delta > 0$ *there exists constants* z_0, K^*, ω^*, $K^* > 0$, $\omega^* > 0$ *such that the solution of problem* (P) *satisfies*

(5.8) $$|u(x,t) - U(x-c_0 t-z_0)| < K^* e^{-\omega^* t}, \quad \text{uniformly} \quad x \geq \bar{x}(t) + \delta.$$

PROOF. Apply Lemma 1 to the domain Q^*. The role of the boundary function will be played by the function $u(x,t)$ itself. We learn from (5.3) that the behaviour of u along the t-dependent boundary of Q^* fulfills the conditions of Lemma 1. The fact that the lower bound of x of the domain Q^* depends on t does not matter in the proof. The only point to check is whether the argument of the sub- and supersolutions along this boundary tends to minus infinity. For the subsolution this argument reads $\bar{x}(t) - c_0 t + s(t)$. In view of the behaviour of $s(t)$, the fact that $c_1 < c_0$ and the equality

(5.9) $$\bar{x}(t) - c_0 t + s(t) = (c_1-c_0)t + s(t), \quad t \geq 0,$$

this argument runs to minus infinity as $t \to \infty$. An analogous result holds for the supersolution. For the corresponding β value (see (4.12)) we have to take

(5.10) $$\beta^* < \min(\beta_1(c_0-c_1), \gamma^*, \mu) = \gamma^*,$$

so we know $\omega^* < \beta^*$. The equality sign in (5.10) follows from the calculations in the Appendix. □

Now fix the number $\delta > 0$ in Lemma 3 and consider the complement of Q^* in Q.

DEFINITION 5. Define the following subdomain of Q

(5.11) $$Q^{**} = \{(x,t) \mid 0 < x < \bar{x}^*(t) = \bar{x}(t)+\delta = c_1 t+\delta,\ t > 0\}.$$

From Definitions 4, 5 it follows that $Q\backslash Q^* \subset Q^{**}$ and $Q\backslash Q^{**} \subset Q^*$.

We give the following a priori lower bounds for $u(x,t)$.

(5.12) $$u(x,t) \geq \underline{m}(x,t-T_0), \quad (x,t) \in Q^{**},\ t \geq T_0,$$

(5.13) $$u(x,t) \geq U(x-c_0 t-z_0) - K^* e^{-\omega^* t}, \quad (x,t) \in Q\backslash Q^{**},\ t \geq 0.$$

Estimate (5.12) follows from the construction of the subsolution $\underline{m}$ in the Appendix. Estimate (5.13) follows from the result of Lemma 3, where the domain of convergence was $Q\backslash Q^{**}$: u converges to a travelling wave. Next we shall prove that u converges to V_θ for $(x,t) \in Q^{**}$. Therefore we define the following function $\tilde{u}(x,t)$ for the fixed chosen $\delta > 0$,

(5.14) $$\tilde{u}(x,t) = \begin{cases} u(x,t), & 0 \leq x \leq \bar{x}^*(t),\ t \geq 0, \\ u(x,t) + \chi\left(\frac{2x-2\bar{x}^*(t)-\delta}{\delta}\right)(V_\theta(x)-u(x,t)), & \\ & \bar{x}^*(t) \leq x \leq \bar{x}^*(t)+\delta,\ t \geq 0, \\ V_\theta(x), & \bar{x}^*(t)+\delta \leq x,\ t \geq 0. \end{cases}$$

See (4.21) for $\chi(x)$. From the definition (5.14) and the estimates (5.12) and (5.13) we obtain for some $C_1 > 0$

(5.15) $$1 - \tilde{u}(x,t) \leq C_1 e^{-\tau x}, \quad x \geq 0,\ t \geq 0,$$

where $\tau = \min(\nu,\omega^*/c_1)$. According to the A Priori Estimate Theorem we have for $Q_{\delta_1} = \{(x,t) \mid x > \delta_1,\ t > \delta_1\}$, δ_1 a small fixed number, for some $C_2 > 0$

(5.16) $$|\tilde{u}_x|,\ |\tilde{u}_{xx}|,\ |\tilde{u}_t| \leq C_2 e^{-\tau x}, \quad (x,t) \in Q_{\delta_1},$$

(5.17) $$H(\tilde{u}_{xx};\alpha;Q_{\delta_1}),\ H(\tilde{u}_t;\alpha;Q_{\delta_1}) \le C_2 e^{-\tau x}.$$

In view of the fact that $u \in C^{2,\alpha}(\overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+})$ (see Theorem 1) we know that there exists some number $M > 0$ such that

$$|\tilde{u}_x|,\ |\tilde{u}_{xx}|,\ |\tilde{u}_t|,\ H(\tilde{u}_{xx};\alpha;\bar{Q}),H(\tilde{u}_t;\alpha;\bar{Q}) \le M,$$

so that we can extend (5.16), (5.17) up to the boundary $x = 0$, $t = 0$. Thus for some $C_3 > 0$

(5.18) $$|\tilde{u}_x|,\ |\tilde{u}_{xx}|,\ |\tilde{u}_t| \le C_3 e^{-\tau x}, \qquad (x,t) \in \tilde{Q} = \{(x,t) \mid x \ge 0, t \ge 0\},$$

(5.19) $$H(\tilde{u}_{xx};\alpha;\tilde{Q}),\ H(\tilde{u}_t;\alpha;\tilde{Q}) \le C_3 e^{-\tau x}.$$

We use these estimates to prove that $\lim_{t\to\infty} \tilde{u}(x,t) = V_\theta(x)$, uniformly $x \in \overline{\mathbb{R}^+}$. As in Lemma 1 we can make the following statements

1. $\{\tilde{u}(\cdot;t) \mid t \ge 0\}$ is relatively compact in $C^2(\overline{\mathbb{R}^+})$.

