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NERVE IMPULSE PROPAGATION IN A BRANCHING NERVE SYSTEM: A SIMPLE MODEL

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Nerve impulse propagation in a branching nerve system: a simple model*)

by

J.P. Pauwelussen

ABSTRACT

Local spatial changes of nerve axon geometry such as diameter increase and branching, may cause that action potential waves approaching a region of geometric change fail to propagate beyond it.

In this paper, this effect will be examined for a special kind of nonuniformity, within the framework of a simple model: an initial value problem for a single nonlinear diffusion equation on an unbounded domain.

KEY WORDS & PHRASES: *nerve pulse propagation, nonlinear diffusion equation, equilibrium solutions, comparison principle, stability*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we shall investigate the initial value problem

$$(1.1)^1 \quad (1.1)^2 \quad (I)\begin{cases} u_t = e_\varepsilon(x)u_{xx} + f(u)\ , & x \in \mathbb{R}\backslash\{0\}, \quad t > 0 \\ u(x,0) = \chi(x)\ , \quad \chi \in C(\mathbb{R})\ , & 0 \le \chi \le 1 \\ u \text{ and } u_x \text{ continuous at } x = 0 & \end{cases}$$

where

$$(1.2) \qquad e_\varepsilon(x) = \begin{cases} 1 & ;\ x < 0 \\ \varepsilon \in (0,1] & ;\ x > 0 \end{cases}$$

and

$$(1.3) \qquad f(u) = u(1-u)(u-a) \qquad , \ 0 < a < \tfrac{1}{2}.$$

Our motivation for studying (1.1) originates from the problem of propagation of electric excitation along the cylindrical branches of a tree shaped unmyelinated nerve axon. Assume the branching system to be of infinite extension where the variable x measures the distance along the conductor. If we restrict ourselves to the situation of only one branching point at $x = 0$, one branch of radius 1 for $x < 0$ and k branches of radius r at the part $x > 0$ (fig 1.1), this situation can be described by a reaction diffusion system of the form [6],[11],

$$(1.4) \qquad \begin{aligned} u_t &= r(x)u_{xx} + F(u,w) \\ w_t &= G(u,w) \qquad , \qquad x \in \mathbb{R}\backslash\{0\}, \quad t > 0 \end{aligned}$$

where t denotes time, (u,w) takes on values in $\mathbb{R} \times \mathbb{R}^n$ for some $n > 0$ and $r(x)$ is the diameter of a branch of the nerve at place x, i.e. $r(x) = 1$

for $x < 0$ and $r(x) = r$ for $x > 0$. In (1.4), u represents

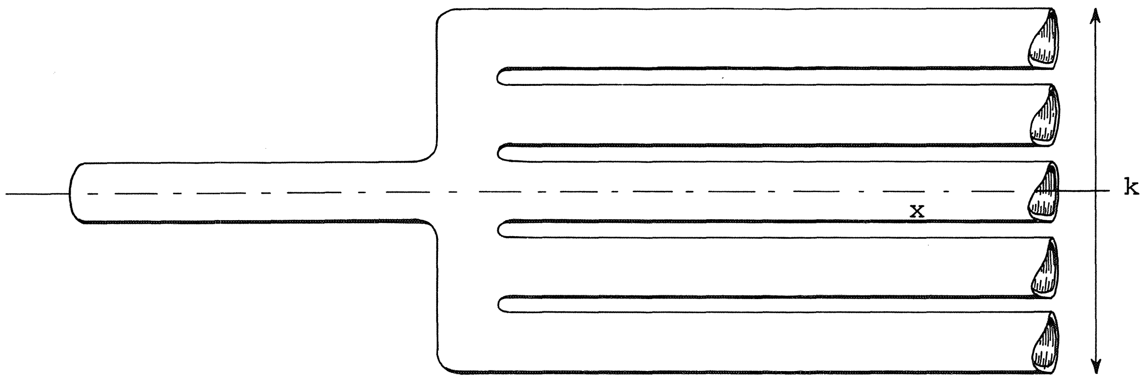

fig. 1.1.

the transmembrane potential minus the rest potential while the auxiliary variable $w = (w_1, \ldots, w_n)^T$ describes the transport of certain ions (K^+, Na^+, Cl^-) through the membrane which covers the neuron. At $x = 0$, the transmembrane potential u is continuous as well as the internal current which is proportional to the gradient of this potential times the surface area. Hence, at the branching point $x = 0$, u satisfies

(1.5) $$u_x(0-,t) = kr^2 u_x(0+,t).$$

To remove this discontinuity in u_x we replace x for $x > 0$ by $\frac{x}{r^2 k}$. Then, in terms of this rescaled variable, the system (1.4) transforms into

(1.6) $$\begin{aligned} u_t &= e_\varepsilon(x) u_{xx} + F(u,w), \\ w_t &= G(u,w), \qquad x \in \mathbb{R}\setminus\{0\}, \quad t > 0 \end{aligned}$$

with $e_\varepsilon(x)$ given by (1.2) where $\varepsilon = r^{-3}k^{-2}$.

In the special situation that $\varepsilon = 1$ and thus $e_\varepsilon(x) \equiv 1$, the relevant examples of (1.6) [1],[6],[8] all have the property that they allow travelling wave solutions, i.e. non-constant solutions which are functions of the single argument $z = x - ct$ for some constant c. For $\varepsilon \neq 1$, it was shown numerically by RINZEL [14] for an example in which $n = 1$, and by GOLDSTEIN and RALL [6] who treated an example in which $n = 2$, that action potential waves approaching the region of geometric change from the left continue to propagate beyond this region if the increase of radius or the amount of branching is sufficiently small. However, if r or k is large then action potential waves may fail to propagate beyond the branching point. In a subsequent paper [11] we shall demonstrate this qualitative behaviour of solutions of the system (1.6) under conditions on F and G, covering under some conditions, the examples in [14] and [6].

In the present paper we shall, as a first step, analyse the simplified version (1.1) of (1.6) which is a degenerate case of the system treated in [14].

Problem I is also of interest in its own right as it arises in population genetics [10]. In fact a solution u of Problem I can be interpreted as the frequency of alleles of one type, A say, amongst the total number of alleles in a population of individuals of possible genotypes AA, Aa and aa, living in a one-dimensional habitat where the heterozygote is underdominate, and with different migration rates on either side of the point $x = 0$.

Let us first consider the case of a uniform axon, i.e. $\varepsilon = 1$. Then, travelling wave solutions will be understood as solutions $u(x,t) = w(z)$, $z = x - ct$, of $(1.1)^1$ for some $c \in \mathbb{R}$ such that $w(-\infty) = 1$ and $w(+\infty) = 0$. As a consequence, w satisfies the equation

$$w'' + cw' + f(w) = 0, \quad z \in \mathbb{R}. \tag{1.7}$$

Note that any translate of w also satisfies (1.7). It can be proved that w is strictly decreasing and that the wave speed c is unique and positive, see [3] and [7] (fig. 1.2).

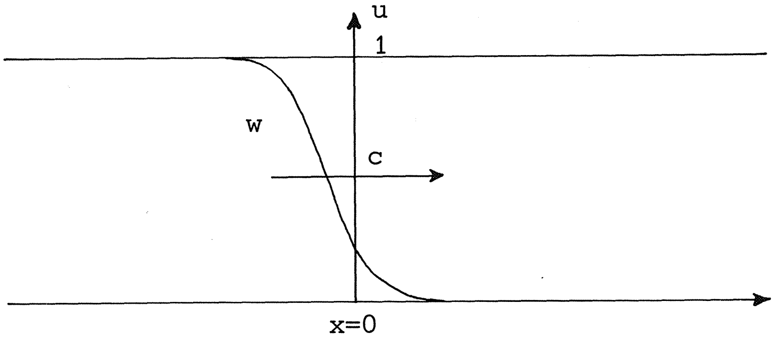

fig. 1.2.

Obviously, if we take for the initial function $\chi(x) \equiv w(x)$ then $u(x,t) \equiv w(x-ct)$. This is a particular case of the conditions

$$\liminf_{x \to -\infty} \chi(x) > a, \tag{1.8}$$

$$\limsup_{x \to +\infty} \chi(x) < a \tag{1.9}$$

given by FIFE & McLEOD [3], who showed that under these conditions the solution $u(x,t)$ of (1.1) for $\varepsilon = 1$, converges to $w(x-ct+x_0)$ exponentially, for some $x_0 > 0$ as $t \to \infty$.

If $\varepsilon < 1$ but $1 - \varepsilon$ small then one might expect that the solution of Problem I, where χ still satisfies (1.8) and (1.9), behaves as a wave, travelling from the region where $e_\varepsilon(x) = 1$ towards the region where $e_\varepsilon(x) = \varepsilon$. Note that $w^*(z) \equiv w(z\sqrt{\varepsilon})$ satisfies the equation

$$\varepsilon w'' + c^* w' + f(w) = 0, \qquad z \in \mathbb{R}, \tag{1.10}$$

where $c^* = c\sqrt{\varepsilon}$. We shall find that if χ satisfies (1.8) and (1.9), there exists an $\varepsilon^* \in (0,1)$ such that for $\varepsilon \in (\varepsilon^*,1]$, the solution $u(x,t)$ of (1.1) converges to $w^*(x-c^*t+x_0)$ exponentially for some $x_0 \in \mathbb{R}$, as $t \to \infty$.

If $\varepsilon \in (0,1)$ but ε small then it turns out that stationary solutions $q(x)$ (i.e. $q_t \equiv 0$) of (1.1)[1] exist which satisfy the boundary conditions $q(-\infty) = 1$ and $q(+\infty) = 0$. These solutions are strictly decreasing, see FIFE & PELETIER [5]. In fact, for $\varepsilon < \varepsilon^*$ there exist exactly two such solutions $q_-(x)$ and $q_+(x)$ where $q_- < q_+$, where for $\varepsilon > \varepsilon^*$, such solutions do not exist. A sketch of the corresponding bifurcation diagram is given in fig. 1.3.

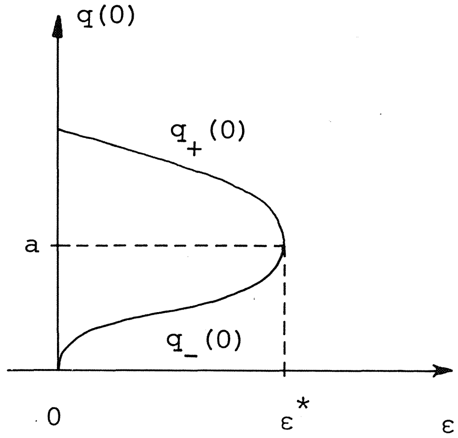

fig. 1.3. q(0) vs. ε.

These functions $q_-(x)$ and $q_+(x)$ will act as a blockade for solutions of (1.1). In fact, we shall find among other things that $u(x,t)$ converges to $q_-(x)$ if (1.8) is satisfied and if $\chi(x) < q_+(x)$. This last condition surely implies (1.9) whence we find that the value of ε^* is critical.

The plan of the paper is as follows. In Section 2 we shall show the existence of the number ε^*. In Section 3 we shall formulate a result about existence and uniqueness of a solution $u(x,t)$ of Problem I. Section 4 will be devoted to a comparison principle which we shall use in Section 5 to derive some preliminary stability properties of the functions $q_-(x)$ and $q_+(x)$. We shall find that $q_-(x)$ is stable and $q_+(x)$ is unstable. Thus the lowest branch in the bifurcation diagram fig. 1.3 is the stable one while the upper branch is the unstable one. Moreover we shall show the convergence of u towards the travelling wave w^* if $\varepsilon > \varepsilon^*$, and if $\varepsilon \leq \varepsilon^*$ under the condition that $\chi(x)$ is large enough.

At the expense of a much more complicated analysis these results will be improved in Section 6. Finally, in Section 7 we give some numerical results.

REMARK. The results in this paper also apply to the case of the more general $f \in C^1(\mathbb{R})$, satisfying

$$f(0) = f(a) = f(1) = 0, \qquad 0 < a < 1,$$
$$f(u) < 0 \text{ for } 0 < u < a,\ f(u) > 0 \text{ for } a < u < 1$$
$$f'(0) \neq 0 \neq f'(1),$$
$$\int_0^1 f(u)\,du > 0.$$

2. STATIONARY SOLUTIONS

Stationary solutions $q = q(x)$ of $(1.1)^1$ satisfy the equation

$$(2.1) \qquad 0 = e_\varepsilon(x)q'' + f(q) \qquad x \in \mathbb{R},$$

together with the regularity condition

$$(2.2) \qquad q \text{ and } q_x \text{ continuous at } x = 0.$$

We shall mostly be concerned with solutions of (2.1) which satisfy the boundary conditions

$$(2.3) \qquad q(-\infty) = 1,\ q(+\infty) = 0$$

and we shall write shortly Problem II for (2.1)-(2.3). Let us first consider equation (2.1) in case $e_\varepsilon(x)$ = constant, $e_\varepsilon(x) \equiv \varepsilon$ say. Introducing formally the function $P(q) = q_x(q)$ in this case we find $P(q)$ to satisfy the equation

$$(2.4) \qquad P.P_q + \frac{f}{\varepsilon} = 0.$$

The trajectories for (2.1), given by $(q,P(q))$ where P satisfies (2.4) are shown in figure 2.1. (We have used the fact that $\int_0^1 f(u)\,du > 0$).

