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SOME PROBLEMS IN CONNECTION WITH THE INCOMPLETE GAMMA FUNCTIONS

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Some problems in connection with the incomplete gamma functions

by

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ABSTRACT

In this note three problems on the incomplete gamma functions are considered. Some of the results are based on earlier results of the author on uniform asymptotic expansions for these functions.

KEY WORDS & PHRASES: *incomplete gamma functions, asymptotic expansions, inversion of incomplete gamma functions, convexity of incomplete gamma functions*

INTRODUCTION

The incomplete gamma functions are

$$(1) \qquad P(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t}\, dt, \qquad Q(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1} e^{-t}\, dt.$$

We suppose that x and a are real, $x \geq 0$, $a > 0$. In this note we consider three problems on these functions.

(i) The first problem is concerned with the function

$$(2) \qquad e(s) = \frac{1}{\Gamma(s+1)} \int_s^\infty t^s e^{-t}\, dt,$$

which in terms of $Q(a,x)$ introduced above is

$$(3) \qquad e(s) = Q(s+1,s).$$

It is known that $e(0) = 1$ and that $\lim_{s\to\infty} e(s) = \frac{1}{2}$. J. VAN DE LUNE [1] investigated the behaviour of e(s) and he proved that e(s) tends decreasingly to ½. From further investigations he conjectured that e(s) is convex and possibly log convex. In this note we prove the convexity of e(s). That is to say, we give a representation of e(s) from which the convexity follows as a special property.

(ii) (Problem 6265, American Mathematical Monthly, 1979, p.311).

Let

$$(4) \qquad Q_n(x) = e^{-x}\left(1+x+\ldots+\frac{x^n}{n!}\right), \qquad n = 0,1,\ldots .$$

Prove or disprove the following assertion:

If $x = s_n$ is the solution to the equation

$$(5) \qquad Q_n(x) = \tfrac{1}{2}$$

then $s_n - n$ approaches 2/3 as $n \to \infty$. (This statement will appear to be true).

(iii) (Problem 6271, American Mathematical Monthly, 1979, p.509).

For positive integers n define

$$(6)\qquad \begin{aligned} a_n &= \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \dots + \frac{(n-1)!}{n^{n-1}} \\ b_n &= \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \dots + \frac{n^{n-1}}{(n+1)\dots(2n-1)} \quad . \end{aligned}$$

(A) Prove that, for all $n \geq 1$, $0 < b_n - a_n < 1$.

(B) Prove or disprove that

$$\lim_{n\to\infty} (b_n - a_n) = 2/3$$

and that

$$b_n - a_n - 2/3 = \mathcal{O}(1/n).$$

(The statements in (B) will appear to be true.)

1. ON THE CONVEXITY OF e(s)

The convexity of e(s) follows from the representation

$$(1.1)\qquad e(s) = \tfrac{1}{2} + \int_{-\pi}^{\pi} e^{-s\phi(\theta)} f(\theta)\,d\theta$$

where ϕ en f are non-negative functions on $(-\pi,\pi)$. This formula will be proved in the present section and the functions ϕ, f will be given explicitly. From this representation once again the monotonicity of e(s) follows. It shows also that e(s) is convex and log convex.

The starting point for the proof of (1.1) is the well-known series for the incomplete gamma function, viz.

$$(1.2)\qquad \gamma(a,x) = e^{-x}\Gamma(a) \sum_{n=0}^{\infty} \frac{x^{a+n}}{\Gamma(a+n+1)} \quad .$$

It converges for all finite $a, x \in \mathbb{C}$ (except for $a = 0,-1,-2,\dots$). It follows from the integral

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt = e^{-x} \int_0^x e^{\tau} (x-\tau)^{a-1} \, d\tau$$

and by expanding the function e^{τ} in powers of τ. Next we use Hankel's integral for the reciprocal gamma function (see WHITTAKER & WATSON ([4,p.245])

(1.3) $$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^t t^{-z} \, dt,$$

where the path of integration starts at $-\infty$ on the real axis, encircles the origin in the positive direction and returns to the starting point.