The proof runs along the same lines as in Lemma 1. We denote the limit function by $w(x)$.

2. The limit function satisfies

$$\max(0,2V_0(x)-1) \le w(x) \le 1.$$

This estimate is trivial after the discussion above. To prove that $w(x) = V_\theta(x)$ we need again a Lyapunov functional.

3. The Lyapunov function V.

Define the functionals

$$V[\tilde{u}] = \int_0^\infty \{\tfrac{1}{2}\tilde{u}_x^2 - F(\tilde{u}) + F(1)\}dx,$$

$$Q[\tilde{u}] = \int_0^\infty \{\tilde{u}_{xx} + f(\tilde{u})\}^2 dx,$$

which are well defined by (5.15) and (5.16). As Lemma 1 we find

$$w_{xx} + f(w) = 0, \quad x \in [0,L] \text{ for every } L,$$

$$w(0) = \lim_{t\to\infty} \tilde{u}(0,t) = \lim_{t\to\infty} h(t) = \theta,$$

$$\lim_{x\to\infty} w(x) = 1.$$

From the uniqueness in the phase plane we conclude $w(x) = V_\theta(x)$, $x \geq 0$. By the uniquness of the limit in $C^2(\overline{\mathbb{R}^+})$ it follows that $\tilde{u}(x,t)$ converges to $V_\theta(x)$ not only along sequences but for all t, thus

$$\text{(5.20)} \qquad \lim_{t\to\infty} \tilde{u}(x,t) = V_\theta(x), \text{ uniformly } x \in \overline{\mathbb{R}^+}.$$

4. The rate of convergence is exponential.

For this property we cannot directly refer to the proof of FIFE & McLEOD [4], because of the influence of of the function $h(t)$, but after some appropriate changes we can still use the basic idea underlying their proof. Define

$$\text{(5.21)} \qquad k(x,t) = \tilde{u}(x,t) - V_\theta(x-\alpha(t)),$$

where $\alpha(t)$ has been chosen so that (recall $\tilde{u}(0,t) = u(0,t) = h(t)$)

$$k(0,t) = h(t) - V_\theta(-\alpha(t)) = 0.$$

So $\lim_{t\to\infty} \alpha(t) = 0$, because $\lim_{t\to\infty} h(t) = \theta$. It follows from the implicit function theorem that $\alpha(t)$ exists and that it is continuously differentiable. In view of the fact that $h(t) - \theta = O(e^{-\gamma t})$, as $t \to \infty$, we find $\theta - V_\theta(-\alpha(t)) = O(e^{-\gamma t})$, as $t \to \infty$ and even $\alpha(t) = O(e^{-\gamma t})$, as $t \to \infty$.

From the definition of $\tilde{u}(x,t)$ (see (5.14)) we learn that $\tilde{u}$ satisfies

$$\text{(5.22)} \qquad \tilde{u}_t = \tilde{u}_{xx} + f(\tilde{u}) + r,$$

where

$$r(x,t) = \begin{cases} = 0, & 0 \le x < \bar{x}^*(t),\ \bar{x}^*(t) + \delta < x,\ t \ge 0, \\ \ne 0, & \bar{x}^*(t) \le x \le \bar{x}^*(t)+\delta,\ t \ge 0, \end{cases}$$

and for some number $C > 0$

$$|r(x,t)| \le Ce^{-\tau x}, \quad \bar{x}^*(t) \le x \le \bar{x}^*(t)+\delta, \quad t \ge 0$$

as a consequence of (5.18) and (5.19). So $k(x,t)$ satisfies

$$k_t(x,t) = k_{xx}(x,t)+\alpha'(t)V_{\theta_x}(x-\alpha(t))+f(V_\theta(x-\alpha(t))+k(x,t))$$
$$- f(V_\theta(x-\alpha(t)))+r(x,t).$$

By (Hf1) ($f \in C^1([0,1])$ this can be written as

(5.23) $$k_t(x,t) = k_{xx}(x,t)+f'(V_\theta(x))k(x,t)+s(x,t)$$
$$+ \alpha'(t)V_{\theta_x}(x-\alpha(t))+r(x,t),$$

where $s(x,t) = o(k(x,t))$ as $t \to \infty$.