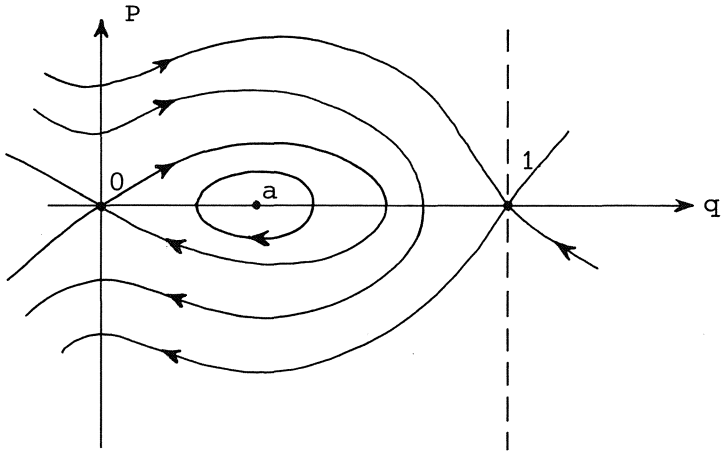

fig. 2.1.

The points $(q,P) = (0,0)$ and $(q,P) = (1,0)$ are saddle points. The stable manifolds will be denoted as "stable ε - manifolds". Similarly we may introduce "unstable ε-manifolds". We shall mostly be concerned with the case $P \leq 0$, $0 \leq q \leq 1$. In this region there is only one stable manifold going to $(0,0)$ and one unstable manifold coming from the point $(1,0)$. Therefore, if no confusion is possible, the *stable ε-manifold* refers to the one going to $(0,0)$ in the lower half plane and the *unstable ε-manifold* denotes the manifold coming from $(1,0)$, pointing into the lower half-plane.

Integration of (2.4) over $[0,q_0]$ with respect to q yields

$$\tfrac{1}{2}P^2(q_0) = -\frac{1}{\varepsilon}\int_0^{q_0} f(q)\,dq + \tfrac{1}{2}P^2(0). \tag{2.5}$$

Consider the case $P(0) = 0$. From (2.5) we see that the value of $P(q_0) < 0$ on the stable ε-manifold is equal to some constant, only depending on q_0, times $1/\sqrt{\varepsilon}$. By integration of (2.4) over $[q_0,1]$ we can show the same property for the value of $P(q_0) < 0$ on the unstable ε-manifold. Using this result we shall prove the following Lemma.

LEMMA 2.1. *Let*

$$(2.6) \qquad \varepsilon^* = \frac{-\int_0^a f(q)\,dq}{\int_a^1 f(q)\,dq}$$

Then

(i) *If* $\varepsilon > \varepsilon^*$ *there exists no solution of Problem II.*

(ii) *If* $\varepsilon = \varepsilon^*$ *there exists a unique solution of Problem II.*

(iii) *If* $0 < \varepsilon < \varepsilon^*$ *there exist exactly two solutions* q_- *and* q_+ *of Problem II which are decreasing and strictly separated and for which the inequalities* $q_-(0) < a < q_+(0)$ *hold.*

PROOF. Since on the stable ε-manifold the value of $-P(q)$ grows with ε^{-1} as $\varepsilon^{-\frac{1}{2}}$, this manifold will intersect the unstable 1-manifold for sufficiently small ε. Observe that by the fact that the unstable 1-manifold lies strictly below the q-axis for $q < 1$, if this manifold intersects the stable ε-manifold nontangentially, it must do so at least twice.

Now suppose that such intersections occur at $q = a_-$ and $q = a_+$, $a_- < a_+$. Integration over both manifolds of $P(q).P'(q)$ on (a_-,a_+) yields

$$2\int_{a_-}^{a_+} \frac{f(q)}{\varepsilon}\,dq = P^2(a_-) - P^2(a_+) = 2\int_{a_-}^{a_+} f(q)\,dq, \qquad \varepsilon < 1.$$

This is only possible if

$$\int_{a_-}^{a_+} f(q)\,dq = 0.$$

Hence $a_- < a_+$ and there are at most two points of intersection of the stable ε-manifold and the unstable 1-manifold. The first time these manifolds intersect they must do so at $q = a$ where by (2.5) the value of $P(a)$ is given by

$$P(a) = -\Big(-\frac{2}{\varepsilon}\int_0^a f(q)\,dq\Big)^{\frac{1}{2}} \qquad .$$

Integration over the unstable 1-manifold on (a.1) yields

$$P(a) = -(2\int_a^1 f(q)\,dq)^{\frac{1}{2}}$$

and thus $\varepsilon = \varepsilon^*$.

A solution $q(x)$ of Problem II corresponds with the stable ε-manifold for $x > 0$ and with the unstable 1-manifold for $x < 0$ where these manifolds must match, by (2.2), at some point. As a consequence, (i) and (ii) hold and for $\varepsilon < \varepsilon^*$ there exist two solutions q_- and q_+ of (2.1)-(2.3) with $q_-(0) = a_- < a_+ = q_+(0)$.

Since q_+ and q_- both correspond to the stable ε-manifold for $0 < q < a_-$ there exists a number $x_0 > 0$ such that $q_+(x+x_0) = q_-(x)$ for $x > 0$. Thus we have that $q_-(x) < q_+(x)$ if $x \geq 0$ and similarly that $q_-(x) < q_+(x)$ if $x \leq 0$.

Note that for $\varepsilon = \varepsilon^*$, $q_-(x)$ and $q_+(x)$ coincide where $q_-(0) = q_+(0) = a$. The functions $q_-(x)$ and $q_+(x)$ are sketched in figure 2.2 below.

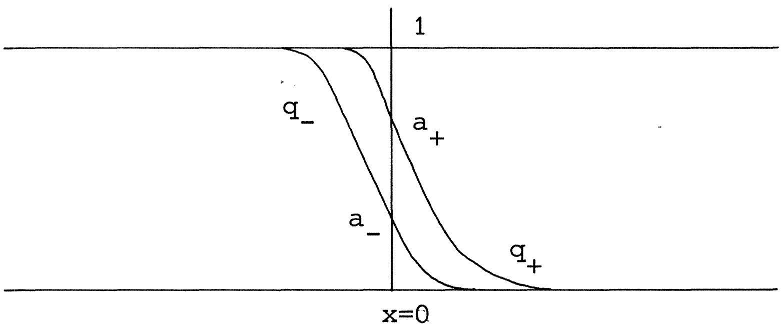

fig. 2.2.

We remark that $q_-(x)$ and $q_+(x)$ are not the only bounded solutions of (2.1) and (2.2). For example, the unstable 1-manifold also intersects at any $q = a_0 \in (a_-,a_+)$ a closed trajectory in the ε-phase plane, enclosing the point $(a,0)$. Hence the corresponding solution $q(x)$ of (2.1) and (2.2) where $q(0) = a_0$, is strictly decreasing for $x < 0$, it approaches the value

1 for $x \to -\infty$ and it is periodic for $x > 0$ with a period depending on the solution at hand.

By similar reasoning we can also find stationary solutions which approach 0 for $x \to -\infty$ and which are periodic for $x > 0$ as well as solutions which are periodic for both $x < 0$ and for $x > 0$ but with different periods. These last two types of solution exist for all $\varepsilon \in (0,1)$.

REMARK. Since at $q = a_-$, the stable ε-manifold and the unstable 1-manifold intersect we have, using (2.5) and the corresponding expression for P on the unstable 1-manifold, that

$$-\int_0^{a_-} f(q)\,dq = \varepsilon \int_{a_-}^{1} f(q)\,dq.$$

Differentiation with respect to a_- yields

$$\frac{d\varepsilon}{da_-} = \frac{(\varepsilon-1)f(a_-)}{\int_{a_-}^{1} f(q)\,dq}.$$

It follows that $\frac{d\varepsilon}{da_-} = 0$ for $a_- = 0$(i.e. $\varepsilon = 0$) and for $a_- = a$ (i.e. $\varepsilon = \varepsilon^*$). and this was used in the bifurcation diagram figure 1.3. Similarly it can be proved that $\frac{d\varepsilon}{da_+} = 0$ if $\varepsilon = \varepsilon^*$ and $\frac{d\varepsilon}{da_+} < 0$ if $\varepsilon = 0$.

3. EXISTENCE AND UNIQUENESS FOR PROBLEM I

In [11] we have treated in full detail the existence-uniqueness problem for a general system of equations of which $(1.1)^1$ is a particular case. Therefore, in this section we shall only give the result and we refer to [11] for further details.

According to [11], a function $u: \mathbb{R} \times [0,T) \to \mathbb{R}$ is a solution of Problem I on $[0,T)$ if and only if

(i) $u_{xx}, u_t \in C(\mathbb{R}\setminus\{0\}x(0,T) \to \mathbb{R})$,
$u_x \in C(\mathbb{R}x(0,T) \to \mathbb{R})$,
$u \in BC(\mathbb{R}x\, [0,T) \to \mathbb{R})$,

(ii) u satisfies (1.1) on $\mathbb{R}\setminus\{0\}x(0,T)$.

THEOREM 3.1. *Let* $T > 0$. *Then Problem I has a unique solution* $u(x,t)$. *Moreover for arbitrary* $\delta \in (0,T)$ *and any x-interval* J, *not including an open neighbourhood of* $x = 0$ *we have for any* $\alpha \in (0,1)$

$$u \in C^{2+\alpha,1+\alpha/2}\ (Jx(\delta,T))$$

where $C^{2+\alpha,1+\alpha/2}(Jx(\delta,T))$ *means the space of functions* $u = u(x,t)$, $x \in J$, $t \in (\delta,T)$ *where* u_{xx} *and* u_t *are Hölder-continuous with respect to* x *and* t, *respectively, and the corresponding Hölder coefficients are* α *and* $\alpha/2$.

4. A COMPARISON PRINCIPLE

Since the diffusion coefficient $e_\varepsilon(x)$ in (1.1) is not continuous and therefore u_{xx} and u_t need not be bounded or continuous at $x = 0$, the results of [12] do not apply directly to the present situation. Still a comparison principle can be derived.

First we shall prove a maximum principle for functions which are smooth for $x \neq 0$ and $t > 0$. However in practical applications it often occurs that comparison functions are used which are not smooth along certain curves in the (x,t)-plane. In order to deal with these kind of functions we require an extended maximum principle which we shall use to prove a comparison principle in sufficient generality to cover all the applications occurring in this paper.

To begin with we give a definition.

DEFINITION. Let E be a region in the (x,t)-plane with boundary ∂E. Let (x_0,t_0) be a point on ∂E. Then we say that E satisfies the *interior circle condition* at (x_0,t_0) if at (x_0,t_0) a circle tangent to ∂E can be constructed such that the radial direction from the center of the circle is not parallel to the t-axis and the set of points (x,t) inside or on this circle, with

$t < t_0$ is nonempty and lies entirely in E.

THEOREM 4.1. *Let for* $T > 0$

(4.1) $\quad \phi \in BC(\mathbb{R} \times (0,T] \to \mathbb{R}) \cap C^{2,1}(\mathbb{R}\setminus\{0\} \times (0,T] \to \mathbb{R})$

where ϕ *satisfies the inequality*

(4.2) $\quad L\phi \equiv \phi_t - e_\varepsilon(x)\phi_{xx} - c\phi_x + h(x,t)\phi \le 0, \quad x \in \mathbb{R}\setminus\{0\},\ t \in (0,T)$

where c *is a constant,* $e_\varepsilon(x)$ *is given by* (1.2) *and* $h(x,t)$ *is nonnegative bounded. Suppose* $\phi_x(0\pm,t)$ *exist and*

(4.3) $\quad \phi_x(0+,t) \ge \phi_x(0-,t).$

Moreover suppose that

(4.4) $\quad \phi(x,0) \le 0, \quad x \in \mathbb{R}.$

Then $\phi(x,t) \le 0$ *for* $x \in \mathbb{R}$, $t \in (0,T]$ *and if* $\phi(x,0) < 0$ *on some open subset of* $\mathbb{R}$ *then* $\phi(x,t) < 0$ *for all* $x \in \mathbb{R}$, $t \in (0,T]$.

PROOF. Introduce the regions $V^{\pm} = \{(x,t) \mid \pm x > 0,\ 0 < t \le T\}$. The proof of this theorem is based on a maximum principle of KRYZAŇSKI [9] on parabolic operators in an (n+1)-dimensional, unbounded domain in $\mathbb{R}^n \times \mathbb{R}^+$. In our reformulation of this result we shall restrict ourselves to the one-dimensional parabolic operator (n=1).

We consider the general parabolic inequality

(4.5) $\quad \mathcal{L}u \equiv u_t - au_{xx} - bu_x + Hu \le 0$

on an unbounded domain D in the (x,t)-plane of which the boundary consists of the straight lines $t = 0$ and $t = T$ and possibly a curve S which is nowhere perpendicular to the t-axis. The functions a, b and H are assumed to be bounded on D where $a(x,t)$ is positive and bounded away from zero and $H(x,t) \ge 0$. Denote by Γ the union of S and the part of ∂D where $t = 0$.

Let u satisfy (4.5) where $u \in C^{2,1}(D \to \mathbb{R}) \cap C(D \cup \Gamma \to \mathbb{R})$. Moreover suppose that there exist positive numbers K_0, K_1 such that

$$(4.6) \qquad |u(x,t)| \le K_0 \exp [K_1 x^2], \qquad (x,t) \in D \cup \Gamma.$$

Then if for some nonnegative M, $u(x,t) \le M$ on Γ it follows that $u(x,t) \le M$ on D.