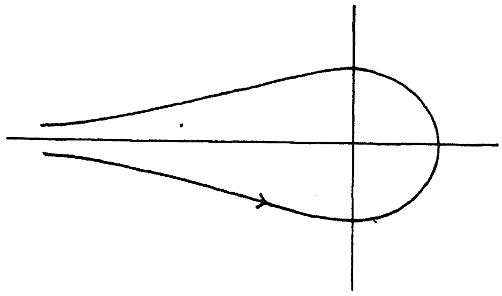

Figure 1. Contour for (1.3) and (1.5).

There is a branch cut along the negative real axis, $t^{-z} = e^{-z \ln t}$ and $\ln t$ is purely real when t is on the positive part of the real axis. When using (1.3) in (1.2) we first choose the contour so that for all t-values on it we have $|t| > x$ (we consider positive x values in (1.2)). Then we have for $a > 0$

(1.4) $$\frac{\gamma(a,x)}{\Gamma(a)} = \lim_{m\to\infty} e^{-x} \sum_{n=0}^{m} \frac{x^{a+n}}{\Gamma(a+n+1)} =$$

$$= \lim_{m\to\infty} \frac{e^{-x}}{2\pi i} \sum_{n=0}^{m} x^{a+n} \int_{-\infty}^{(0^+)} e^t t^{-a-n-1} dt$$

$$= \lim_{m\to\infty} \frac{e^{-x}}{2\pi i} \int_{-\infty}^{(0^+)} e^t (x/t)^a \, t^{-1} \sum_{n=0}^{m} (x/t)^n \, dt$$

$$= \lim_{m\to\infty} \frac{e^{-x}}{2\pi i} \int_{-\infty}^{(0^+)} e^t (x/t)^a t^{-1} \, \frac{1-(x/t)^{m+1}}{1-x/t} \, dt$$

$$= \frac{e^{-x}}{2\pi i} \int_{-\infty}^{(0^+)} e^t (x/t)^a \, \frac{dt}{t-x} \, ,$$

since $|x/t| < 1$ on the contour. The contour cuts the positive real axis in a point t_0 satisfying $0 < x < t_0$. Next we take $t = xv$ and we obtain

$$(1.5) \qquad \frac{\gamma(a,x)}{\Gamma(a)} = \frac{e^{-x}}{2\pi i} \int_{-\infty}^{(0^+)} e^{xv}\, v^{-a} \frac{dv}{v-1} ,$$

where the new contour is as in Fig.1; it cuts the real axis at the right of $v = 1$. Now we return to $e(s)$. We know that

$$e(s) = Q(s+1,s) = 1- \frac{\gamma(s+1,s)}{\Gamma(s+1)} .$$

Hence we take $x = s$, $a = s+1$ and we write (1.5) as

$$(1.6) \qquad 1 - e(s) = \frac{\gamma(s+1,s)}{\Gamma(s+1)} = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^{s(v-1-\ln v)} \frac{dv}{v(v-1)} .$$

We now take a special contour of integration. Cauchy's theorem gives us a lot óf freedom in selecting a new contour. We want to take it through $v = 1$ (the saddle point of the exponential function in the integral of (1.6)). Therefore we introduce $\varepsilon > 0$ and we write

$$(1.7) \qquad 1 - e(s) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^{s(v-1-\ln v)} \frac{dv}{v(v-1+\varepsilon)} .$$

Guided by the methods of asymptotic analysis we take as the new contour (a saddle point contour, on which the phase of $v-1-\ln v$ is a constant)

$$L = \{v = \rho e^{i\theta} \mid -\pi < \theta < \pi,\ \rho = \frac{\theta}{\sin\theta}\}.$$