Define the operator M

$$Mk = -k_{xx} - f'(V_\theta(x))k, \quad k(0) = 0.$$

M is symmetric, bounded below on $C_0^\infty(0,\infty) \cap L^2(0,\infty)$, so according Friedrichs Theorem M possesses a self-adjoint extension with $\mathcal{D}(M) \subset L^2(0,\infty)$. We study the behaviour of $k(x,t)$ for large values of x. Fix $t > 0$, then

$$\lim_{x\to\infty} k(x,t) = \lim_{x\to\infty} V_\theta(x)-V_\theta(x-\alpha(t)) = \lim_{x\to\infty} \alpha(t)V_{\theta_x}(\xi(x)) = 0,$$

where ξ represents some function with $\xi(x) > x$. In view of the fact that $V_{\theta_x}(\xi) = O(e^{-\nu\xi})$, $\xi \to \infty$, and $\alpha(t) \le \sup_{0\le t<\infty} \alpha(t) = A^*$, we find that

$$k(x,t) \le Ae^{-\nu x}, \quad x \ge \bar{x}^*(t) + \delta,$$

so that

$$\bar{k}(t) = \int_0^\infty k(x,t)dx \le \int_0^{\bar{x}^*(t)+\delta} k(x,t)dx + A/\nu.$$

Because $\lim_{t\to\infty} k(x,t) = 0$ uniformly in x, we know that given $\varepsilon > 0$, for t larger than some number T_ε

(5.24) $$\bar{k}(t) \le (\bar{x}^*(t)+\delta)\varepsilon + A/\nu = C_1 t + C_2,$$

with

$$C_1 = \varepsilon c_1, \quad C_2 = (x_0+\delta)\varepsilon + A/\nu.$$

From the theory of singular Sturm-Liouville problems we learn (see TITCHMARSH [4])) that the spectrum $\sigma(M)$ of M consists of a continuum in the interval $[\bar{\lambda},\infty)$, where $\bar{\lambda} = \lim_{x\to\infty} - f'(V_\theta(x)) = -f'(1) > 0$ and possible a discrete part in $(-\lambda(\theta),\bar{\lambda})$, where $\bar{\lambda}(\theta) = \sup_{0\le x<\infty} f'(V_\theta(x))$. Now we shall prove that the smallest eigenvalue of M, λ_0, is positive (if it exists). Let ℓ represent the eigenfunction belonging to the eigenvalue of λ_0, so

(5.25) $$\ell_{xx} + f'(V_\theta(x))\ell = -\lambda_0\ell, \ \ell(0) = 0, \ \ell(x) > 0, \ x \in (0,\infty).$$

We differentiate the expression for V_θ $(V_{\theta_{xx}} + f(V_\theta) = 0)$.

(5.26) $$V_{\theta_{xxx}} + f'(V_\theta)V_{\theta_x} = 0.$$

Multiply (5.25) by $(-V_{\theta_x})$ and (5.26) by ℓ, add both expressions and integrate over $(0,\infty)$:

$$\lambda_0 \int_0^\infty \ell V_{\theta_x} dx = \int_0^\infty (\ell V_{\theta_{xxx}} - \ell_{xx} V_{\theta_x})dx = \ell_x(0).V_{\theta_x}(0).$$

We know that $V_{\theta_x}(0) > 0$ and also that $\ell_x(0) \neq 0$ (otherwise ℓ should be identical zero), so $\int_0^\infty \ell V_{\theta_x} dx > 0$ and $\ell_x(0).V_{\theta_x}(0) > 0$, and hence $\lambda_0 > 0$.

Multiply (5.23) by k and integrate over $(0,\infty)$, then $(\|k\|^2 = \int_0^\infty k^2 dx)$

$$(5.27)\qquad \frac{1}{2}\frac{d}{dt}\|k\|^2 = (-Mk,k) + \int_0^\infty s.k dx + \alpha'(t)\int_0^\infty V_{\theta_x}(x-\alpha(t))k(x,t)dx + \int_0^\infty r.k dx,$$

where

$$(5.28)\qquad \left|\int_0^\infty s.k dx\right| = o(1)\ \|k\|^2, \quad t \to \infty,$$

$$(5.29)\qquad \left|\alpha'(t)\int_0^\infty V_{\theta_x}(x-\alpha(t))k(x,t)dx\right| = O(\alpha'(t)).O(\bar{k}(t)) = = O((C_1 t + C_2)e^{-\gamma t}), \quad t \to \infty,$$

$$(5.30)\qquad \left|\int_0^\infty r.k dx\right| \le \int_{\bar{x}^*(t)}^{\bar{x}^*(t)+\delta} Ce^{-\tau x}dx < \delta Ce^{-\tau \bar{x}^*(t)} = C_3 e^{-\tau c_1 t}, \ t \to \infty.$$

For $k \in \mathcal{D}(M)$ we have $(-Mk,k) \le (-\lambda_0)\|k\|^2$, so from (5.28), (5.29), (5.30) there exists for arbitrary $\rho, \frac{1}{2} \le \rho < 1$ a number T_0 such that for some number $C_4 > 0$

$$(5.31)\qquad \frac{1}{2}\frac{d}{dt}\|k\|^2 \le (-\lambda\rho_0)\|k\|^2 + C_4 e^{-\bar{\gamma}t}, \quad t \ge T_0,\ \bar{\gamma} = \min(\gamma, \tau c_1).$$

Finally we find

$$(5.32)\qquad \|k\|^2 \le \frac{2C_4}{2\lambda_0\rho-\bar{\gamma}}e^{-\bar{\gamma}t} + (\|k\|^2_{t=T_0} - \frac{2C_4}{2\lambda_0\rho-\bar{\gamma}})e^{-2\lambda_0\rho t}, \quad t \ge T_0.$$

Thus if

i) $\quad 2\lambda_0\rho > \bar{\gamma},\ \|k\|^2 = O(e^{-\bar{\gamma}t}), \qquad t \to \infty,$

ii) $\quad 2\lambda_0\rho < \bar{\gamma},\ \|k\|^2 = O(e^{-2\lambda_0\rho t}), \qquad t \to \infty.$

If $\lambda_0 > \bar{\gamma}/2$, then there exists a ρ, $\frac{1}{2} \le \rho < 1$ such that $2\lambda_0\rho > \bar{\gamma}$, so we have case i); if $\lambda_0 \le \bar{\gamma}/2$, then $2\lambda_0\rho < \bar{\gamma}$, and we have case ii). So the decay is $O(e^{-\tilde{\mu}t})$, where $\tilde{\mu}$ is arbitrarily close to $\min(2\lambda_0,\gamma,\tau c_1)$, which is in turn arbitrarily close to $\min(2\lambda_0,\gamma,c_1\nu,\omega^*)$ because $\tau = \min(\nu,\omega^*/c_1)$.