This is KRYZAŇSKI'S result and specifying it to $D = V^+$ or $D = V^-$ this yields that $\phi(x,t)$ can only attain a nonnegative maximum on $\overline{V^+ \cup V^-}$ (the closure of $V^+ \cup V^-$) at a point $(0,t_0)$ for some $0 < t_0 \le T$ or at a point $(x_0,0)$ for some $x_0 \in \mathbb{R}$. If we suppose that this maximum is positive, the second possibility is excluded by (4.4).

Let us assume that ϕ attains a positive maximum M at a point $(0,t_0)$ for some $0 < t_0 \le T$. If ϕ attains this maximum also at some point (x_1,t_1) of V^+ or V^- then it is proved in Theorem 2 in [12; p.168] that $\phi \equiv M$ on each segment of a line $t = t_2 \in [0,t_1]$ which lies in V^+ or V^- and contains the point (x_1,t_2) (The Interior Point Theorem). Thus $\phi(x,0) > 0$ for some $x \neq 0$ which contradicts (4.4). We conclude that

$$(4.7) \qquad \phi(x,t) < M, \quad (x,t) \in V^+ \cup V^-.$$

Observe that for all $t_0 \in (0,T]$, V^+ as well as V^- satisfies the interior circle conditon at $(0,t_0)$. We shall now proceed as in the proof of Theorem 3 in [12; p. 170]. We construct a disk K of radius R, tangent to the t-axis at the point $(0,t_0)$. We denote the coordinates of the center of the circle ∂K (the boundary of K) by (x_0,t_0) which we shall assume to lie inside V^+ (see fig. 4.1).

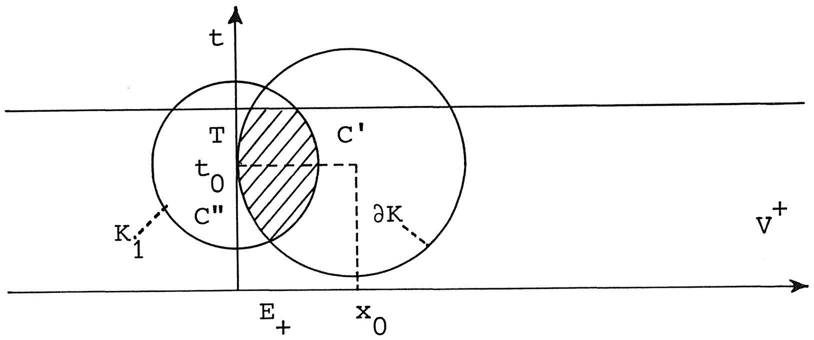

fig. 4.1.

We also construct a disk K_1 with center at $(0,t_0)$ and radius $< R$. Let C' be the portion of the boundary ∂K_1 contained in $\bar{K}$ and let C'' be the open arc of ∂K in K_1. The arcs C' and C'' form the boundary of a lens-shaped region E and we assume R to be so small that $\bar{E}$ and the x-axis are disjoint. Denoting by E^+ the part of E inside V^+ we have by (4.7) that $\phi < M$ on $\overline{E^+}$ except at $(0,t_0)$.

In [12] the circle ∂K and therefore E is assumed to lie entirely inside the domain under consideration, V^+ in this case. However, the same arguments can be used here yielding that $\frac{\partial\psi}{\partial\nu}(0,t_0) > 0$ where $\partial/\partial\nu$ is any derivative in a direction pointing out of E^+ and thus in particular

$$(4.8) \qquad \phi_x(0+,t_0) < 0.$$

We shall refer to this result as the Boundary Point Theorem. In a similar way for V^- instead of V^+ we obtain that

$$(4.9) \qquad \phi_x(0-,t_0) > 0$$

and by (4.3) we have a contradiction.

It remains to prove the statement about strict inequalities. If $\phi(x,0) < 0$ on some open interval which, for example has a nonempty intersection with the positive x-axis then, by the Interior Point Theorem, $\phi(x,t) < 0$ in V^+. If $\phi(0,t_0) = 0$ for some $0 < t_0 \le T$ then similar to the above analysis this yields that $\phi_x(0+,t_0) < 0$. However, since $\phi(x,t) \le 0$ in $V^+ \cup V^-$ we must have $\phi_x(0-,t_0) \ge 0$. By (4.3) we have a contradiction. Hence $\phi(0,t) < 0$ and therefore $\phi(x,t) < 0$ in points in V^-, close to the t-axis. Application of the Interior Point Theorem finally yields that $\phi(x,t) < 0$ in V^- as well.

<u>THEOREM 4.2.</u> (Extended maximum principle).

Let for $T > 0$

$$(4.10) \qquad \phi \in BC(\mathbb{R}\times(0,T] \to \mathbb{R}) \cap C^{2,1}((\mathbb{R}\setminus\{0\}\times(0,T])\setminus D \to \mathbb{R})$$

where D *is the union of finitely many, continuous curves in the* (x,t) *plane,* *given by* $x = x_j(t)$ *say, for* $t > 0$ *and* $j = 1,2,\dots,N$, *such that on both sides of these curves, the region* $(\mathbb{R}\setminus\{0\}\times(0,T])\setminus D$ *satisfies the interior circle condition. Let* ϕ *satisfy the inequality*

$$(4.11) \qquad L\phi \le 0, \quad x \in \mathbb{R}\setminus\{0\},\ t \in (0,T),\ x \ne x_j(t),\ j = 1,\dots,\ ,$$

where L *is defined in* (4.2).

Suppose ϕ *satisfies the conditions* (4.3) *and* (4.4) *and the additional condition that*

$$(4.12) \qquad \phi_x(x_j(t)+,t) \ge \phi_x(x_j(t)-,t).$$

Then the conclusion of Theorem 4.1 *holds.*

PROOF. This proof is similar to the one of the preceeding Theorem if we handle points of D in the same fashion as points $(0,t)$, $0 < t \le T$.

THEOREM 4.3. (Comparison principle).

Let for $T > 0$, u *and* v *be two functions such that* u-v *satisfies the conditions (4.3), (4.4), (4.10) and (4.12) for the curves* $x = x_j(t)$ *as given in Theorem 4.2.*

Moreover suppose that u *and* v *satisfy the inequality*

$$(4.13) \qquad Nu \le Nv, \qquad x \in \mathbb{R}\setminus\{0\},\ t \in (0,T),\ x \ne x_j(t)$$

where N *is the nonlinear differential operator*

$$(4.14) \qquad Nu \equiv u_t - e_\varepsilon(x)u_{xx} - cu_x - F(x,t,u)$$

with $e_\varepsilon(x)$ *and* c *as in Theorem 4.1 and for a given function* $F \in C^{0,0,1}(\mathbb{R} \times [0,T] \times \mathbb{R})$. *Then*

$$(4.15) \qquad u(x,t) \le v(x,t), \qquad x \in \mathbb{R},\ t \in (0,T].$$

Moreover if $u(x,0) < v(x,0)$ *on some open subset of* $\mathbb{R}$ *then* $u(x,t) < v(x,t)$ *for all* $x \in \mathbb{R}$, $t \in (0,T]$.

PROOF. The proof of this theorem is, in a technical sense, similar to the proof of Proposition 2.1 in [1]. Define for $\lambda > 0$ a new function w by

$$w(x,t) = e^{\lambda t}(u(x,t) - v(x,t)). \tag{4.16}$$

Choosing λ sufficiently large (cf. [1]) we can find a bounded positive function $F_1(x,t)$ such that

$$w_t - e_\varepsilon(x)w_{xx} - cw_x + F_1(x,t)w \le 0. \tag{4.17}$$

The function w satisfies therefore all the conditions of the extended maximum principle and consequently (4.15) holds.

DEFINITION 4.1. Consider the general differential operator N, given by (4.14). We shall call a function ϕ a *lower solution* of the equation

$$Nu = 0 \tag{4.18}$$

for $0 < t < T$ if ϕ satisfies the smoothness conditions (4.10) and (4.12) together with (4.3) and $N\phi \le 0$ for $x \in \mathbb{R}\setminus\{0,x_1(t),\dots,x_N(t)\}$, $t \in (0,T)$ with the curves $x_j(t)$ as given in Theorem 4.2.

A function ϕ is called *upper solution* of (4.18) if it satisfies the same conditions with all the inequality signs reversed.

NOTATION. We shall write $u(x,t;\chi)$ for the solution of (1.1).

By the above comparison principle we can prove an a priori estimate for $u(x,t;\chi)$.

THEOREM 4.4.

$$0 \le u(x,t;\chi) \le 1, \quad x \in \mathbb{R}, \ t \ge 0. \tag{4.19}$$

PROOF. The function $\phi(x,t) \equiv 0$ and $\psi(x,t) \equiv 1$ are lower and upper solution for equation (1.1), respectively. Moreover $0 \le u(x,0;\chi) = \chi(x) \le 1$, by assumption, and consequently application of Theorem 4.3 yields (4.19).

The stationary solutions $q_+(x)$ and $q_-(x)$ introduced in Section 2, are upper as well as lower solution for equation (1.1). Hence, application of Theorem 4.3 yields a first "blocking result".

<u>THEOREM 4.5</u>. *Let* $0 < \varepsilon < \varepsilon^*$ *and assume*

$$0 \le \chi(x) \le q_+(x) \qquad (\chi(x) \le q_-(x)).$$

Then

$$0 \le u(x,t;\chi) \le q_+(x) \qquad (u(x,t;\chi) \le q_-(x))$$

for all $x \in \mathbb{R}$, $t > 0$.

5. STABILITY AND PROPAGATION, A FIRST IMPRESSION.

In this Section we shall for $\varepsilon \in (0,\varepsilon^*]$, give some first results on the stability and instability of the stationary solutions $q_-(x)$ and $q_+(x)$, respectively. In particular we shall estimate the region of attraction of $q_-(x)$, and examine the behaviour of the solution of Problem I, starting high above $q_-(x)$. If $\varepsilon \in (\varepsilon^*,1)$ we shall give meaning to the statement that the solution u of Problem I travels away from the point $x = 0$, and give conditions on χ that u does so.

In the derivation of our results we shall make use of results of ARONSON & WEINBERGER [1], [2] and VELING [16].

To begin with we introduce a few upper and lower solutions. Consider for $\lambda \ge 0$ and $\varepsilon < \varepsilon^*$ the function

$$q_\lambda(x) \equiv q_+(x+\lambda), \tag{5.1}$$

the translation of q_+ to the left over a distance λ. Let the number $\lambda_0 > 0$ be such that

$$q_{\lambda_0}(0) = a. \tag{5.2}$$

Since $q_0(0) = a_+ > a$, $q_\infty(0) = 0 < a$ and $q_\lambda(x)$ is strictly decreasing with respect to λ ($q_+'(x) < 0$ for all $x \in \mathbb{R}$!) the number λ_0 is well defined. We shall restrict ourselves to $\lambda < \lambda_0$. Then $q_\lambda(x) > a$ for $x \leq 0$ and therefore

$$(5.3) \qquad q_\lambda''(x) = -\frac{f(q_\lambda)}{e_\varepsilon(x+\lambda)} < 0.$$

Hence

$$(5.4) \qquad e_\varepsilon(x)q_\lambda''(x) + f(q_\lambda) = [e_\varepsilon(x) - e_\varepsilon(x+\lambda)]q_\lambda''(x) \leq 0.$$

Thus $q_\lambda(x)$ is an upper solution of (1.1) for $0 \leq \lambda < \lambda_0$. In the same way as we have shown in Section 2 that $q_+(x) > q_-(x)$ for all $x \in \mathbb{R}$ we can show that $q_\lambda(x) > q_-(x)$ for all $x \in \mathbb{R}$, $\lambda < \lambda_0$. Similarly $q_\lambda(x)$ is a lower solution of (1.1) for $\lambda < 0$.

Another lower solution can be constructed as follows. The stable 1-manifold in the (u,u_x)-plane intersects the u-axis at some point $u = u^* \in (0,1)$ (see figure 2.1). In fact u^* is given by the relation $\int_0^{u^*} f(u)du = 0$ which follows from (2.5) for $\varepsilon = 1$. Trajectories of the equation

$$(5.5) \qquad u_{xx} + f(u) = 0$$

intersecting the u-axis at a point $u_0 \in (u^*,1)$ correspond with solutions $u(x;u_0)$ of (5.5) where for some x_0 $u(x_0;u_0) = u_0$, which are symmetric with respect to the line $x = x_0$, and which are nonnegative, only on a finite interval. We define a function $Q = Q(x;x_0,u_0)$ by

$$Q(x;x_0,u_0) = \begin{cases} u(x,u_0) & \text{if } u(x,u_0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We shall always choose the point x_0 such that $Q(x;x_0,u_0)$ vanishes for $x \geq 0$.

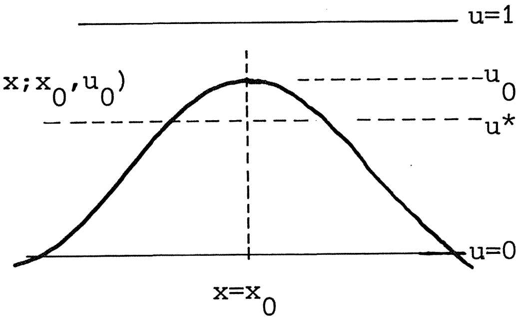

fig. 5.1.

Using these upper and lower solutions we are able to examine the asymptotic behaviour of the solutions of (1.1) as $t \to \infty$. Thereby we make use of the following Lemma which is an extension of a result of ARONSON & WEINBERGER [1].