For $v \in L$ we have

$$v-1-\ln v = \rho e^{i\theta} - 1 - \ln\rho - i\theta =$$

$$= \frac{\theta}{\sin\theta}(\cos\theta + i\sin\theta) - 1 - \ln\frac{\theta}{\sin\theta} - i\theta$$

$$\overset{D}{=} -\phi(\theta).$$

It appears that ϕ is real on $(-\pi,\pi)$ and that ϕ is non-negative there. For $|\theta| < \pi$ we have the expansion

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1} (2n+1)}{n(2n)!} B_{2n} \theta^{2n}$$
$$= \frac{1}{2}\theta^2 + \frac{1}{36}\theta^4 + \frac{1}{405}\theta^6 + \ldots$$

(all coefficients are positive, B_{2n} are Bernoulli numbers).

The integral in (1.7) is with this contour

$$(1.8) \qquad \frac{1}{2\pi i} \int_L e^{-s\phi(\theta)} \frac{dv}{v(v-1+\varepsilon)},$$

where θ and v are related by $v = \theta \operatorname{ctg}\theta + i\theta$. Each point of L corresponds with a single point of the interval θ, and vice versa. It follows that we can integrate with respect to θ; using $\rho(\theta) = \theta/\sin\theta$,

$$\frac{1}{v}\frac{dv}{d\theta} = \frac{d \ln v}{d\theta} = \frac{d(\ln \rho + i\theta)}{d\theta} = \frac{\rho'(\theta)}{\rho(\theta)} + i = 1/\theta - \operatorname{ctg}\theta + i,$$

and (1.7) is written (via (1.8)) as

$$(1.9) \qquad 1 - e(s) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{-s\phi(\theta)} \frac{1/\theta - \operatorname{ctg}\theta + i}{\theta \operatorname{ctg}\theta + i\theta - 1 + \varepsilon} d\theta .$$

When separating real and imaginairy parts of the integrand we write

$$\frac{(\frac{1}{\theta} - \operatorname{ctg}\theta + i)(\theta \operatorname{ctg}\theta - i\theta - 1 + \varepsilon)}{(\theta \operatorname{ctg}\theta - 1 + \varepsilon)^2 + \theta^2} =$$
$$\frac{(\frac{1}{\theta} - \operatorname{ctg}\theta)(\theta \operatorname{ctg}\theta - 1 + \varepsilon) + \theta + i(2\theta \operatorname{ctg}\theta - 2 + \varepsilon)}{(\theta \operatorname{ctg}\theta - 1 + \varepsilon)^2 + \theta^2} .$$

The real part is odd (as a function of θ) and thence it does not contribute in the integral. Hence (1.9) is written as a real integral

$$1 - e(s) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-s\phi(\theta)} (2\theta \operatorname{ctg}\theta - 2 + \varepsilon)}{(\theta \operatorname{ctg}\theta - 1 + \varepsilon)^2 + \theta^2} d\theta$$

and finally we have to pass to the ε-limit. To do so we consider the two functions

$$F_1(\varepsilon) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{-s\phi(\theta)}(\theta\ \mathrm{ctg}\ \theta-1)}{(\theta\ \mathrm{ctg}\ \theta-1+\varepsilon)^2+\theta^2}\, d\theta$$

$$F_2(\varepsilon) = \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-s\phi(\theta)}}{(\theta\ \mathrm{ctg}\ \theta-1+\varepsilon)^2+\theta^2}\, d\theta$$

and it will appear that both limits $\lim_{\varepsilon\to 0} F_i(\varepsilon)$ exist. The integrand of $F_1(\varepsilon)$ is bounded in the θ,ε variables, $\theta \in (-\pi,\pi)$, $\varepsilon \geq 0$, it is not continuous in $\varepsilon = 0$, $\theta = 0$. To see this we write