Now we use [5], Lemma 5.1. For $f \in C^1(\overline{\mathbb{R}^+}) \cap L^2(\mathbb{R}^+)$

$$|f|_0^3 \le \frac{2}{3}\,|f|_1.\|f\|^2,$$

so $|k(\cdot,t)|_0 \le C_5 e^{-\tilde{\mu}/3t}$ for some number $C_5 > 0$, because $|k(\cdot,t)|_1$ can be estimated uniformly by a constant (we recall that $u \in C^{2,\alpha}(\bar{Q})$). So

$$\sup_{0\le x<\infty} |\tilde{u}(x,t)-V_\theta(x)| = \sup_{0\le x<\infty} |\tilde{u}(x,t)-V_\theta(x-\alpha(t))-\alpha(t)V_{\theta_x}(\xi(x))|$$

$$\le |k(\cdot,t)|_0 + C_6|\alpha(t)| \le C_5 e^{-\tilde{\mu}/et} + C_6 e^{-\gamma t}.$$

Thus there exists a $\tilde{K}$, $\tilde{\omega}$, $\tilde{K} > 0$, $\tilde{\omega} > 0$, $\tilde{\omega} = \frac{1}{3}\tilde{\mu}$ such that

$$|\tilde{u}(x,t) - V_\theta(x)| < \tilde{K}e^{-\tilde{\omega}t}, \text{ uniformly } x \in \overline{\mathbb{R}^+}. \tag{5.33}$$

If we restrict (5.33) to the domain Q^{**} and we use (5.8) for $Q\backslash Q^{**}$ we find $(C_7,C_8 > 0)$

$$\begin{aligned} &|u(x,t) - U(x-c_0t - z_0) - V_\theta(x) + 1| \\ &\le |u(x,t) - V_\theta(x)| + |1-U(x-c_0t-z_0)| \\ &\le \tilde{K}e^{-\tilde{\omega}t} + C_7e^{-\beta_1(c_0-c_1)t}, \qquad (x,t) \in Q^{**}, \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} &|u(x,t) - U(x-c_0t - z_0) - V_\theta(x) + 1| \\ &\le |u(x,t) - U(x-c_0t-z_0)| + |1-V_\theta(x)| \\ &\le K^*e^{-\omega^*t} + C_8e^{-\nu x} \\ &\le K^*e^{-\omega^*t} + C_8e^{-\nu(\delta+c_1t)}, \qquad (x,t) \in Q\backslash Q^{**}. \end{aligned} \tag{5.35}$$

Taking (5.34),(5.35) together, we learn that there exists constants z_0, K, ω, $K > 0$, $\omega = \min(\tilde{\omega},\omega^*,\beta_1(c_0-c_1),\nu c_1) = \frac{1}{3}\tilde{\mu} > 0$ such that

$$(5.36) \qquad |u(x,t) - U(x-c_0t-z_0) - V_\theta(x) + 1| < Ke^{-\omega t}, \text{ uniformly } x \in \overline{\mathbb{R}^+},$$

which was the first statement of Theorem 2.

6. THE CASE $\theta = 1$

As was observed in section 4 it was not possible to extend the uniform convergence domain up to $x = 0$, because the a priori estimates did bot hold there. Nevertheless it is possible to prove the same result (5.36) for $\theta = 1$. We know already that the solution converges to 1 for $\bar{x}^*(t) = c_1 t$ (even for $x = \delta$) in an exponential way. So we can apply the techniques of section 5 directly with the only difference that we take $u = 1$ instead of $u = V_\theta(x)$. Finally we find that $\lim_{t\to\infty} u(x,t) = 1$ uniformly $0 \le x \le \bar{x}^*(t)$, with decay $O(e^{-\tilde{\omega}t})$, with $\tilde{\omega} = \frac{1}{3}\tilde{\mu}$ and $\tilde{\mu}$ arbitrarily close to $\min(-2f'(1),\gamma,c_1\nu,\omega^*)$. This result together with (5.8) gives the second statement of Theorem 2, with $\omega = \frac{1}{3}\tilde{\mu}$.