LEMMA 5.1. *Let*

$$\chi \in C(\mathbb{R} \to [0,1]) \cap C^2(\mathbb{R}\backslash\{0,x_1,\ldots,x_N\} \to [0,1])$$

satisfy the differential inequality

$$e_\varepsilon(x)\chi'' + f(\chi) \le 0, \qquad x \in \mathbb{R}\backslash\{0,x_1,\ldots,x_N\} \tag{5.6}$$

for real numbers 0 *and* $x_i \neq 0$, $i = 1,\ldots,N$.

Suppose $\chi'(x\pm)$, $\chi''(x\pm)$ *exist and* $\chi'(x+) - \chi'(x-) \le 0$ *at* $x \in \{0,x_1,\ldots,x_N\}$. *Then* $u(x,t;\chi)$ *is for each* x *a nonincreasing function of* t. *Moreover*

$$\lim_{t\to\infty} u(x,t;\chi) = q(x), \tag{5.1}$$

uniformly in each compact interval, where $q(x)$ *is the largest stationary solution of* (1.1) *satisfying the inequality* $q(x) \le \chi(x)$.

PROOF. The proof of this Lemma is given in the same way as the proof of

Proposition 2.2 in [1] using however this time the comparison principle Theorem 4.3 instead of the comparison principle Proposition 2.1 in [1]. (see also Proposition 2.2 in [2]).

Since $\chi(x)$ is an upper solution for equation (1.1) it follows by Theorem 4.3 that

$$u(x,h;\chi) \le u(x,0;\chi) = \chi(x) \tag{5.8}$$

for all $h > 0$. Now $u(x,t;\chi)$ is an upper solution of equation (1.1) which is smooth for $t > 0$, except at $x = 0$. Application of Theorem 4.3 yields

$$u(x,t+h;\chi) \le u(x,t;\chi), \qquad h > 0,\ x \in \mathbb{R}. \tag{5.9}$$

Thus for each x, the function $u(x,t;\chi)$ is nonincreasing in t and bounded below by zero (Theorem 4.4). Therefore

$$\lim_{t\to+\infty} u(x,t:\chi) \equiv \tau(x) \tag{5.10}$$

exists. Now following the proof of Proposition 2.2 in [1] or Proposition 2.2 in [2] one can show that $\frac{\partial^n u}{\partial x^n}(\cdot,t,\chi) \to \tau^{(n)}(\cdot)$ for $n = 0,1$, uniformly on bounded intervals not including $x = 0$, and that $\tau(x)$ satisfies equation (2.1) and is a stationary solution of (1.1). Since $u_t(x,t;\chi) \le 0$ and $u(x,t;\chi)$ is uniformly bounded (cf. Theorem 4.4), there exists an M such that for all $x \neq 0$ and $t > 0$, $u_{xx} = (u_t-f(u))/e_\varepsilon(x) \le M$. Hence for any pair (x,x_0) with $x > x_0$ we find by integration of u_{xx} that

$$u_x(x,t:\chi) - \tau'(x) \le (u_x(x_0,t:\chi) - \tau'(x_0)) +$$

$$+ (\tau'(x_0) - \tau'(x)) + M(x-x_0).$$

By this inequality, the continuity of τ' and the uniform convergence of $u_x(\cdot,t;\chi)$ towards τ' outside $x = 0$ we can find for any $\delta > 0$, numbers ρ_δ and t_δ such that for $t > t_\delta$ and $|x| < \rho_\delta$, $u_x(x,t:\chi) - \tau'(x) \le \delta$. Similarly exchanging x and x_0 we find that, $u_x(x,t:\chi) - \tau'(x) \ge -\delta$ for x near the origin and sufficiently large t. As a consequence, the convergence of $u_x(\cdot,t;\chi)$ to τ' for $t \to \infty$ is uniform on all compact intervals. In particular $u_x(\cdot;t,\chi)$ is bounded on any compact interval and integration of u_x leads in the same way as above to the uniform convergence of $u(\cdot,t;\chi)$ to τ for $t \to \infty$ on all compact intervals.

Finally we note that for every stationary solution $\sigma(x)$ of (1.1) where $\sigma(x) \le q(x)$ it follows by Theorem 4.3 that $\tau(x) \ge \sigma(x)$.

REMARK 5.2. Similarly one can show that if $e_\varepsilon(x)\chi'' + f(\chi) \ge 0$ for $x \in \mathbb{R}\setminus\{0,x_1,\dots,x_N\}$ where $x_i \ne 0$ and $\chi'(x+) - \chi'(x-) \ge 0$ at $x = x_i$ or 0 then $u(x,t;\chi)$ is nondecreasing in t for each x. Moreover as $t \to \infty$, $u(x,t;\chi)$ tends to the smallest stationary solution of (1.1), which is greater than or equal to $\chi(x)$.

The following theorem gives a first asymptotic stability result for the stationary solution $q_-(x)$ and, at the same time estimates its region of attraction. We shall use the functions $Q(x;x_0,u_0)$ and $q_\lambda(x)$, introduced earlier in this section.

Recall that $Q(x;x_0,u_0) \ne 0$, only on a finite interval in $(-\infty,0)$, that the maximum of Q, u_0 lies in the interval $(u^*,1)$ where u^* is defined by

$$\int_0^{u^*} f(v)\,dv = 0$$

and $\lambda < \lambda_0$ where λ_0 is given by (5.2).

THEOREM 5.1. *Let* $0 < \varepsilon \le \varepsilon^*$.

Suppose there exist numbers $u_0 \in (u^*,1)$ *and* x_0 *such that*

$$Q(x;x_0,u_0) \le \min\{\chi(x),\ q_-(x)\} \tag{5.11}$$

and there exists $\lambda \in (0,\lambda_0)$ *such that*

$$(5.12)^1 \qquad \chi(x) \le q_\lambda(x) \qquad \text{if } \varepsilon < \varepsilon^*$$

$$(5.12)^2 \qquad \chi(x) \le q_-(x) \qquad \text{if } \varepsilon = \varepsilon^*.$$

Then

$$(5.13)\qquad \lim_{t\to\infty} u(x,t;\chi) = q_-(x),$$

uniformly on closed intervals.

PROOF. As we have seen above, $Q(x;x_0,u_0)$ is a lower solution and $q_\lambda(x)$ for $\varepsilon < \varepsilon^*$ and $\lambda \in (0,\lambda_0)$, and $q_-(x)$ for $\varepsilon = \varepsilon^*$ are upper solutions of equation (1.1). Except for the decreasing stationary solutions of (1.1), all the nonconstant stationary solutions of (1.1) have periodic parts for $x > 0$ or for $x < 0$, where in the last case the maximum over $(-\infty,0)$ is less than u^*. Therefore, the only stationary solution of (1.1), lying between $Q(x;x_0,u_0)$ and $q_\lambda(x)$ for $\lambda \in (0,\lambda_0)$ and $\varepsilon < \varepsilon^*$ is $q_-(x)$ and (5.13) is now a consequence of Lemma 5.1. If $\varepsilon = \varepsilon^*$ we take $q_-(x)$ instead of $q_\lambda(x)$ leading to the same result.

For $\varepsilon < \varepsilon^*$, a sketch of a domain of attraction of q_- is given in fig 5.2. (In the sense that functions which take values in the shaded region belong to the domain of attraction).

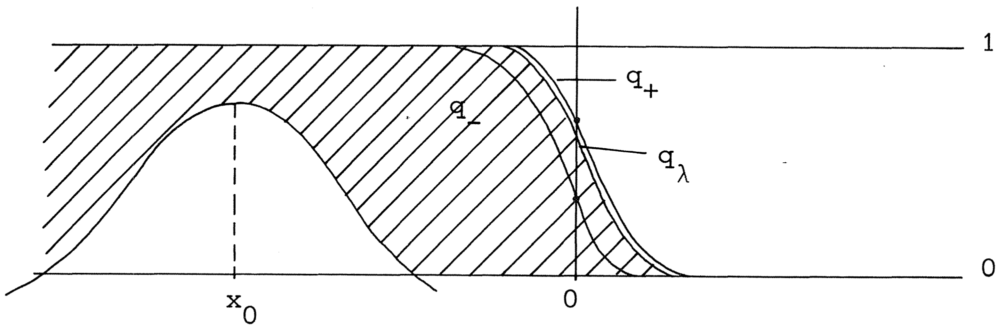

fig. 5.2.

Solutions of (1.1) where $\chi(x) \geq q_\lambda(x)$ for some $\lambda < 0$ are attracted by $u = 1$. This is shown in Theorem 5.2 below.

THEOREM 5.2. *Let* $0 < \varepsilon < \varepsilon^*$.

Suppose there exists a number $\lambda < 0$ *such that*

$$(5.14)\qquad \chi(x) \geq q_\lambda(x), \qquad x \in \mathbb{R}.$$

Then

(5.15) $\lim_{t\to\infty} u(x,t:\chi) = 1$

uniformly on closed intervals.

PROOF. The only stationary solution of (1.1), taking values in [0,1] and above $q_+(x)$ is the function $u = 1$. Relation (5.15) now follows by Lemma 5.1.

If $\varepsilon > \varepsilon^*$ then no decreasing stationary solutions of (1.1) exist. Moreover there exist in this case no stationary solutions lying entirely above $Q(x;x_0,u_0)$ for any, $u_0 \in (u^*,1)$, x_0 and below $u = 1$. (cf. Proof Theorem 5.1).

Therefore, if $\chi(x) \geq Q(x;x_0,u_0)$ then the solution of (1.1) must tend to 1 as $t \to \infty$, by Lemma 5.1. Thus we have proved the following Theorem.

THEOREM 5.3. *Let* $\varepsilon^* < \varepsilon \leq 1$.

Suppose there exist numbers $u_0 \in (u^*,1)$, x_0, *such that*

(5.16) $Q(x;x_0,u_0) \leq \chi(x)$.

Then

(5.17) $\lim_{t\to\infty} u(x,t;\chi) = 1,$

uniformly on closed intervals.

Both under the assumptions of Theorem 5.2 and those of Theorem 5.3 we have seen that $u(x,t;\chi) \to 1$ as $t \to \infty$. What we really expect is that u "travels away" from the point $x = 0$.

In order to state this precisely, recall that there exists a unique wave speed $c = c^*$ for which the equation

(5.18) $v_t = \varepsilon v_{xx} + f(v), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+$

has a solution of the form $v(x,t) = w(x-ct)$ with $w(-\infty) = 1$ and $w(+\infty) = 0$. Also sign $c^* =$ sign $\int_0^1 f(u)du > 0$. Moreover, modulo translation there exists precisely one such solution and we can define a unique representative w^* by requiring $w^*(0) = 1/2$.

DEFINITION 5.1. We say that u *travels away* from $x = 0$ if $u(x,t;\chi)$ tends, as $t \to \infty$, to $w^*(x-c^*t+K)$, uniformly for $x \geq x^* > 0$ for some $K > 0$, $x^* > 0$.

To prove that, under certain conditions, u travels away from $x = 0$ we shall use a recent result of VELING [17] which is based on earlier work of FIFE & MCLEOD [3].

LEMMA 5.2. *Let* $u(x,t)$ *be a solution of the problem*

(5.19) $\quad u_t = \varepsilon u_{xx} + f(u), \qquad x \geq 0,\ t \geq 0$

(5.20) $\quad u(0,t) = \Phi(t), \qquad 0 \leq \Phi(t) \leq 1,$

$\quad u(x,0) = \chi(x), \qquad 0 \leq \chi(x) \leq 1,$

where $\Phi(t)$ *and* $\chi(x)$ *satisfy the conditions*

(5.21) $\quad \Phi_t$ *Hölder continuous for* $t \geq 0$

$\quad \chi_{xx}$ *Hölder continuous for* $x \geq 0$

(5.22) $\quad \Phi(0) = \chi(0)$

$\quad \Phi_t(0) = \varepsilon\chi_{xx}(0) + f(\chi(0))$

(5.23) $\quad \limsup_{x\to\infty} \chi(x) < a$

(5.24) $\quad |1-\Phi(t)| \leq Me^{-\gamma t}$ *for some* $M,\gamma > 0$ *and all* $t \geq 0$.

Then there exist constants $K > 0$, $\omega > 0$, z_0 *such that*

(5.25) $\quad |u(x,t) - w^*(x-c^*t-z_0)| < Ke^{-\omega t}$, *uniformly for* $x \geq 0$.

Since the conditions (5.21), (5.22) on $u(0,t)$ and $u(x,0)$ in our case, are in general not satisfied we shall frequently apply Lemma 5.2 for

$x \geq x^* > 0$, $t \geq t^* > 0$. As a result (5.25) holds in that case for $t \geq t^*$, uniformly in $x \geq x^*$. However in order to apply Lemma 5.2 we have to show that $u(x^*,t;\chi)$ tends to 1 exponentially as $t \to \infty$ for some $x^* > 0$, and this does not follow immediately from the last two theorems.

Under different but still satisfactory conditions on $\chi(x)$ we shall establish the convergence of $u(\cdot,\cdot;\chi)$ to w^* in Section 6.

In the present Section we shall indicate, by a simple example how we shall use Lemma 5.2 in the treatment of this behaviour of $u(x,t;\chi)$ once it tends to $u = 1$.