$$\frac{e^{-s\phi(\theta)}(\theta\ \mathrm{ctg}\ \theta-1)}{(\theta\ \mathrm{ctg}\ \theta-1+\varepsilon)^2+\theta^2} = e^{-s\phi(\theta)}\ \frac{\theta\ \mathrm{ctg}\ \theta-1}{\theta^2}\ \frac{\theta^2}{(\theta\ \mathrm{ctg}\ \theta-1+\varepsilon)^2+\theta^2}$$

and it is clear that only the last fraction matters, of which the boundedness for small θ and ε values is easily verified. It follows that

$$\lim_{\varepsilon\to 0} F_1(\varepsilon) = F_1(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{-s\phi(\theta)}(\theta\ \mathrm{ctg}\ \theta-1)}{(\theta\ \mathrm{ctg}\ \theta-1)^2+\theta^2}\, d\theta .$$

The second function needs more concern. For small θ the integrand may be compared with $1/(\varepsilon^2+\theta^2)$. We write

$$F_2(\varepsilon) = \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-s\phi(\theta)}}{\varepsilon^2+\theta^2}\, d\theta + \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} e^{-s\phi(\theta)}\left\{\frac{1}{(\theta\mathrm{ctg}\theta-1+\varepsilon)^2+\theta^2} - \frac{1}{\theta^2+\varepsilon^2}\right\} d\theta .$$

The second integrand is uniformly bounded and hence the second term vanishes in the limit $\varepsilon \to 0$. The first one becomes by substituting $\theta = \varepsilon t$

$$\frac{1}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} e^{-s\phi(\varepsilon t)}\ \frac{dt}{1+t^2} = \frac{1}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{dt}{1+t^2} + \frac{1}{2\pi}\int_{\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{-s\phi(\varepsilon t)}-1}{1+t^2}\, dt,$$

and by some routine analysis it follows easily that for $\varepsilon \to 0$ we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \frac{1}{2} .$$

Hence

$$\lim_{\varepsilon\to 0} F_2(\varepsilon) = \tfrac{1}{2} \quad \text{and} \quad 1 - e(s) = \tfrac{1}{2} + F_1(0)$$

or

$$e(s) = \tfrac{1}{2} + \int_{-\pi}^{\pi} e^{-s\phi(\theta)} f(\theta)d\theta,$$

$$\phi(\theta) = \ln \frac{\theta}{\sin\theta} + 1 - \theta \operatorname{ctg}\theta$$

$$f(\theta) = \frac{1}{\pi} \frac{1-\theta \operatorname{ctg}\theta}{(1-\theta \operatorname{ctg}\theta)^2+\theta^2} .$$

It is easily seen that $\phi(\theta) \geq 0$, $f(\theta) \geq 0$ on $(-\pi,\pi)$. This proves representation (1.1).

The above analysis follows in some parts earlier work of the author (see [2],[3]) were representations of the incomplete gamma functions as (1.5) were used for obtaining the asymptotic expansion of these functions.

2. ON THE SOLUTION OF $Q_n(x) = \tfrac{1}{2}$

$Q_n(x)$ is a special case of the incomplete gamma function $Q(a,x)$. We have

(2.1) $$Q_n(x) = Q(n+1,x),$$

and we know that

$$Q_n(0) = 1, \quad Q_n(\infty) = 0, \quad \frac{d}{dx} Q_n(x) = -\frac{1}{n!} x^n e^{-x}.$$

Hence, the equation $Q_n(x) = \tfrac{1}{2}$ has a unique solution $x = s_n$.