7. APPENDIX

In this Appendix we shall construct a subsolution $\underline{m}$ for problem (P")

$$(\text{P}'') \qquad \begin{cases} m_t = m_{xx} + f(m), & (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+), \\ m(x,0) = \underline{u}(x,T_0), & x \in \overline{\mathbb{R}^+}, \\ m(0,t) = 0, & t \in \overline{\mathbb{R}^+}. \end{cases}$$

We know that it is possible to choose T_0 such that for any given x-interval $(0,X]$ and for any given $\varepsilon > 0$

$$(7.1) \qquad 0 < V_0(x) - \underline{u}(x,T_0) < \varepsilon, \quad x \in (0,X].$$

Our object is to prove that there exists positive constants c_1, $\tilde{C}$, γ^* such that

$$(7.2) \qquad 1 - \underline{m}(c_1t+c_1T_0,t) < \tilde{C}e^{-\gamma^* t}, \quad t > 0.$$

The subsolution $\underline{m}(x,t)$ will be composed of three functions $\underline{m}_i$, $i = 1,2,3$

$$(7.3) \qquad \underline{m}_1(x,t) = 0,$$

$$(7.4) \qquad \underline{m}_2(x,t) = V_0(x-r(t)) - p_0 e^{\beta x - \alpha t},$$

$$(7.5) \qquad \underline{m}_3(x,t) = U(x-c_0 t+s(t)) - q_0 e^{-\gamma t},$$

where $r(t)$ and $s(t)$ are defined by

$$(7.6) \qquad r(t) = R(1-e^{-\delta t}),$$

$$(7.7) \qquad s(t) = s(0) + S(1-e^{-\gamma t}).$$

We define $V_0(x)$ for negative x as the natural continuation of $V_0(x)$ for positive x, then $V_0(x)$ is negative for negative x.

In the sequel we have to specify the positive parameters p_0, β, α, R, δ, q_0, γ, S and the parameter $s(0)$. We define the subsolution $\underline{m}(x,t)$ as follows

$$(7.8) \qquad \underline{m}(x,t) = \begin{cases} \max(\underline{m}_1,\underline{m}_2), & 0 \le x \le x_1(t), \\ \max(\underline{m}_2,\underline{m}_3), & x_1(t) \le x \le x_2(t), \\ & \qquad\qquad \underline{m}_{2_x}(x_2(t),t) < 0, \\ \max(\underline{m}_3,\underline{m}_1), & x_2(t) \le x \le x_3(t), \\ \underline{m}_1, & x_3(t) \le x. \end{cases}$$

The function $\underline{m}(x,t)$ has been pictured in figure 1.

First we shall calculate under which conditions on the parameters the functions $\underline{m}_i$ satisfy the differential inequalities

$$(7.9) \qquad L[\underline{m}_i] = \underline{m}_{i_{xx}} + f(\underline{m}_i) - \underline{m}_{i_t} \ge 0, \quad x_{i-1}(t) < x < x_i(t),\ t > 0,$$

$$i = 2,3.$$

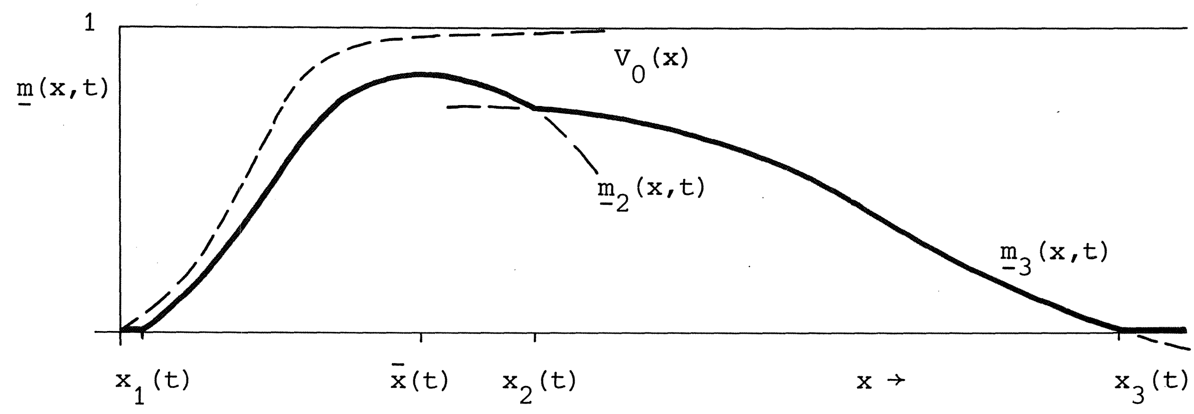

fig. 1

We remark that $\underline{m}_1(x,t) = 0$ is a trivial subsolution.

We evaluate $L[\underline{m}_2]$:

(7.10) $$L[\underline{m}_2] = f(V_0-p) - f(V_0) - \beta^2 p - \alpha p + \dot{r}\, V_{0_x},$$

where we have used the shorthand notation $V_0 = V_0(x-r(t))$ and $p = p_0 e^{\beta x-\alpha t}$. In the same way as in section 4 we split the range of V_0 up in parts: $0 \le V_0 \le 1-\delta_2$ and $1-\delta_2 < V_0 \le 1$. But before we can proceed we have to bound the function $x_2(t)$. It is possible to prove that $x_2(t) = c_2 t + O(1)$, $t \to \infty$, for some positive constant c_2, which will determined later (see (7.56)). Now we can bound $p = p(x,t)$ as follows

(7.11) $$p = p_0 e^{\beta x-\alpha t} \le p_0^* e^{(\beta c_2-\alpha)t}, \quad x \le x_2(t),\ t \ge 0.$$

We choose

(7.12) $$\delta = -(\beta c_2-\alpha),$$

and by a proper choice of α,β we can insure that δ is positive. Choose now ρ an arbitrary small number and let $\delta_2 = \delta_2(\rho)$, and $p_2 = p_2(\rho)$ be such that the third inequality in (4.4) holds:

