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HETEROCLINIC WAVES OF THE FITZHUGH-NAGUMO EQUATIONS

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Heteroclinic waves of the Fitz-Hugh-Nagumo equations *)

by

J.P. Pauwelussen

ABSTRACT

In this paper we study the existence and analyze the stability of heteroclinic wave solutions of a piecewise linear version of the Fitz-Hugh-Nagumo system of reaction-diffusion equations.

KEY WORDS & PHRASES: *FitzHugh-Nagumo equations, heteroclinic waves, stability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In the mathematical theory of nerve impulse propagation, the system of reaction diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) - w,$$

(1.1)

$$\frac{\partial w}{\partial t} = \sigma u - \gamma w, \quad x \in \mathbb{R}, \ t > 0,$$

referred to as the FitzHugh-Nagumo equations has attracted a great deal of attention. Here σ and γ are positive parameters and f is usually chosen to be the cubic $f(u) = u(1-u)(u-a)$ with $a \in (0,\frac{1}{2})$.

The equations (1.1) were first suggested by FitzHugh [14] and Nagumo et.al. [17], for $\gamma = 0$, as a simplification of the Hodgkin-Huxley equations [15]. These authors were interested in homoclinic travelling wave solutions of (1.1), i.e. bounded solutions of the form $(u(x,t),w(x,t)) = (U(x+ct,W(x+ct))$ for some $c \in \mathbb{R}$ with $(U(z),W(z)) \to (0,0)$ as $z \to \pm\infty$.

If

(1.2) $$\frac{\sigma}{\gamma} > \frac{(1-a)^2}{4}$$

then the point (0,0) in the (u,w)-plane is the only equilibrium state. In this case travelling wave solutions are of a very special kind. They consist of one or more repeated so-called pulses of which one specific example is shown in fig. 1.1 and which are all very much alike.

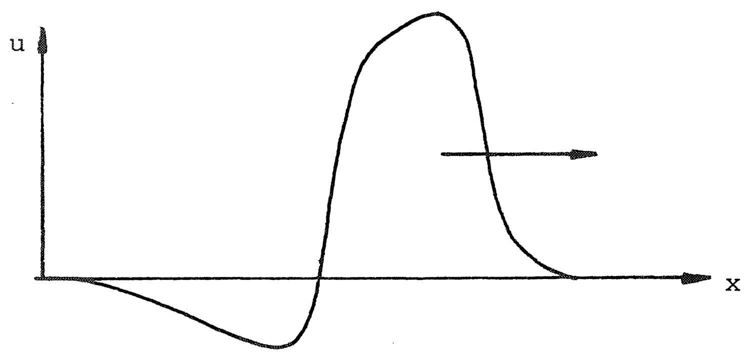


fig. 1.1: A single pulse.

In fact the results of Feroe [11],[12] and Evans, Fenichel, Feroe [9] indicate that a wide class of such wave trains is possible such as homoclinic waves which consist of exactly n of these pulses for any $n \in \mathbb{N}$. Moreover a large variety of infinite wave-trains including periodic wave solutions, is possible.

If the restriction (1.2) is dropped, other kinds of wave solutions may arise. For example, if we take $\sigma = 0$, the only bounded solution w of $(1.1)^2$ is $w \equiv 0$ and the system (1.1) reduces to