From (1) and (2.1) it follows that for $n > 0$

$$Q_n(x) = \frac{1}{n!} \int_x^\infty e^{-t} t^n dt = \frac{n}{n!} \int_\lambda^\infty e^{-n(\tau-\ln(\tau n))} d\tau$$

$$= \sqrt{\frac{n}{2\pi}} \frac{1}{\Gamma^*(n)} \int_\lambda^\infty e^{-n(\tau-1-\ln\tau)} d\tau$$

with $\lambda = x/n$, $\Gamma^*(n) = \Gamma(n) n^{-n+\frac{1}{2}} (2\pi)^{-\frac{1}{2}} e^n$. Next we introduce η: $\mathbb{R}^+ \to \mathbb{R}$ by defining

$$(2.2) \qquad \eta(\tau) = (\tau-1)\left[\frac{2(\tau-1-\ln\tau)}{(\tau-1)^2}\right]^{\frac{1}{2}}, \qquad \tau > 0,$$

where the square root is positive for positive values of its argument. It is easily seen that $\eta \in C^\infty(\mathbb{R}^+)$. A different way of defining η is via the equation

$$(2.3) \qquad \tfrac{1}{2}\eta^2(\tau) = \tau - 1 - \ln\tau, \quad \text{sign}(\eta(\tau)) = \text{sign}(\tau-1).$$

With (2.3) we obtain

$$(2.4) \qquad Q_n(x) = \sqrt{\frac{n}{2\pi}}\,\frac{1}{\Gamma^*(n)} \int_{\zeta(\lambda)}^{\infty} e^{-\frac{1}{2}n\eta^2} \frac{d\tau}{d\eta}\, d\eta,$$

where $\zeta(\lambda)$ is defined as η, viz.

$$(2.5) \qquad \tfrac{1}{2}\zeta^2(\lambda) = \lambda - 1 - \ln\lambda, \; \lambda = x/n, \; \text{sign}(\zeta(\lambda)) = \text{sign}(\lambda-1).$$

An expression for $d\tau/d\eta$ follows from (2.3), i.e.,

$$(2.6) \qquad \eta\,\frac{d\eta}{d\tau} = 1 - 1/\tau, \qquad \frac{d\tau}{d\eta} = \frac{\eta\tau}{\tau-1}\,.$$

It is not possible to write this explicitly in terms of η. For small η we have (this follows from (2.6) by substitution)

$$(2.7) \qquad \tau = 1 + \eta + \frac{1}{3}\eta^2 + \frac{1}{36}\eta^3 + \ldots\,.$$

In (2.4) we write the function $d\tau/d\eta$ as $d\tau/d\eta - 1 + 1$, giving

$$(2.8) \qquad Q_n(x) = \frac{1}{\Gamma^*(n)}\left[\tfrac{1}{2}\,\text{erfc}(\zeta\sqrt{n/2}) + R_n(\zeta)\right]$$

where

$$(2.9) \qquad R_n(\zeta) = \sqrt{\frac{n}{2\pi}} \int_{\zeta}^{\infty} e^{-\frac{1}{2}n\eta^2} \left(\frac{d\tau}{d\eta} - 1\right) d\eta.$$

By a partial integration we obtain, writing $f(\eta) = \frac{1}{\eta}(d\tau/d\eta - 1)$,

$$R_n(\zeta) = \frac{-1}{\sqrt{2\pi n}} \int_\zeta^\infty f(\eta)\, de^{-\frac{1}{2}n\eta^2} =$$

$$= \frac{1}{\sqrt{2\pi n}} \{f(\zeta) e^{-\frac{1}{2}n\zeta^2} + \int_\zeta^\infty f'(\eta) e^{-\frac{1}{2}n\eta^2}\, d\eta\},$$

and using (2.3), (2.5) and (2.6) we infer that

$$(2.11) \qquad f(\eta) = \frac{\tau}{\tau-1} - \frac{1}{\eta}, \quad f(\zeta) = \frac{\lambda}{\lambda-1} - \frac{1}{\zeta}, \quad f'(\eta) = \frac{1}{\eta^2} - \frac{\eta\tau}{(\tau-1)^3}.$$

From (2.7) it follows that

$$(2.12) \qquad f(\eta) = \frac{2}{3} + \frac{1}{12}\eta + \ldots, \quad f(0) = \frac{2}{3}, \quad f'(0) = \frac{1}{12}.$$