The technique is that we search for a lower solution $u(x,t)$ of (1.1) which is equal to one of the functions $w^1(x-ct)$ and $w^\varepsilon(x-ct)$, satisfying the equation

$$(5.26) \qquad E w^E_{zz} + c w^E_z + f(w^E) = 0, \qquad z = x-ct$$

for $E = 1$ and $E = \varepsilon$, respectively. In this Section we choose $E = 1$ for $z < 0$ and $E = \varepsilon$ for $z > 0$ and we require that $w^1(-\infty) = 1$, $w^\varepsilon(+\infty) = 0$. Thus we are looking for a function $u(x,t) = w(z)$ where $w(z)$ satisfies

$$(5.27) \qquad e_\varepsilon(z) w_{zz} + c w_z + f(w) = 0.$$

Let us first consider equation (5.27) for $e_\varepsilon(z) =$ constant, $e_\varepsilon(z) = \varepsilon$ say as we have done for the case $c = 0$ in Section 2. Introducing formally $P(w) = w_z(w)$ we find that $P(w)$ satisfies the equation

$$(5.28) \qquad PP_w + \frac{c}{\varepsilon} P + \frac{f}{\varepsilon} = 0.$$

Again the points $(w,P) = (0,0)$ and $(w,P) = (1,0)$ are saddle points and only one stable manifold of $(0,0)$ and one unstable manifold of $(1,0)$ lie in the region $P < 0$, $0 \leq w \leq 1$. As these manifolds now depend on c we shall refer to them as *stable* (ε,c)-*manifold* and *unstable* (ε,c)-*manifold* or, more generally, if we consider different functions f, *stable* (ε,f,c)-*manifold* and *unstable* (ε,f,c)-*manifold*.

Now suppose $\varepsilon^* < \varepsilon < 1$. Then by the continuity of trajectories of (5.28), and thus also of $P_w(w)$ with respect to c we have that for c sufficiently small the stable (ε,c)-manifold and the unstable $(1,c)$-manifold have no points in common and furthermore, there exist unique numbers $w_c, w_c^* \in (0,1)$ such that the stable (ε,c)-manifold and the unstable $(1,c)$-manifold have horizontal slope at $w = w_c$ and $w = w_c^*$, respectively. Denote these points of horizontal slope by (w_c, P_c) and (w_c^*, P_c^*). Since, for $c > 0$ sufficiently small, $P_c^* < P_c$ and $f_w > 0$ in an interval including a, w_c and w_c^*, it follows by taking $P_w = 0$ in (5.28) that

(5.29) $\quad a < w_c < w_c^*.$

The manifolds under consideration are sketched in figure 5.3.

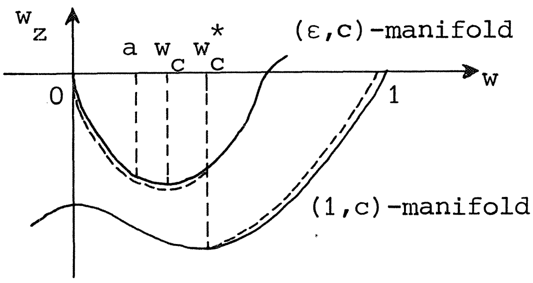

fig. 5.3.

The solution of (5.27) we are looking for and which we shall denote by $\underline{w}(z)$ is now chosen such that it corresponds with the unstable $(1,c)$-manifold for $z < 0$ and $\underline{w} > w_c^*$, and with the stable (ε,c)-manifold for $z > 0$ and $\underline{w} < w_c^*$ (the dotted line in fig. 5.3). We mention that since for the solution P(w) of (5.28) we have $P_w(w) = w_{zz}/w_z$, positive slope of the trajectories correspond to negative w_{zz}. Thus

(5.30) $\quad \underline{w}_{zz} < 0, \qquad z < 0.$

REMARK 5.3. It is well known (see for example [3]), that $\underline{w}(z)$ tends to 1, exponentially as $z \to -\infty$.

REMARK 5.4. Along the line $x = ct$ in the (x,t)-plane we have, by definition of $\underline{w}$ that

$$(5.31) \qquad \underline{w}_x(x-ct+) \geq \underline{w}_x(x-ct-)$$

and thus we can apply Theorem 4.3, the comparison principle, in case $\underline{w}$ is a lower solution.

We are now able to make a first attempt to give sufficient conditions on $\chi(x)$ such that the solution $u(x,t;\chi)$ of (1.1) travels away from $x = 0$.

THEOREM 5.4. *Let* $\varepsilon^* < \varepsilon < 1$.

Suppose $\chi(x)$ *satisfies the conditions*

$$(5.32) \qquad \chi(x) > \underline{w}(x), \qquad x \in \mathbb{R}$$

$$(5.33) \qquad \limsup_{x \to \infty} \chi(x) < a.$$

Then u *travels away from* $x = 0$.

PROOF. By (5.27) and (5.30) it follows for $u(x,t) = \underline{w}(x-ct)$ that

$$(5.34) \qquad u_t(x,t) - e_\varepsilon(x)u_{xx}(x,t) - f(u(x,t))$$
$$= [e_\varepsilon(x-ct)-e_\varepsilon(x)]w_{zz}(x-ct) = \begin{cases} 0 & , x < 0 \text{ or } x > ct \\ (1-\varepsilon)w_{zz}(x-ct) < 0, & 0 < x < ct. \end{cases}$$

Thus $u(x,t)$ is a lower solution of (1.1) and therefore, using (5.32) it follows that

$$(5.35) \qquad u(x,t:\chi) \geq \underline{w}(x-ct), \qquad x \in \mathbb{R}, \ t \in \mathbb{R}^+.$$

Since for $t \to \infty$, $\underline{w}(x-ct)$ for fixed x ($x = x^* > 0$, say) tends to 1 exponentially and $u(x,t:\chi) \leq 1$ it follows that $u(x^*,t:\chi)$ tends to 1 exponentially. Application of Lemma 5.2 now yields the desired result.

The above theorem is of course not such a strong result but all the techniques we need in the next Section to prove stronger results on convergence to w^* are present here. In that sense, this theorem gives a first impression of the proofs of some of the theorems in Section 6.

6. STABILITY AND PROPAGATION

In Section 5 we established some results on the qualitative behaviour of the solution $u(x,t;\chi)$ of (1.1), using simple upper and lower solutions. In this Section we shall extend these results. To be precise we shall prove the following theorems.

For the definitions of the function $Q(x:x_0,u_0)$ and the numbers x_0,u_0,u^*, occurring in the theorems we refer to the preceding Section.

THEOREM 6.1. (Extension of Theorem 5.1)

Let $0 < \varepsilon \leq \varepsilon^*$.

Suppose

(6.1) $\qquad \chi(x) < q_+(x), \qquad x \in \mathbb{R}$

and one of the following conditions is satisfied.

$(6.2)^1 \qquad \liminf_{x \to -\infty} \chi(x) > a$

$(6.2)^2 \qquad Q(x;x_0,u_0) \leq \min\{\chi(x),q_-(x)\}$ *for some* $u_0 \in (u^*,1), \quad x_0$.

Then

(6.3) $\qquad \lim_{t\to\infty} u(x,t;\chi) = q_-(x)$

uniformly on closed intervals.

THEOREM 6.2. (Extension of Theorem 5.2).

Let $0 < \varepsilon < \varepsilon^*$.

Suppose

(6.4) $\chi(x) > q_+(x), \qquad x \in \mathbb{R}.$

Then

(6.5) $$\lim_{t\to\infty} u(x,t;\chi) = 1$$

uniformly on intervals $(-\infty,K]$, $K \in \mathbb{R}$.

Moreover if

(6.6) $$\limsup_{x\to\infty} \chi(x) < a$$

then u *travels away from* $x = 0$.

THEOREM 6.3. (Extension of Theorem 5.4).

Let $\varepsilon^* < \varepsilon < 1$.

Suppose

(6.7) $$\liminf_{x\to-\infty} \chi(x) > a.$$

Then

(6.8) $$\lim_{t\to\infty} u(x,t;\chi) = 1$$

uniformly on intervals $(-\infty,K]$, $K \in \mathbb{R}$. *Moreover if*

(6.9) $$\limsup_{x\to\infty} \chi(x) < a$$

then u *travels away from* $x = 0$.

Note that this theorem does not imply Theorem 5.3. In [3] it was proved that the solution of (1.1) for $\varepsilon = 1$ tends to a travelling wave if the conditions (6.7) and (6.9) are satisfied. In that sense, this result is extended by Theorem 6.3.

Finally we prove the exponential stability of $q_-(x)$ in the supremum norm.

THEOREM 6.4. *Let* $0 < \varepsilon < \varepsilon^*$.
Then there exist positive constants δ, μ, K *such that* $\|\chi - q_-\|_\infty \le \delta$ *implies*

(6.10) $$\|u(\cdot,t;\chi)-q_-\|_\infty \le Ke^{-\mu t}, \qquad t \ge 0.$$

In the proofs of the above theorems we shall frequently make use of phase plane properties of equations of the form

(6.11) $$Ew_{zz} + cw_z + g(w) = 0, \qquad z \in \mathbb{R}$$

where $E \in \{\varepsilon, 1\}$, $c \ge 0$ and g is a third order polynomial having zeroes $\alpha, \beta, 1$, $\alpha < \beta < 1$ where $g'(\alpha) < 0$. A useful tool in phase plane considerations is the following Lemma, due to ARONSON & WEINBERGER [2, Lemma 4.1].

LEMMA 6.1. *For* $j = 1$ *and* 2 *let* $p_j(q)$ *denote a real-valued continuous function defined on* $[a_1,a_2]$ *which satisfies the differential equation*

(6.12) $$p_j' = F_j(q,p_j),$$

in (a_1,a_2). *If* $p_1(a_1) > p_2(a_1)$ *and if either*

(6.13) $$F_1(q,p_1(q)) > F_2(q,p_1(q))$$

or

(6.14) $$F_1(q,p_2(q)) > F_2(q,p_2(q))$$

in (a_1,a_2) *then* $p_1(q) \ge p_2(q)$ *in* $[a_1,a_2]$.

Using this Lemma one can study how the (E,g,c)-manifolds leaving $(1,0)$ (the unstable one) or going to $(\alpha,0)$ (the stable one) in the phase plane of (6.11), lying below the w-axis, vary with g or with c. The results are stated in Lemma 6.2 and Lemma 6.3.

LEMMA 6.2. *Consider two cubics* $F_i(w)$, $i = 1,2$ *with zeroes* α_i, β_i, 1, $\alpha_i < \beta_i < 1$ *where* $F_i'(\alpha_i) < 0$. *Suppose*

$$(6.18) \qquad F_1(w) < F_2(w), \qquad \alpha_1 < w < 1.$$

Let $E \in (0,1]$.

Then for $c \in \mathbb{R}$, *the stable* (E,F_1,c)*-manifold lies below the stable* (E,F_2,c)*-manifold as long as the latter lies below the* w*-axis in the* (w,w')*-plane. The unstable* (E,F_1,c)*-manifold lies above the unstable* (E,F_2,c)*-manifold as long as the former lies below the* w*-axis.*

PROOF. Introducing formally $P(w) = w_z(w)$ where w satisfies the equation (6.11) we find that $P(w)$ satisfies the equation

$$(6.16) \qquad EP_w + c + \frac{g(w)}{P(w)} = 0.$$

If $\alpha_1 < \alpha_2$ then the first part of this lemma follows by application of Lemma 6.1 to equation (6.16) for $g(w) = F_j(w)$, $j = 1,2$. If $\alpha_1 = \alpha_2$ then the slope of the stable (E,F_1,c)-manifold at $w = \alpha_1$ is less than that of the stable (E,F_2,c)-manifold at $w = \alpha_1$ and application of Lemma 6.1 on an interval $[\alpha_1 + \varepsilon_0, 1]$ for some sufficiently small positive ε_0 again yields the first part of this lemma.

The other part is proved analogously.

LEMMA 6.3.

Let

$$(6.17) \qquad c_1 < c_2$$

Then the stable (E,g,c_2)*-manifold lies below the stable* (E,g,c_1)*-manifold as long as the latter lies below the* w*-axis. The unstable* (E,g,c_2)*-manifold lies above the unstable* (E,g,c_1)*-manifold as long as it lies below the* w*-axis.*

PROOF. Similar to the proof of Lemma 6.2 using the fact that the slopes of both the stable (E,g,c)-manifold at $(\alpha,0)$ and the unstable (E,g,c)-

manifold at (1,0) are monotone functions of c.

PROOF OF THEOREM 6.1. We consider first the case $\varepsilon \in (0,\varepsilon^*)$.
As in the proof of Theorem 5.1 we shall prove Theorem 6.1 by means of upper and lower solutions, satisfying (6.1) and (6.2) instead of χ, and which enclose $\chi(x)$. Since the case $\min\{\chi(x),q_-(x)\} \geq Q(x;x_0,u_0)$ has already been treated in Theorem 5.1 we shall only consider the situation that $\liminf_{x\to-\infty} \chi(x) > a$.