(7.13) $$f(u-p) - f(u) \geq \mu_2 p, \quad 0 \leq p \leq p_2 < 1-a, \ 1-\delta_2 \leq u \leq 1,$$

where

(7.14) $$\mu_2 = \nu^2(1-\rho).$$

As in section 4, let

(7.15) $$k = \inf_{0 \leq V_0 \leq 1-\delta_2} \frac{d}{dx} V_0(x).$$

Choose further

(7.16) $$R = (K + \beta^2 + \alpha)p_0^*/\delta k,$$

see (4.3) for K and let

(7.17) $$\beta^2 + \alpha = \mu_2.$$

We consider (7.10)

1. $0 \leq V_0 \leq 1-\delta_2$,
   $$L[\underline{m}_2] \geq -Kp - (\beta^2+\alpha)p + \delta R e^{-\delta t} k$$
   $$\geq e^{-\delta t}[-Kp_0^* - (\beta^2+\alpha)p_0^* + \delta Rk] = 0, \quad \text{by (7.16)}.$$
2. $1-\delta_2 < V_0 \leq 1$,
   $$L[\underline{m}_2] \geq \mu_2 p - (\beta^2+\alpha)p = 0, \quad \text{by (7.10)}.$$

We note that this inequality holds only for

(7.18) $$p(x,t) \leq p_2.$$

We shall discuss this condition later on. We evaluate $L[\underline{m}_3]$.

(7.19) $$L[\underline{m}_3] = f(U-g) - f(U) - \gamma q - \dot{s}U_x,$$

where we have used the shorthand notation $U = U(x-c_0t+s(t))$ and $q = q_0e^{-\gamma t}$. In exactly the same way as in section 4 we prove that $L[\underline{m}_3] \geq 0$ by choosing

(7.20) $$\ell = \sup_{\delta_1 \leq U \leq 1-\delta_2} \frac{d}{dz} U(z),$$

(7.21) $$\gamma \leq \min(\mu_1, \mu_2),$$

(7.22) $$S = (K+\gamma)q_0/\gamma(-\ell),$$

where δ_1 is an arbitrary small number and μ_1 follows from (4.4). In view of (7.13) we have to choose $q_0 < p_2$. We recall the asymptotic behaviour of $V_0(x)$ and $U(z)$, see (2.3) and (2.8)

(7.23) $$1 - V_0(x) = a_1e^{-\nu x}(1+o(1)), \quad x \to \infty, \ \nu^2 = -f'(1),$$

(7.24) $$1 - U(z) = a_2e^{\beta_1 z}(1+o(1)), \quad z \to -\infty, \ \beta_1 = -\tfrac{1}{2}[c_0-\sqrt{c_0^2+4\nu^2}],$$

where a_1 and a_2 are some positive numbers; see for analogous results UCHIYAMA [10, Thm.2.1]. We shall use the relation

(7.25) $$(\nu+\beta_1)(\nu-\beta_1) = c_0\beta_1,$$

which follows from the quadratic equation $\beta^2 + c_0\beta - \nu^2 = 0$ for β_1.

Next we determine the behaviour of the function $\bar{x}(t)$ defined by $\frac{d}{dx} \underline{m}_2(\bar{x}(t),t) = 0$:

(7.26) $$\frac{d}{dx} V_0(x-r(t)) - \beta p_0 e^{\beta x-\alpha t}\Big|_{x=\bar{x}(t)} = 0.$$

By the relation

(7.27) $$\frac{d}{dx} V_0(x) = \nu(1-V_0(x))(1+o(1)), \quad x \to \infty,$$

and by (7.23) we find

(7.28) $$\frac{d}{dx} V_0(x) = \nu a_1 e^{-\nu x}(1+o(1)), \quad x \to \infty.$$

Using (7.26) and (7.28) we find with the aid of the implicit function theorem for $\bar{x}(t)$ the following expression

(7.29) $$\bar{x}(t) = c_3 t + \bar{x}_0 + o(1), \quad t \to \infty, \quad c_3 = \frac{\alpha}{\nu+\beta},$$

and for $\underline{m}_2(\bar{x}(t),t)$

(7.30) $$\underline{m}_2(\bar{x}(t),t) = V_0(\bar{x}(t)-r(t)) - p_0 e^{\beta\bar{x}(t)-\alpha t}$$

$$= V_0(\bar{x}(t) - r(t)) - \frac{1}{\beta}\frac{d}{dx} V_0(\bar{x}(t)-r(t)).$$

In view of (7.27) we know the existence of a positive constant A such that

(7.31) $$\frac{d}{dx} V_0(x) \leq A(1-V_0(x)), \quad x \geq 0,$$

so we can estimate $\underline{m}_2(\bar{x}(t),t)$ as follows

(7.32) $$\underline{m}_2(\bar{x}(t),t) \geq (1 + \frac{A}{\beta})V_0(\bar{x}(t)-r(\infty)) - \frac{A}{\beta}.$$

Let us compare $\underline{m}_2(\bar{x}(t),t)$ with $\underline{m}_3(\bar{x}(t),t)$. An easy estimate for $\underline{m}_3(\bar{x}(t),t)$ reads

(7.33) $$\underline{m}_3(\bar{x}(t),t) \leq \sup_{-\infty<x<\infty} \underline{m}_3(x,t) = 1 - q_0 e^{-\gamma t}.$$