By using the asymptotic behaviour of η as $\tau \to 0$ or $\tau \to \infty$ (which follows from (2.3)) it can be seen that f and f' are bounded on $\mathbb{R}^+$ and that consequently

$$(2.13) \qquad R_n(\zeta) = \mathcal{O}(n^{-\frac{1}{2}} e^{-\frac{1}{2}n\zeta^2}) \quad \text{as} \quad n \to \infty,$$

uniformly for all $\zeta \in \mathbb{R}$ (or, equivalently, for all $\lambda \in \mathbb{R}^+$). Furthermore we need an asymptotic expansion of $R_n(0)$. From (2.10) and (2.12) we obtain

$$(2.14) \qquad R_n(0) = \frac{1}{\sqrt{2\pi n}} \{\frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} + \mathcal{O}(n^{-1})\}, \qquad n \to \infty.$$

The function $\Gamma^*(n)$ has the well-known expansion (this follows from Stirling's expansion)

$$(2.15) \qquad \Gamma^*(n) = 1 - \frac{1}{12n} + \mathcal{O}(n^{-2}), \qquad n \to \infty.$$

After these preparations we return to the problem of solving $Q_n(x) = \frac{1}{2}$. $Q_n(x)$ is replaced by (2.8) and we solve the new equation for ζ. Since erfc is monotone (erfc$(-\infty) = 2$, erfc$(0) = 1$, erfc$(\infty) = 0$), we conclude (by using (2.13) and (2.15)) that for large n-values the ζ-solution of the equation $Q_n(x) = \frac{1}{2}$ occurs near $\zeta = 0$ (or $\lambda = 1$, or $x = n$). The argument of the error function in (2.8) should be small, hence $\zeta\sqrt{n} \to 0$ as $n \to \infty$. $R_n(\zeta)$ is analytic in a neighbourhood of $\zeta = 0$ and the error function is an entire function of ζ. Hence (upon expanding the functions in ζ in (2.8) in $\zeta = 0$) we have to solve the equation in ζ (where we suppose large values of n)

$$\text{(2.16)} \qquad \tfrac{1}{2} + \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{\zeta^k}{k!}\frac{d^k}{d\zeta^k}\operatorname{erfc}(\zeta\sqrt{n/2}) + \sum_{k=0}^{\infty}\frac{\zeta^k}{k!}\frac{d^k}{d\zeta^k}R_n(\zeta) = \tfrac{1}{2}\Gamma^*(n)$$

where the derivatives are evaluated at $\zeta = 0$. From (2.7) and (2.9) it follows that $R_n'(0) = 0$ and furthermore we have

$$\tfrac{1}{2}\frac{d}{d\zeta}\operatorname{erfc}(\zeta\sqrt{n}/2) = -\sqrt{\frac{n}{2\pi}}\,e^{-\frac{1}{2}n\zeta^2} = -\sqrt{\frac{n}{2\pi}} \text{ (at } \zeta = 0).$$

Hence, from (2.16) we obtain for $\zeta \to 0$ (using the implicit function theorem)

$$-\sqrt{\frac{n}{2\pi}}\,\zeta + R_n(0) + \mathcal{O}(\zeta^2) = \tfrac{1}{2}(\Gamma^*(n)-1)$$

and using (2.14) and (2.15) we obtain

$$\zeta = -\tfrac{1}{2}\sqrt{\frac{2\pi}{n}}\left(-\frac{1}{12n} + \mathcal{O}(n^{-2})\right) + \frac{1}{n}\left(\frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} + \mathcal{O}(n^{-1})\right) + \mathcal{O}(\zeta^2)$$