We shall construct functions $\psi(x) \not\equiv q_+(x)$, $\phi(x) \not\equiv 0$, satisfying for some $t_0 > 0$

$$\text{(6.18)}\qquad \begin{aligned} &\max\{\chi(x),q_-(x)\} < \psi(x) \leq q_+(x) \\ &0 \leq \phi(x) < \min\{u(x,t_0;\chi),\ q_-(x)\} \end{aligned}$$

and

$$\text{(6.19)}\qquad \begin{aligned} &e_\varepsilon(x)\psi_{xx} + f(\psi) \leq 0, \qquad x \in \mathbb{R}\setminus\{0,x_1,x_2\} \\ &e_\varepsilon(x)\phi_{xx} + f(\phi) \geq 0, \qquad x \in \mathbb{R}\setminus\{0,x_3,x_4\} \end{aligned}$$

for some x_1–$x_4 \in \mathbb{R}$ where $\psi \in C(\mathbb{R} \to [0,1]) \cap C^2(\mathbb{R}\setminus\{0,x_1,x_2\} \to [0,1])$ and $\phi \in C(\mathbb{R} \to [0,1]) \cap C^2(\mathbb{R}\setminus\{0,x_3,x_4\} \to [0,1])$. Moreover $\psi'(x\pm)$ and $\psi''(x\pm)$ exist and $\psi'(x+) - \psi'(x-) \leq 0$ for $x \in \{0,x_1,x_2\}$ and $\phi'(x\pm)$, $\phi''(x\pm)$ exist and $\phi'(x+) - \phi'(x-) \geq 0$ for $x \in \{0,x_3,x_4\}$. Then the only stationary solution of (1.1) between $\phi(x)$ and $\psi(x)$ is $q_-(x)$ and application of Lemma 5.1 yields

$$\text{(6.20)}\qquad \lim_{t\to\infty} u(x,t;\psi) = \lim_{t\to\infty} u(x,t:\phi) = q_-(x).$$

Since $u(x,t_0;\chi) \in (\phi(x),\ \psi(x))$ we must also have

$$\text{(6.21)}\qquad \lim_{t\to\infty} u(x,t;\chi) = q_-(x).$$

Construction $\psi(x)$.

Consider for $\delta > 0$ the function

(6.22) $\quad g(u) = f(u) + \delta u(1-u),$

where, in order to have $g'(0) < 0$

(6.23) $\quad -a + \delta = f'(0) + \delta < 0.$

To begin with we shall find for δ sufficiently small a function ψ such that

(6.24) $\quad e_\varepsilon(x)\psi_{xx} + g(\psi) = 0,$

(6.25) $\quad \psi(-\infty) = 1, \ \psi(+\infty) = 0.$

Clearly ψ satisfies $(6.19)^1$.

Denote a_+^δ as the largest point of intersection of the stable $(\varepsilon,g,0)$-manifold and the unstable $(1,g,0)$-manifold.

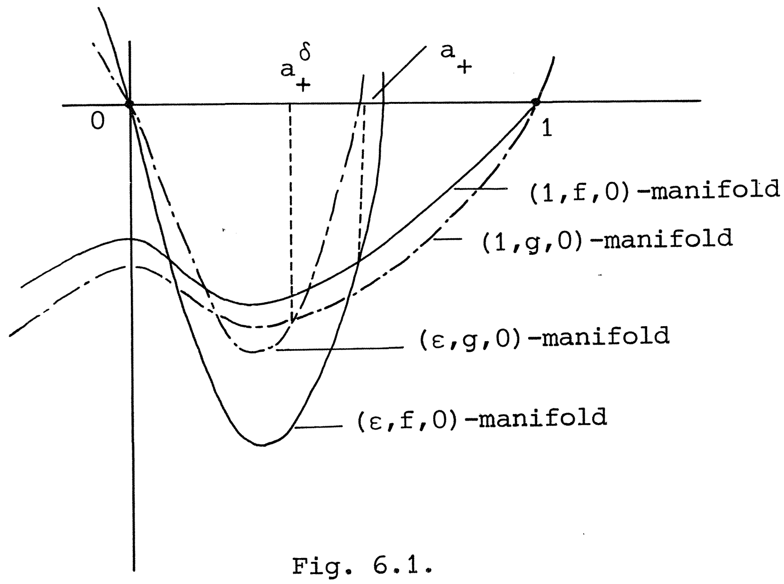

Fig. 6.1.

For δ small enough there exists a solution $q(x) \in C^1(\mathbb{R})$ of

$$
(6.26) \quad \begin{aligned}
&e_\varepsilon(x)q'' + g(q) = 0, \qquad x \neq 0 \\
&q(0) = a_+^\delta \\
&q(-\infty) = 1, \quad q(+\infty) = 0.
\end{aligned}
$$

The values of q_+' and q' at $q_+ = u$ and $q = u$ will be denoted by $R_+(u)$ and $R(u)$, respectively. These functions satisfy the equations (cf. (2.4))

$$(6.27)\qquad \begin{aligned} R_+ R_{+,u} &= \begin{cases} -f/\varepsilon & ; \quad 0 < u < a_+, \\ -f & ; \quad a_+ < u < 1, \end{cases} \\ R\,R_u &= \begin{cases} -g/\varepsilon & ; \quad 0 < u < a_+^{\delta} \\ -g & ; \quad a_+^{\delta} < u < 1. \end{cases} \end{aligned}$$

The corresponding manifolds are sketched in fig 6.1 (cf. Lemma 6.2).

We shall first confine ourselves to a treatment of the $(\varepsilon,f,0)$ and the $(\varepsilon,g,0)$ manifolds. On the $(\varepsilon,f,0)$-manifold we have

$$(6.28)\qquad R_+(u) = -[-\frac{2}{\varepsilon} F(u)]^{\frac{1}{2}} \quad ; \qquad 0 < u < a_+$$

where

$$(6.29)\qquad F(u) = \int_0^u f(\tau)d\tau.$$

(Integrate (6.27)). On the $(\varepsilon,g,0)$-manifold we find similarly

$$(6.30)\qquad R(u) = -[-\frac{2}{\varepsilon} \{F(u) + \delta u^2(\frac{1}{2} - \frac{1}{3}u)\}]^{\frac{1}{2}}, \qquad 0 < u < a_+^{\delta}.$$

Consider the inverse functions $\xi_+(u)$, $\xi(u)$ of $q_+(x)$, $q(x)$, respectively for $0 < u < a_+^{\delta}$ (see fig. 6.2).

Define

$$(6.31)\qquad \Phi(u) = \xi_+(u) - \xi(u)$$

Then

$$(6.32)\qquad \Phi(a_+^{\delta}) > 0.$$

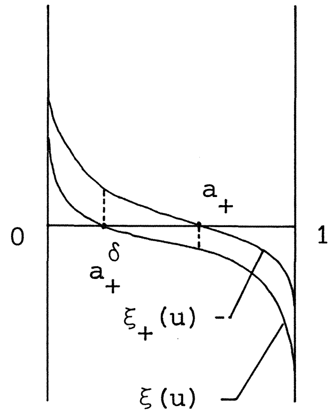

fig. 6.2.

We shall now show that $\Phi_u(u) > 0$ for $u \in (0,a_+^\delta)$ and $\Phi(u) < 0$ for $0 < u < \hat{u}$ for some $\hat{u}$ not depending on $\delta \in (0,\delta_0)$ for δ_0 sufficiently small. Also we shall show that $a_+ - a_+^\delta = o(\delta)$. Then for $\rho_0 > 0$, by reasons of continuity there exists a number $M > 0$ such that for any $\rho \in (0,\rho_0)$ there exists a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ a solution ψ of

(6.33) $$e_\varepsilon(x)\psi'' + g(\psi) = 0\,, \qquad x \in \mathbb{R},$$
$$\psi(-\infty) = 1, \quad \psi(+\infty) = 0$$

satisfies

(6.34) $$\begin{aligned} &q_+(0) - \rho \le \psi(0) < q_+(0) \\ &q_+(x) - \rho \le \psi(x)\,, \qquad 0 \le x < M \\ &q_+(x) \le \psi(x)\,, \quad x \ge M. \end{aligned}$$

Let us first estimate $a_+ - a_+^\delta$.
The number a_+ follows from

(6.35) $$\int_0^{a_+} f(\tau)d\tau + \varepsilon \int_{a_+}^{1} f(\tau)d\tau = 0$$

and a_+^δ follows from

$$(6.36)\qquad \int_0^{a_+^\delta} [f(\tau) + \delta\tau(1-\tau)]d\tau + \varepsilon \int_{a_+^\delta}^1 [f(\tau) + \delta\tau(1-\tau)]d\tau = 0.$$

Subtracting these two relations we find

$$(6.37)\qquad (1-\varepsilon) \int_{a_+^\delta}^{a_+} f(\tau)d\tau = \delta[\int_0^{a_+} \tau(1-\tau)d\tau + \varepsilon \int_{a_+^\delta}^1 \tau(1-\tau)d\tau].$$

For δ_1 sufficiently small we have

$$(6.38)\qquad \int_{a_+^\delta}^{a_+} f(\tau)d\tau \geq r[a_+-a_+^\delta], \qquad r > 0 \text{ independent of } \delta \in (0,\delta_1).$$

Combining (6.37) and (6.38) we find that there exists a number $\lambda > 0$, not depending on $\delta \in (0,\delta_1)$ such that

$$(6.39)\qquad a_+ - a_+^\delta \leq \lambda\delta.$$

Next we estimate $\Phi_u(u)$.
Using the relation

$$(6.40)\qquad (1-h)^{-\frac{1}{2}} \geq 1 + \tfrac{1}{2}h, \qquad 0 \leq h < 1$$

it follows for $0 < u < a_+^\delta$ and δ small enough that

$$\frac{1}{R(u)} = \frac{1}{R_+(u)} \left\{1 + \frac{\delta u^2(\frac{1}{2} - \frac{1}{3}u)}{2F(u)}\right\}^{-\frac{1}{2}} \leq$$

(6.41)

$$\leq \frac{1}{R_+(u)} + \frac{\delta u^2(\frac{1}{2} - \frac{1}{3}u)}{4F(u)[-\frac{2}{3}F(u)]^{\frac{1}{2}}}$$

and using

$$(6.42) \qquad F(u) = -\tfrac{1}{2}au^2(1+O(u)), \qquad u \to 0$$

it is obvious that there exist numbers δ_2, $K > 0$, such that for $0 < \delta \le \delta_2$

$$(6.43) \qquad \Phi_u(u) = \frac{1}{R_+(u)} - \frac{1}{R(u)} \ge \frac{K\delta}{u} \qquad 0 < u \le a_+^{\delta}.$$

Finally we show the existence of $\hat{u}$.
For sufficiently small δ_3 we have

$$(6.44) \qquad \Phi(a_+^{\delta}) \le -2\,\frac{[a_+ - a_+^{\delta}]}{R_+(a_+)} < -\frac{2\lambda\delta}{R_+(a_+)}, \qquad 0 < \delta \le \delta_3$$

Since by (6.43), $\Phi_u(u) > 0$, a sufficient condition for $\Phi(\hat{u}) < 0$ is

$$(6.45) \qquad \int_{\hat{u}}^{a_+^{\delta}} \Phi_u(u)\,du = \Phi(a_+^{\delta}) - \Phi(\hat{u}) > \Phi(a_+^{\delta})$$

and therefore also

$$(6.46) \qquad \int_{\hat{u}}^{a_+^{\delta}} \Phi_u(u)\,du > \frac{-2\lambda\delta}{R_+(a_+)}$$

implies $\Phi(\hat{u}) < 0$. Now choose

$$(6.47) \qquad \hat{u} < a_+^{\delta} \exp\left[2\,\frac{\lambda}{KR_+(a_+)}\right],$$

then, by (6.43) this yields (6.46) and thus $\Phi(\hat{u}) < 0$. In a similar way we can treat the $(1,f,0)$- and $(1,g,0)$-manifolds (the case $x < 0$) and as a result we may extend (6.33), (6.34) to:

Let $\rho_0 > 0$. Then there exist numbers $M,N > 0$ such that for any $\rho \in (0,\rho_0)$ there exists a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ a solution ψ of (6.33) satisfies

$$
\begin{aligned}
&q_+(0) - \rho \le \psi(0) < q_+(0) \\
(6.48) \qquad &q_+(x) - \rho \le \psi(x) \qquad , \; -N < x < M, \\
&q_+(x) \le \psi(x) \qquad , \quad x \le -N,\ x \ge M.
\end{aligned}
$$

Replacing $\psi(x)$ by $q_+(x)$ if $\psi(x) \ge q_+(x)$, we find that $\psi(x)$ satisfies the desired conditions for some x_1, x_2, if we take ρ_0 sufficiently small.

Construction $\phi(x)$.

The unstable $(1,f,0)$-manifold corresponds with the decreasing solution of

$$
(6.49) \qquad \begin{aligned} &q''(x) + f(q) = 0, \qquad x \le -L \\ &q(-L) = 0, \quad q(-\infty) = 1 \end{aligned}
$$

where $L \in \mathbb{R}$. Take $L > 0$. Similar to the proof of Lemma 4.1 in [3] one can find functions $\mu(t)$, $\xi(t)$, $\mu(t) \downarrow 0$, $\xi(t) \to \xi_0 > 0$, $(t\to\infty)$ such that

$$
(6.50) \qquad v(x,t) \equiv q(x+\xi(t))-\mu(t)
$$

satisfies

$$
(6.51) \qquad \begin{aligned} &v_t - v_{xx} - f(v) \le 0, \qquad x \le -L-\xi(t) < 0 \\ &v(x,0) < \min\{\chi(x),\ q-(x)\}. \end{aligned}
$$

The stable $(1,f,0)$-manifold intersects the line $q' = 0$ for $q = u^* \in (0,1)$ (see Section 5). The trajectory, intersecting the q-axis at a value $q = u_0 \in (u^*,1)$ corresponds with a solution $Q(x,x_0,u_0)$ of equation (6.49), vanishing at points $x = x_3$ and $x = x_4$ for some $x_3 < x_4 < 0$ say, taking at most the value $q = u_0$ at $x = x_0$. Then since $\xi(t)$ is bounded and $\mu(t) \to 0$, if $x_4 \ll -L$ there exists a time t_0 such that

$$
(6.52) \qquad Q(x,x_0,u_0) \le \min\{v(x,t_0),\ q_-(x)\}.
$$

By Theorem 4.3 we have $u(x,t_0;\chi) \ge v(x,t_0)$ which yields $(6.18)^2$ for

$\psi(x) = Q(x;x_0,u_0)$.