$$(1.3) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u).$$

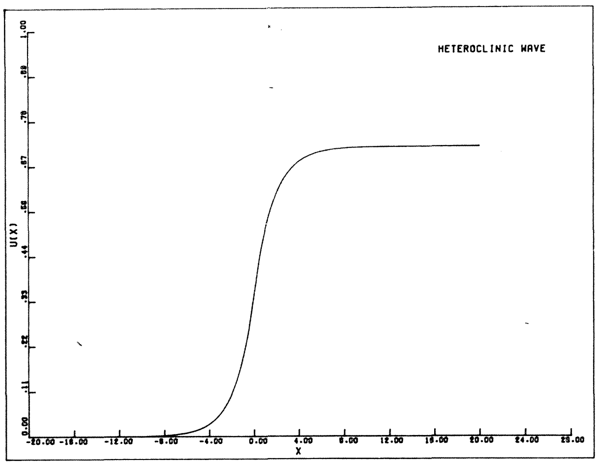
This equation, which also appears in population genetics [2] has two stable equilibrium states $u = 0$ and $u = 1$. It is well known that (1.3) has wave solutions $u(x,t) = U(z)$, $z = x+ct+x_0$ for $x_0 \in \mathbb{R}$ and a unique speed c such that $U(-\infty) = 0$ and $U(+\infty) = 1$. These waves are stable in an appropriate sense, see Fife & Mcleod [13]. Moreover c is positive and except for the freedom of translation, U is unique and strictly increasing, see [13] and Aronson & Weinberger [2]. In view of this result it is reasonable to expect for small $\sigma > 0$ the existence of such heteroclinic waves connecting both the two stable equilibrium states present in this case. Indeed, by a singular perturbation approach, Carpenter [3] proved this to be the case if both σ and γ are small parameters and of the same order.

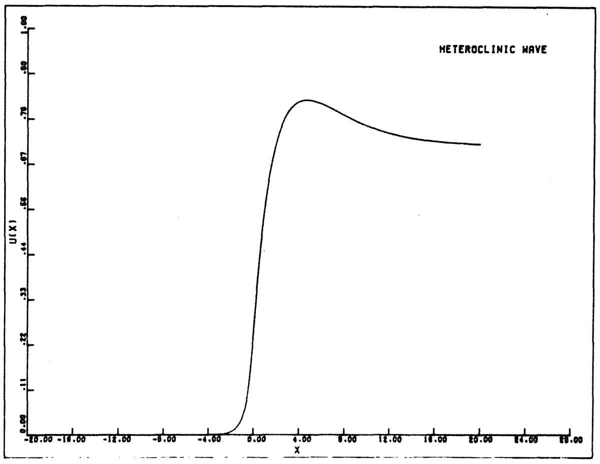
In their paper "Bistable transmission lines" [18], Nagumo et.al. introduced an interesting electrical analogue of the system (1.1). They considered bistable lines which are constructed by cascading many two terminal circuits of a specific type through interstage coupling resistance. The occurrence in (1.1) of the nonlinear term f(u) is due to the presence of tunnel diodes in the basic circuit. Nagumo et.al. distinguished between two types of transmission lines corresponding to equation (1.1) and (1.3) respectively, and they investigated the existence of heteroclinic waves, numerically as well as experimentally.

In this paper we shall follow Mckean [16] and replace f by a piecewise linear function

(1.4) $\qquad f(u) = F[-u + H(u-a)]$

where H is the Heavyside step function. With this choice of f we can actually calculate the wave solutions of (1.1) explicitly as sums of exponentials in $x + ct$ for some $c \in \mathbb{R}$ for $u < a$ and for $u > a$ and which are matched at the points where $u = a$. We shall specialize to those wave solutions for which u takes on the value a only once. Then we find three typical wave forms of which the u-components are shown in fig 1.2. They all satisfy boundary conditions of the kind $u(-\infty) = 0$, $u(+\infty) = \hat{u}$ where $\hat{u}$ will be defined in the next section. The u-component of the wave solution of the first type is strictly decreasing and we shall denote this solution by *wave of type* A. For the second type u grows from $u = 0$ at $x = -\infty$ to a maximum value and then decreases towards $u = \hat{u}$ at $x = +\infty$. We shall refer to this solution as *wave of type* B. Finally u may oscillate round $u = \hat{u}$, approaching $\hat{u}$ at $x = \infty$ and this solution will be denoted by *wave of type* C.