If we can prove that

(7.34) $$q_0 e^{-\gamma t} > (1 + \frac{A}{\beta})(1-V_0(\bar{x}(t)-r(\infty))), \quad t \geq 0,$$

then, by (7.32) and (7.33) the consequence is

(7.35) $$\underline{m}_2(\bar{x}(t),t) > \underline{m}_3(\bar{x}(t),t), \quad t \geq 0.$$

In view of (7.23) there exists a constant $\tilde{a}_1 > a_1$ such that

(7.36) $$1 - V_0(x) \le \tilde{a}_1 e^{-\nu x}, \quad x \ge 0.$$

Using (7.36) the inequality (7.34) is implied by

(7.37) $$\gamma < \nu c_3 = \nu \frac{\alpha}{\nu+\beta},$$

together with

(7.38) $$q_0 > (1 + \frac{A}{\beta}) \tilde{a}_1 e^{-\nu(\bar{x}_0 - r(\infty))}.$$

By reducing p_0 we enlarge $\bar{x}_0$, so the righthand side of (7.38) can be made arbitrarily small and we can choose $q_0 < p_2$, which was necessary in view of (7.13).

Next we define the functions $x_4(t)$, $x_5(t)$ by

(7.39) $$\underline{m}_2(x_4(t),t) = 1-p_2, \quad \underline{m}_{2x}(x_4(t),t) < 0,$$

(7.40) $$\underline{m}_3(x_5(t),t) = 1-p_2.$$

Our object is to prove that $x_4(t) < x_5(t)$, $t \ge 0$. This inequality gives together with (7.35) the existence of the function $x_2(t)$ for all time, where $x_2(t)$ is defined by

(7.41) $$\underline{m}_2(x_2(t),t) = \underline{m}_3(x_2(t),t), \quad \underline{m}_{2x}(x_2(t),t) < 0.$$

It is clear that the solution $\bar{x}_4(t)$ of

(7.42) $$1 - p_0 e^{\beta \bar{x}_4(t) - \alpha t} = 1-p_2$$

satisfies $x_4(t) < \bar{x}_4(t)$ and analogously that the solution $\underline{x}_5(t)$ of

(7.43) $$U(\underline{x}_5(t) - c_0 t + s(\infty)) - q_0 = 1-p_2$$

satisfies $\underline{x}_5(t) < x_5(t)$. It is easy to determine $\bar{x}_4(t)$ and $\underline{x}_5(t)$

(7.44) $\quad \bar{x}_4(t) = \frac{\alpha}{\beta} t + \frac{1}{\beta} \ell n \left(\frac{p_2}{p_0}\right),$

(7.45) $\quad \underline{x}_5(t) = c_0 t - s(\infty) + U^{-1}(1-p_2+q_0),$

U^{-1} is the inverse of U. If we choose

(7.46) $\quad c_4 = \frac{\alpha}{\beta} < c_0,$

(7.47) $\quad \frac{1}{\beta} \ell n \left(\frac{p_2}{p_0}\right) < - s(\infty) + U^{-1}(1-p_2+q_0),$

then we have proved the desired inequalities

(7.48) $\quad x_4(t) < \bar{x}_4(t) < \underline{x}_5(t) < x_5(t).$

We note that $s(\infty) = s(0) + S = s(0) + (K+\gamma)q_0/\gamma(-\ell)$. So we can achieve (7.47) by choosing $s(0)$ small enough, possibly negative. Once we have found the existence of $x_2(t)$ we can determine its asymptotic behaviour. First we specify α, β and γ

(7.49) $\quad \alpha = \beta c_0 - \sqrt{\rho},$

(7.50) $\quad \gamma < \min(\mu_1, \mu_2, \nu c_3).$

The condition (7.17) together with (7.49) gives

(7.51) $$\beta = \frac{1}{2}\left[-c_0 + \sqrt{c_0^2 + 4\nu^2}\right] - \frac{1}{2}\left[\sqrt{c_0^2 + 4\nu^2} - \sqrt{c_0^2 + 4\nu^2 + 4\sqrt{\rho} - 4\rho\nu^2}\right]$$

$$= \beta_1 + \frac{\sqrt{\rho}}{\sqrt{c_0^2 + 4\nu^2}} + O(\rho).$$

Note that the choice (7.49) implies the condition (7.46). We calculate νc_3

(7.52) $$\nu c_3 = \nu \frac{\alpha}{\nu+\beta} = \nu \frac{\beta c_0 - \sqrt{\rho}}{\nu+\beta} =$$

$$\nu \frac{\beta_1 c_0}{\nu+\beta_1} + O(\sqrt{\rho}) = \nu(\nu-\beta_1) + O(\sqrt{\rho}),$$

by (7.49), (7.51) and (7.25). We note that $\gamma < \nu c_3$ implies also

(7.53) $$\gamma < \beta_1\sqrt{c_0^2 + 4\nu^2} + O(\sqrt{\rho}),$$

because

(7.54) $$\nu(\nu-\beta_1) < \beta_1\sqrt{c_0^2 + 4\nu^2}.$$

Inequality (7.54) can be proved as follows

$$\nu(\nu-\beta_1) = \beta_1^2 + c_0\beta_1 - \nu\beta_1 = \beta_1^2 + \beta_1(-2\beta_1 + \sqrt{c_0^2 + 4\nu^2} - \nu) =$$

$$- \beta_1^2 - \nu\beta_1 + \beta_1\sqrt{c_0^2 + 4\nu^2} < \beta_1\sqrt{c_0^2 + 4\nu^2}.$$