If $\varepsilon = \varepsilon^*$ then $q_+(x) = q_-(x)$. If we choose $\psi(x) = q_-(x)$ then clearly $\psi(x)$ is an upper solution. The remaining part of the proof in this case is identical to the corresponding part for $\varepsilon < \varepsilon^*$.

PROOF OF THEOREM 6.2.

Consider for $\delta > 0$ the function

$$g(u) = f(u) - \delta u(1-u) \tag{6.53}$$

where δ is so small that

$$\int_0^1 g(u)\,du > 0. \tag{6.54}$$

Similar to the proof of Theorem 6.1 one can show that for $\rho_0 > 0$ there exist numbers M, N > 0 such that for any $\rho \in (0,\rho_0)$ there exists a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ a solution ψ of

$$\begin{aligned} &e_\varepsilon(x)\psi_{xx} + g(\psi) = 0 \\ &\psi(-\infty) = 1,\ \psi(+\infty) = 0 \end{aligned} \tag{6.55}$$

exists which satisfies

$$\begin{aligned} &q_+(0) < \psi(0) \le q_+(0) + \rho \\ &q_+(x) < \psi(x),\ -N < x < M \\ &q_+(x) \ge \psi(x),\ x \le -N, \quad x \ge M. \end{aligned} \tag{6.56}$$

Basic for the proof of Theorem 6.1 are the qualitative properties of the $(1,g,0)$- and $(\varepsilon,g,0)$-manifolds. These properties remain the same for the $(1,g,c)$- and (ε,g,c)-manifolds for $c > 0$ sufficiently small. Therefore, adding a term $c\psi'(x)$ to the left hand-side of equation (6.55) does not affect the above statements about ψ if c is sufficiently small.

To be more precise we have for $\rho_0 > 0$, numbers M, N > 0 such that for any $\rho \in (0,\rho_0)$ there exists a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ and sufficiently small $c > 0$, a solution ψ of

$$(6.57)\qquad \begin{aligned} & e_\varepsilon(x)\psi_{xx} + c\psi_x + g(\psi) = 0 \\ & \psi(-\infty) = 1,\ \psi(+\infty) = 0. \end{aligned}$$

exists which satisfies (6.56).

The function $v(x,t) \equiv \psi(x-ct)$ satisfies $v_{xx} < 0$ if $0 < x < ct$ and therefore for $x \notin \{0,ct\}$ we have

$$\begin{aligned} & v_t - e_\varepsilon(x)v_{xx} - f(v) = \\ & = [e_\varepsilon(x-ct) - e_\varepsilon(x)]v_{xx} + g(\psi) - f(\psi) \le 0. \end{aligned}$$

Choose ρ so small such that $\psi(x) < \chi(x)$.
Application of the comparison principle Theorem 4.3 yields

$$(6.59)\qquad \psi(x-ct) \le u(x,t;\chi), \quad x \in \mathbb{R},\quad t \ge 0.$$

It is well known that $\psi(x-ct)$ tends to 1 exponentially as $x-ct \to -\infty$. If $\chi(x)$ satisfies (6.6) we have by application of Lemma 5.2 that u travels away from $x = 0$.

PROOF OF THEOREM 6.3. Introduce for $\delta > 0$ the cubic

$$(6.60)\qquad g(u) = f(u) - \delta(1-u),$$

which has zeroes 1 and

$$(6.61)\qquad \alpha = \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2+4\delta},$$

$$(6.62)\qquad \beta = \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2+4\delta} \quad .$$

We shall assume δ to be small such that

$$(6.63)\qquad \int_\alpha^1 g(u)\,du > 0.$$

We shall consider solutions of the equations

$$(6.64) \qquad Ew_{zz} + cw_z + g(w) = 0, \qquad E \in \{\varepsilon, 1\}$$

and restrict ourselves to those values of $c > 0$ and δ for which the stable (ε,g,c)-manifold and the unstable $(1,g,c)$-manifold have no points in common for $\alpha \le w \le 1$. We have now the situation sketched in the phase plane picture fig. 6.3.

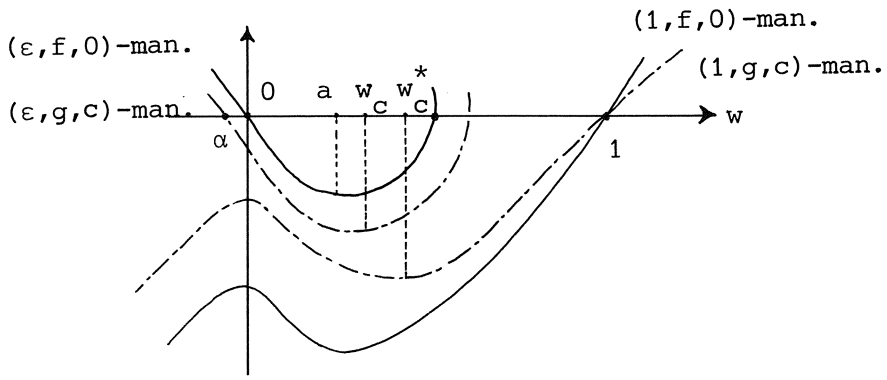

fig. 6.3.

By Lemma 6.1 and 6.2 the stable (ε,g,c)-manifold and the unstable $(1,g,c)$-manifold lie between the stable $(\varepsilon,f,0)$ manifold and the unstable $(1,f,0)$-manifold.

It was demonstrated in Section 5 that for c sufficiently small there exist unique numbers w_c, $w_c^* \in (0,1)$, satisfying $a < w_c < w_c^*$ such that the stable (ε,f,c)-manifold and the unstable $(1,f,c)$-manifold have horizontal slope at $w = w_c$ and $w = w_c^*$, respectively.

The existence of such numbers w_c, w_c^* satisfying $\beta < w_c < w_c^*$, for the stable (ε,g,c)- and the unstable $(1,g,c)$-manifold respectively can be shown in the same way. Denote by $w_\varepsilon(z)$ and $w_1(z)$ the solutions of (6.64) for $E = \varepsilon$, corresponding with the stable (ε,g,c)-manifold and for $E = 1$, corresponding with the unstable $(1,g,c)$-manifold, respectively. Then

$$(6.68)\qquad \begin{aligned} &w_{\varepsilon,zz} > 0, && w \in (\alpha, w_c) \\ &w_{1,zz} < 0, && w \in (w_c{}^*, 1) \\ &w_{\varepsilon,zz}(w=w_c) = w_{1,zz}(w = w_c{}^*) = 0. && \end{aligned}$$

We introduce a new function $U^c(t)$ by

$$(6.66)\qquad U^c(0) = w_c$$

and

$$(6.67)\qquad U^c(z) = \begin{cases} w_1(z), & z < 0, \\ w_c(z), & z > 0. \end{cases}$$

The crucial tool of this proof is the following result.

If conditions (6.7) is satisfied then there exist numbers $K, \mu > 0$, z_0 such that

$$(6.68)\qquad U^c(x-ct-z_0) - Ke^{-\mu t} \le u(x,t;\chi), \qquad t \ge 0,\ x \in \mathbb{R},$$

Before we shall prove this we shall first demonstrate how it enables us to show the convergence of $u(x,t;\chi)$ towards $w^*(z)$. It is well known that $w_1(z)$ tends to 1, exponentially as $z \to -\infty$. Therefore, applying (6.68) for any $x = x^*$ and since $u(x,t;\chi) \le 1$ we find that $u(x^*,t;\chi)$ tends to 1 exponentially as $t \to \infty$. Suppose (6.9) is satisfied. Then, by Lemma 5.2, $u(x,t:\chi)$ travels away from $x = 0$.

<u>Proof of (6.68)</u>. Writing

$$(6.69)\qquad v(z,t) = u(x,t), \quad z = x-ct$$

we find $v(z,t)$ to satisfy

$$(6.70)\qquad \begin{aligned} &v_t = e_\varepsilon(z+ct)v_{zz} + cv_z + f(v) \\ &v(z,0) = \chi(z). \end{aligned}$$

We shall follow the proof of Lemma 4.1 in [3].

Bounded functions $\zeta(t)$ and $q(t)$ ($q(t)$ positive) will be chosen such that

$$(6.71) \qquad \underline{v}(z,t) \equiv \max[\alpha,\ U^c(z+\xi(t)) - q(t)]$$

forms a lower solution for equation (6.70).

The proof of this lemma differs from the proof of Lemma 4.1 in [3] in the sense that here we have to deal with the discontinuous coefficient $e_\varepsilon(z+ct)$.

Choose

$$(6.72) \qquad 1 - q_0 \in (a,\ \liminf_{x \to -\infty} \chi(x)).$$

Take z^* so that $U^c(z+z^*) - q_0 \leq \chi(z)$ for all z. This is possible for sufficiently large z^*, because of (6.7).

Introduce

$$(6.73) \qquad \Phi(u,q) = \begin{cases} [f(u-q) - f(u)]/q, & q > 0, \\ \quad - f'(u) \qquad , & q = 0. \end{cases}$$

Then Φ is continuous for $q \geq 0$, and for $0 < q \leq q_0$ we have $a < 1-q_0 \leq 1-q < 1$, so that $\Phi(1-q) > 0$. Taking $q = 0$ we get $\Phi(1,0) = -f'(1) > 0$.

As a consequence there exists a number $\mu > 0$ such that we have $\Phi(1,q) \geq 2\mu$ for $0 \leq q \leq q_0$ and by continuity, there exists a $\delta_0 > 0$ such that $\Phi(u,q) \geq \mu$ for $1 - \delta_0 \leq u \leq 1$, $0 \leq q \leq q_0$. In this range we have, still following [3]

$$(6.74) \qquad f(u-q) - f(u) \geq \mu q.$$

Set $\zeta = z + \xi(t)$, then we find that, if $\underline{v} > \alpha$,

$$(6.75) \qquad \begin{aligned} N\underline{v} := {} & \underline{v}_t - e_\varepsilon(z+ct)\underline{v}_{zz} - c\underline{v}_z - f(\underline{v}) \\ = {} & U^{c'}(\zeta)\xi(t) - \dot{q}(t) - e_\varepsilon(z+ct)U^{c''}(\zeta) - cU^{c'}(\zeta) - f(U^c(\zeta)-q) \\ = {} & U^{c'}(\zeta)\xi(t) - \dot{q}(t) - [e_\varepsilon(z+ct) - e_\varepsilon(\zeta(t))]U^{c''}(\zeta) + \\ & + [g(U^c(\zeta)) - f(U^c(\zeta))] + [f(U^c(\zeta)) - f(U^c(\zeta) - q)], \quad t \geq 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \neq -\xi(t) \end{aligned}$$

We shall first estimate the term $[e_\varepsilon(z+ct) - e_\varepsilon(\zeta)]U^{c''}(\zeta)$ in (6.75). Let $\zeta_0 > 0$ be defined by the relation

$$U^c(-\zeta_0) \equiv w_1(-\zeta_0) = w_c^*. \tag{6.76}$$

Taking $w_{zz} = 0$ and $w = w_c^*$ in (6.64) we see that $g(w_c^*) = O(c)$, $c \to 0$. Thus we have also $g(w) = O(c)$ for $w_c \le w \le w_c^*$. For $-\zeta_0 < \zeta < 0$ this leads to

$$\begin{aligned}[e_\varepsilon(z+ct) - e_\varepsilon(\zeta)]U^{c''}(\zeta) &= -[e_\varepsilon(z+ct) - e_\varepsilon(\zeta)]\,. \\ &\qquad [cU^{c'}(\zeta) + g(U^c(\zeta))] \\ &= O(c).\end{aligned} \tag{6.77}$$

For $\zeta = z + \xi(t)$ outside $(-\zeta_0,0)$, the inequalities

$$\begin{aligned} e_\varepsilon(z+\xi(t)) - e_\varepsilon(z+ct) &\ge 0, \qquad z \le -\xi(t) \\ e_\varepsilon(z+\xi(t)) - e_\varepsilon(z+ct) &\le 0, \qquad z \ge -\xi(t) \end{aligned} \tag{6.78}$$

together with (6.65) imply

$$[e_\varepsilon(z+ct) - e_\varepsilon(\zeta)]\, U^{c''}(\zeta) \ge 0, \quad \zeta \in \mathbb{R}\setminus(-\zeta_0,0]. \tag{6.79}$$

Using (6.74), (6.77) and (6.79), (6.75) reduces for $U^c \in [1-\delta_0,1]$, $q \in [0,q_0]$, $\zeta \ne 0$ to

$$N\underline{v} \le U^{c'}(\zeta)\,\dot{\xi}(t) - \dot{q}(t) = DcI(\zeta) - \delta[1-U^c(\zeta)] - \mu q \tag{6.80}$$

for some $D > 0$ where

$$I(\zeta) = \begin{cases} 1\,; & w_c < U^c(\zeta) < w_c^* \\ 0\,; & \text{elsewhere.} \end{cases} \tag{6.81}$$

We shall find a function $\xi(t)$ for which $\dot{\xi}(t) \ge 0$. Then we finally obtain from (6.80) for $\zeta \neq 0$

(6.82) $$N\underline{v} \leq -\dot{q} - \mu q, \qquad 1-\delta_0 \leq U^c \leq 1,$$

provided δ_0 is small enough.
Choose $q(t) = q_0 \exp[-\mu t(2)$ which results in

(6.83) $$N\underline{v} \leq -\frac{\mu}{2} q_0 \exp[-\frac{\mu t}{2}], \quad 1-\delta_0 \leq U^c \leq 1.$$

By possibly further reducing the size of μ and δ_0 it follows analogously that

(6.84) $$N\underline{v} \leq -\frac{\mu}{2} q_1 \exp[-\frac{\mu t}{2}], \quad \alpha \leq U^c \leq \alpha + \delta_0,$$

for some $q_1 > 0$.