$$= \frac{2}{3n} + \mathcal{O}(n^{-2}), \qquad n \to \infty,$$

as an approximative solution of (2.16). In terms of λ it reads as $\lambda = 1 + \eta + \mathcal{O}(\eta^2) = 1 + \frac{2}{3n} + \mathcal{O}(n^{-2})$ and in terms of x we have $x = n + \frac{2}{3} + \mathcal{O}(n^{-1})$, as the approximative solution of $Q_n(x) = \frac{1}{2}$. From this we have indeed

$$s_n = n + 2/3 + \mathcal{O}(n^{-1}), \qquad n \to \infty.$$

In the preprint version of [3] the more general inversion problem $Q(a,x) = q$, $0 < q < 1$ is considered. For $a \to \infty$ an asymptotic expansion is constructed in the form

$$x(q,a) \sim a(x_0 + x_1 a^{-1} + x_2 a^{-2} + \ldots)$$

and the coefficients x_0, x_1, x_2, x_3 are given explicitly. For $q = \frac{1}{2}$ we obtained

$$x(\tfrac{1}{2}, a) \sim a\left(1 - \frac{1}{3} a^{-1} + \frac{8}{405} a^{-2} + \ldots\right).$$

With $a = n+1$ this gives for s_n an extra term in the expansion, viz.

$$s_n = n + \frac{2}{3} + \frac{8}{405} n^{-1} + \mathcal{O}(n^{-2}), \qquad n \to \infty.$$

3. ON THE INEQUALITIES FOR a_n AND b_n

(A) $b_n > a_n$ follows from

$$\frac{n^j}{(n+1)\ldots(n+j)} > \frac{(n-1)\ldots(n-j)}{n^j}, \qquad j = 1,\ldots,n-1,$$

which is proved by observing that $n^2 > n^2 - j^2$. The inequality $b_n - a_n < 1$ follows from the results in (B).

(B) It is easily shown that

$$a_n + 2 = n \int_0^\infty e^{-nt}(1+t)^n dt = n!\,n^{-n} e_n(n)$$

$$b_n = n \int_0^1 (1-t)^n e_{n-2}(nt)\,dt, \qquad n \geq 2,$$

where

$$e_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}, \qquad n = 0,1,\ldots\,.$$

We also have, introducing $\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt$, $2 + a_n = n^{-n}\Gamma(n+1,n)$, since $\Gamma(n+1,x) = n!\,e^{-x} e_n(x)$. Let us write

$$b_n = n \int_0^1 (1-t)^n e^{nt} dt - \varepsilon_n$$

$$\varepsilon_n = n \int_0^1 (1-t)^n [e^{nt} - e_{n-2}(nt)]\,dt.$$

Then $b_n = e^n n^{-n} \gamma(n+1,n) - \varepsilon_n$, with

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt = \Gamma(a) - \Gamma(a,x), \qquad a > 0.$$

Hence

$$(3.1) \qquad b_n - a_n = 2 - \varepsilon_n + e^n n^{-n} [\Gamma(n+1) - \Gamma(n+1,n)].$$

It easily follows that

$$\frac{n^n n!}{(2n)!} < \varepsilon_n < \frac{2n^n n!}{(2n)!} .$$

Hence $\varepsilon_n = \mathcal{O}(n^{-k})$, $n \to \infty$, for any k. This follows from Stirling's formula, which can be written as

$$(3.2) \qquad \Gamma(n+1) = n^{n+\frac{1}{2}} \sqrt{2\pi}\, e^{-n+\frac{\theta}{12n}}, \qquad 0 < \theta < 1, \; n > 0.$$

By using more terms in this expression and by using (which will be proved)

$$(3.3) \qquad \Gamma(n+1,n) = n^n e^{-n} [\sqrt{\pi n/2} + \frac{2}{3} + \frac{1}{24} \sqrt{2\pi/n} - \frac{4}{135n} + \mathcal{O}(n^{-3/2})]$$

we obtain for (3.1)

$$b_n - a_n = \frac{2}{3} + \frac{4}{135n} + \mathcal{O}(n^{-2}).$$

To prove (3.3) we write

$$\Gamma(n+1,n) = n^{n+1} e^{-n} \int_1^\infty \exp[-n(t-1-\ell n t)] dt, \qquad n > 0.$$