Next we can determine the asymptotic behaviour of $x_2(t)$. With the aid of the implicit function theorem we find

(7.55) $$x_2(t) = c_2t + x_0 + o(1), \quad t \to \infty,$$

where

(7.56) $$c_2 = \frac{\alpha-\gamma}{\beta}.$$

The expression (7.55) holds only under the hypothesis

(7.57) $$\gamma < \min(-\beta_1(c_2-c_0), \nu c_2).$$

The first condition of (7.57) is equivalent with

$$\gamma < \beta_1(c_0 - \frac{\alpha-\gamma}{\beta}) = \beta_1(\frac{\gamma}{\beta} + \frac{\sqrt{\rho}}{\beta}),$$

so $\gamma < \beta_1\sqrt{\rho}/(\beta-\beta_1)$ and by (7.51)

$$\gamma < \beta_1\sqrt{c_0^2 + 4\nu^2} + O(\sqrt{\rho}),$$

which is satisfied by (7.53). The second condition of (7.57) is implied by (7.50) if we can demonstrate that indeed $c_3 < c_2$, or

$$c_3 = \frac{\alpha}{\nu+\beta} < \frac{\alpha-\gamma}{\beta} = c_2 \Longleftrightarrow \gamma < \nu \frac{\alpha}{\nu+\beta} \Longleftrightarrow \gamma < \nu c_3.$$

Thus the condition (7.50) on γ implies (7.57). Our choice for $\delta = -(\beta c_2-\alpha)$ (see (7.12)) implies

(7.58) $\qquad \delta = \gamma,$

so indeed δ is positive. With respect to the condition (7.18) we note that $x_2(t) < \bar{x}_4(t)$, so the rather conservative bound $p(x_2(t),t) < p(\bar{x}_4(t),t) = p_2$ (see (7.42)) implies (7.18).

At this moment we have specified p_0, β, α, δ, q_0, γ from the set of parameters mentioned in the beginning of this Appendix, while R and S were determined by (7.16) and (7.21) respectively.

Next we consider the boundary condition $\underline{m}(0,t) = 0$. From our construction we find $\underline{m}(0,t) = \max\,(0,\underline{m}_2(0,t))$, where $\underline{m}_2(0,t) = V_0(-r(t)) - p_0e^{-\alpha t} < 0$, so $\underline{m}(0,t) = 0$. Finally we take s(0) so small negative that inequality (7.47) is satisfied: it amounts to saying that we shift the travelling wave U(z) far to the right.

After we have completed the construction of the function $\underline{m}(x,t)$, we can choose T_0 so large that $\underline{u}(x,T_0)$ satisfies by (7.1)

$$\underline{m}(x,0) < \underline{u}(x,T_0) < V_0(x), \quad x \geq 0.$$

We recall that $\underline{m}(x,0) = 0$, $x \geq x_3(0)$. So indeed $\underline{m}(x,t)$ is a subsolution for problem (P"). If we examine the behaviour of $\underline{m}(c_3t+c_3T_0,t)$, then for t large enough $\underline{m} = \underline{m}_2$ and so there exists a positive constant $\tilde{C}$ such that

(7.59) $\qquad 1 - \underline{m}(c_3t+c_3T_0,t) < \tilde{C}e^{-\nu c_3 t}, \quad t > 0,$

It means we have proven (7.2) with $c_1 = c_3$ and $\gamma^* = \nu c_3$.

Acknowledgement

The author would like to thank Prof. L.A. Peletier for suggesting to him the problem and for giving advice during the work.

8. REFERENCES

[1] D.G. ARONSON & H.F. WEINBERGER, *Nonlinear diffusion in population genetics, combustion, and nerve propagation,* in Partial Differential Equations and Related Topics, ed. J.A. Goldstein. Lecture Notes in Mathematics 446, 5-49, Springer, Berlin, 1975.

[2] D.G. ARONSON & H.F. WEINBERGER, *Multidimensional nonlinear diffusion arising in population genetics,* Adv. in Math. 30 (1978) 33-76.

[3] PAUL C. FIFE, *Mathematical Aspects of Reacting and Diffusion Systems,* Lecture Notes in Biomathematics 28, Springer, Berlin, 1979.

[4] PAUL C. FIFE & J.B. McLEOD, *The approach of solutions of nonlinear diffusion equations to travelling front solutions,* Arch. Rat. Mech. Anal. 65 (1977) 335-362.

[5] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type,* Prentice-Hall, Englewood Cliffs, N.J., 1964.

[6] K.P. HADELER & F. ROTHE, *Travelling fronts in nonlinear diffusion equations,* J. Math. Biol. 2 (1975) 251-263.

[7] H.P. McKEAN, *Nagumo's equation,* Adv. in Math. 14 (1970) 209-223.

[8] O.A. OLEINIK & S.N. KRUZHKOV, *Quasi-linear second-order parabolic equations with many independent variables,* Russ. Math. Surv. 16 (1961) 105-146.

[9] E.C. TITCHMARSH, *Eigenfunction Expansions, Associated with Second-order Differential Equations,* Part I, Oxford University Press, Oxford, 1962.

[10] KOHEI UCHIYAMA, *The behaviour of solutions of some non-linear diffusion equations for large time,* J. Math. Kyoto Un. 18 (1978) 453-508.