Now consider the intermediate values $\alpha + \delta_0 \leq U^c \leq 1 - \delta_0$. Here we know that $U^{c'}(\zeta) \leq -B$ for some $B > 0$, only depending on δ_0. Moreover, we have from the boundedness of f_u that $f(U^c) - f(U^c - q) \leq \kappa q$ for some $\kappa > 0$. Thus for $z \neq -\xi(t)$

(6.85) $$\begin{aligned} N\underline{v} &\leq -B\dot{\xi}(t) - q(t) + DcI(\zeta) - \delta(1-U^c(\zeta)) + \kappa q \\ &\leq -B\dot{\xi}(t) - \dot{q}(t) + \kappa q \end{aligned}$$

for c/δ small enough.

Setting

(6.86) $$\xi(t) = -\frac{q_0(\mu+2\kappa)}{2\mu B}(e^{-\frac{\mu}{2}t} - 1) + z^*$$

for some $z^* \in \mathbb{R}$ we find $\xi(t)$ to be increasing and to approach a finite limit as $t \to \infty$. Together with (6.85) this gives

(6.87) $$N\underline{v} \leq -\frac{1}{4} B(\mu+2\kappa)e^{-\frac{\mu}{2}t}.$$

Now we have shown that for every $T > 0$, $x \neq ct - \xi(t)$, $\underline{v} > \alpha$, $v(x-ct,t)$ is a lower solution of equation (1.1) for $0 \le t \le T$. In the point $z = x-ct = -\xi(t)$ we have $\underline{v}_z(-\xi(t)+,t) > \underline{v}_z(-\xi(t)-,t)$. We conclude that

(6.88) $$\underline{v}(x-ct,t) \le u(x,t;\chi), \qquad t \ge 0, \qquad x \in \mathbb{R}.$$

Therefore

$$\begin{aligned} u(x,t;\chi) &\ge U^c(x-ct+\xi(t)) - q(t) \\ &\ge U^c(x-ct+\xi(\infty)) - \max\{q_0,q_1\}e^{-\frac{\mu}{2}t} \end{aligned}$$

for all $x \in \mathbb{R}$, $t \ge 0$ and this completes the proof.

Proof of Theorem 6.4. The first part of this proof is a modification of the corresponding part of the proof of Theorem 5 in [4].

Write

(6.90) $$u(x,t:\chi) = q_-(x) + v(x,t).$$

Then v satisfies the equation

(6.91) $$v_t = e_\varepsilon(x)v_{xx} + f'(q_-(x))v + h(x,v)$$

where

(6.92) $$h(x,v) \equiv f(q_-(x)+v) - f(q_-(x)) - f_u(q_-(x))v.$$

We note that

(6.93) $$\|h(\cdot,v)\|_\infty \le M\|v\|_\infty^2$$

where

(6.94) $$M = \tfrac{1}{2}\sup\{f_{uu}(u)\,|\,u \in [0,1]\} = \tfrac{1}{2}(1+a).$$

Consider first the linear eigenvalue problem in $L^2(\mathbb{R})$

(6.95) $$A_v \equiv v_{xx} + Q(x)v = \lambda v$$

where

$$Q(x) \equiv \frac{f'(q_-(x))}{e_\varepsilon(x)}$$

From [15, Ch. V] and [16. Ch. XVI] we deduce that the spectrum of A consists of a continuum in the interval $(-\infty,\mu^*]$ and a (possibly empty) discrete set of eigenvalues, in the interval $(\mu^*,\bar{\mu})$ where, by definition

(6.96)
$$\mu^* = \max\{Q(+\infty),\ Q(-\infty)\} = \max\{\frac{f'(0)}{\varepsilon},\ f'(1)\}$$
$$\bar{\mu} = \sup\{Q(x) \mid x \in \mathbb{R}\}.$$

(In [15], [16] most results are stated under the assumption that Q is continuous; however, the relevant proofs remain valid if Q is piecewise continuous).

We shall show that the entire spectrum is on the negative half-line and bounded away from zero. Since $\mu^* < 0$ we still have to prove this statement for the discrete spectrum in $(\mu^*,\bar{\mu})$.

Let λ_0 be the largest eigenvalue and let $v_0 \in L^2(\mathbb{R})$ be the corresponding eigenfunction. Since A is self-adjoint, the spectrum $\sigma(A)$ is bounded from above and the operator L, defined by $L\phi \equiv \phi_t - A\phi$ satisfies a strong maximum principle it follows from Theorem XIII.44 in [13] that $v_0(x)$ does not vanish anywhere. We shall take $v_0 > 0$.
Thus we have

(6.97) $$v_0'' + [\frac{f'(q_-(x))}{e_\varepsilon(x)} - \lambda_0]v_0 = 0.$$

The function $q_-(x)$ satisfies the equation

(6.98) $$q_-'' + \frac{f(q_-)}{e_\varepsilon(x)} = 0.$$

Using these relations we find that

$$0 = \int_{\mathbb{R}} [v_0''(x)q_-'(x) + q_-''(x)v_0'(x)]dx$$

(6.99)
$$= -\int_{\mathbb{R}} q_-'(x)v_0(x)[\frac{f'(q_-(x))}{e_\varepsilon(x)} - \lambda_0]dx - \int_{\mathbb{R}} v_0'(x)\ \frac{f(q_-(x))}{e_\varepsilon(x)}\ dx$$

$$= \lambda_0 \int_{\mathbb{R}} q_-'(x)v_0(x)dx - (1 - \frac{1}{\varepsilon})v_0(x)f(a_-).$$

Since $v_0(x) > 0$, $q_-'(x) < 0$, $f(a_-) < 0$ and $\varepsilon \in (0,1)$ it follows that $\lambda_0 < 0$. An immediate corollary is the linearized stability of $q_-(x)$. By means of the comparison principle Th. 4.3, using appropriate comparison functions one can extend this to the exponential stability, expressed by (6.10). However since this part of the proof is similar to the corresponding part of the proof of Theorem 5 in [4] we shall omit it here.

7. NUMERICAL RESULTS

We calculated, numerically, the solution $u(x,t)$ of (1.1) for

(7.1)
$$f(u) = u(1-u)(u-\frac{1}{3})$$

(7.2)
$$\chi(x) = \begin{cases} 8/10 \sin [\frac{18}{100}(x+20)] ; & -20 \le x \le \frac{100}{18}\pi - 20 \\ 0 ; & \text{elsewhere} \end{cases}$$

for several values of ε using an algorithm of VERWER [18]. Note that for this example $\varepsilon^* = 5/32$. For $\varepsilon = 0.1, 0.2, 0.3$ the results are shown in figures 7.1, 7.2 and 7.3, respectively. We plotted $u(x,t)$ for t varying from 0 to 70 with steps 10. From the figures we see that for $\varepsilon = 0.2$ and $\varepsilon = 0.3$, the solution propagates from left to right while for $\varepsilon = 0.1$, the solution is clearly blocked.
Note that for $\varepsilon = 0.2$ the solution can be seen to slow down a little around the point $x = 0$. This effect has also been observed in [6] for a more realistic model in the sense of nerve conduction.

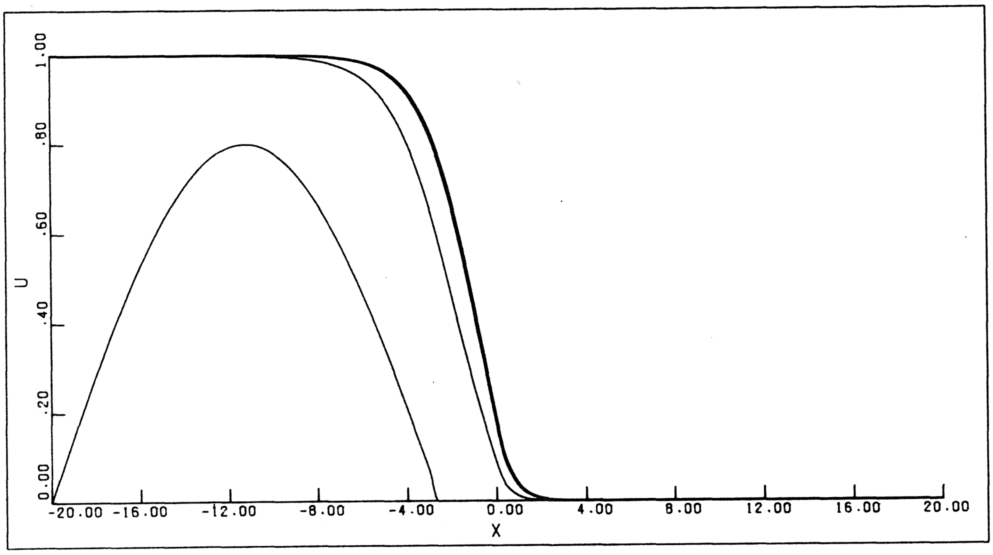

fig. 7.1. Solution u(x,t) of (1.1) for ε = 0.1
and t = 0(10) 70

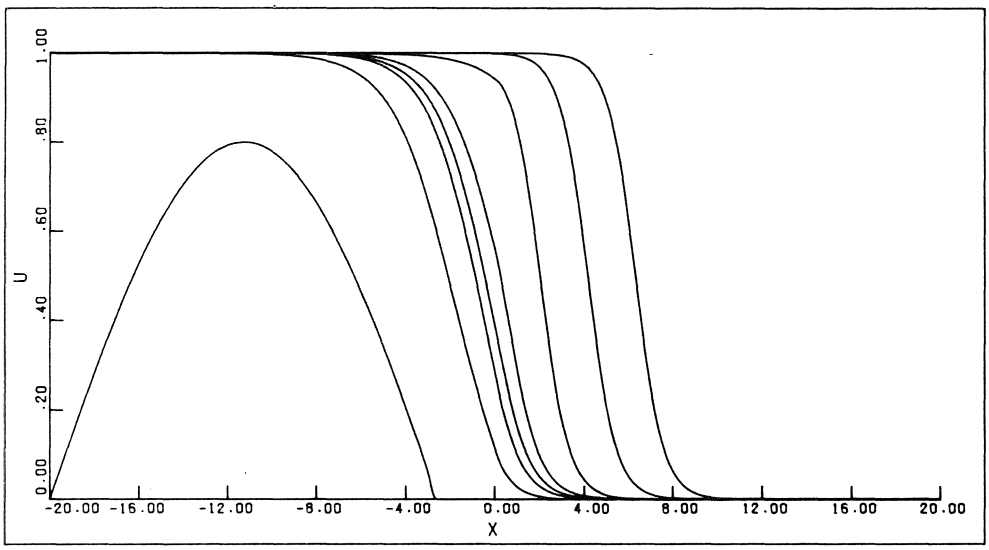

fig. 7.2. Solution u(x,t) of (1.1) for ε = 0.2
and t = 0(10) 70

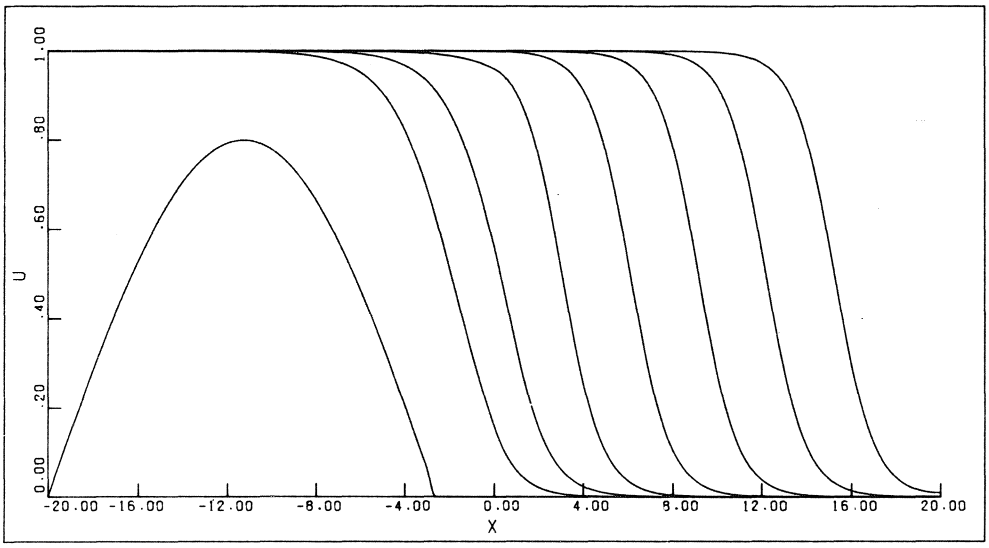

fig. 7.3. Solution $u(x,t)$ of (1.1) for $\varepsilon = 0.3$
and $t = 0(10)\ 70$

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