The transformation $\frac{1}{2} u^2 = t - 1 - \ell n t$, $u \geq 0$, $t \geq 1$, gives

$$(3.4) \qquad \Gamma(n+1,n) = n^{n+1} e^{-n} \int_0^\infty e^{-\frac{1}{2} n u^2} f(u) du,$$

$$f(u) = \frac{dt}{du} = 1 + \frac{2}{3} u + \frac{1}{12} u^2 - \frac{2}{135} u^3 + \mathcal{O}(u^4),$$

where the expansion is valid for $u \to 0$. Standard methods from asymptotic analysis yield (3.3).

Next we show that

;) $$\Gamma(n+1,n) > e^{-n}n^{n}(\sqrt{\pi n/2} + \tfrac{2}{3}), \qquad n > 0.$$

e this we write

$$\Gamma(n+1,n) = e^{-n}n^{n}[\sqrt{\pi n/2} + n\int_{n}^{\infty}\{f(u) - 1\}e^{-\frac{1}{2}nu^{2}}]du$$

$$= e^{-n}n^{n}[\sqrt{n\pi/2} + \tfrac{2}{3} + \int_{0}^{\infty} e^{-\frac{1}{2}nu^{2}}\, g(u)du]$$

partial integration), with $g(u) = \frac{d}{du}\frac{f(u)-1}{u}$. We prove that $g(u) > 0$ 1 > 0.

Since $\frac{dt}{du} = ut/(t-1)$, we easily derive

$$g(u) = \frac{(t-1)^{3}-tu^{3}}{u^{2}(t-1)^{3}}.$$

, $g(u) > 0$ if $(t-1)^{2} > t^{2/3}u^{2}$, or $h(t) > 0$, $h(t) = (t-1)^{2}t^{-2/3}-2(t-1-\ell nt)$, 1. We know that $h(0) = 0$ and it easily follows that $h'(t) > 0$. Hence > 0 on $t > 1$, which implies $g(u) > 0$ on $u > 0$.

This proves (3.5). Next we use (3.2) and (3.5) in (3.1) to obtain

5) $$b_n - a_n < \tfrac{2}{3} + \sqrt{2\pi n}[\exp(\tfrac{1}{12n})-1].$$

expression in n is monotone decreasing for $n > 0$. For $n = 2$ it equals 508... . Hence for $n > 1$ we have

7) $$b_n - a_n < \tfrac{2}{3} + 0.1508... = 0.817... .$$

This inequality can be sharpened by using exact values for $b_n - a_n$ small values of n. Inspection of the first hundred $b_n - a_n$ shows that ximum occurs at $n = 21$ with computed value

$$b_{21} - a_{21} = 0.66907...$$

while our bound gives

$$b_{21} - a_{21} < 0.671151.$$

For n = 100 we computed

$$b_{100} - a_{100} = 0.66725869...$$

while our bound is

$$b_{100} - a_{100} < 0.687564.$$

From these results we conclude that (3.7) can be replaced by

$$b_n - a_n < 0.687564, \qquad n > 1.$$

REFERENCES

[1] LUNE VAN DE, J., *A note on Euler's (incomplete) gamma function,* Report ZN 61 (1975), Mathematisch Centrum, Amsterdam.

[2] TEMME, N.M., *Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function,* Math. Comp. 29 (1975), p.1109-1114.

[3] TEMME, N.M., *The asymptotic expansions of the incomplete gamma functions,* Report TW 165 (1977), Mathematisch Centrum, Amsterdam (preprint of the paper in SIAM J. Math. Analysis 10 (1979), p. 757-766).

[4] WHITTAKER, E.T. & G.N. WATSON, *A course in modern analysis,* Cambridge University Press (1927).