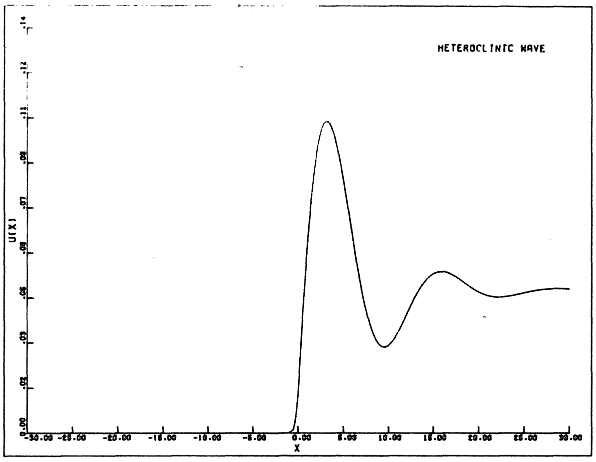


fig. 1.2.: Heteroclinic waves of type A,B and C.

For F = 1, the expression (1.4) for f has been used before by Rinzel & Keller [19] who where interested in homoclinic solitary impulse solutions and periodic solutions. Their results indicate the existence of two single pulse solutions, a fast one and a slow one, and they proved the instability of the slow pulse. Feroe [10], exploring a method due to Evans [7] demonstrated the stability of the fast pulse in an appropriate sense. In this paper we shall derive similar results for the heteroclinic waves. In Section 2 we shall study the existence of waves of the types shown in fig. 1.2. Among other things we show that, in general, waves of type A have a smaller speed than the waves of type B if they exist, with waves of type C having intermediate speed. Also it appears that for small σ/γ^2, all heteroclinic waves are stable in an appropriate sense while if we increase σ/γ^2, unstable waves arise. This is a typical result of Section 3 where we deal with the question of stability.

Finally, to avoid confusion, we distinguish between *theorems* and *assertions* where the latter are based on numerical evidence whereas the validity of the former can be proved rigorously by analytical means.

2. EXISTENCE

With f given by (1.4) and $\sigma/\gamma < (1-a)/a$ the system (1.1) has two equilibrium points $(u,w) = (0,0)$ and $(u,w) = (\hat{u},\hat{w}) \equiv (\gamma F/(\gamma F+\sigma),\ \sigma F/(\gamma F+\sigma))$, see fig. 2.1.

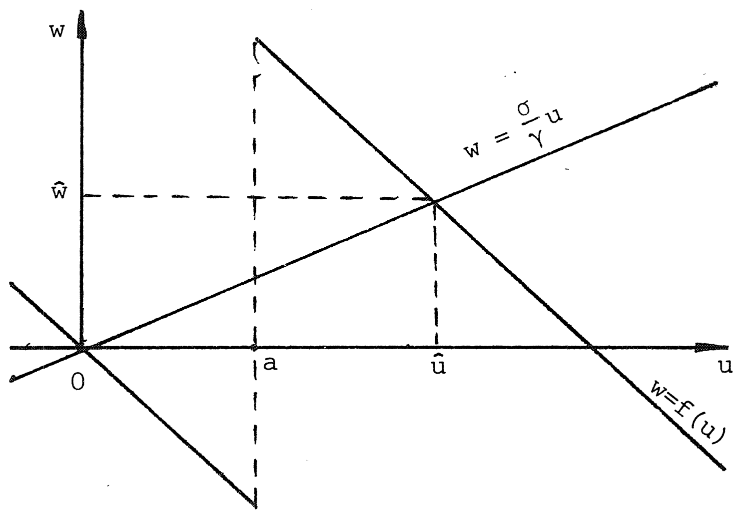


fig. 2.1.

We shall seek travelling wave solutions $(U(z),W(z))$, $z = x + ct$ with

$$(2.1)\qquad \begin{aligned} U(-\infty) &= W(-\infty) = 0, \\ U(+\infty) &= \frac{\gamma}{\sigma} W(+\infty) = \hat{u} \end{aligned}$$

and

$$(2.2)\qquad U(0) = a.$$

Then U satisfies the differential equation

$$(2.3)\qquad \begin{aligned} &U''' - U''(c - \frac{\gamma}{c}) - U'(F+\gamma) - U\frac{\sigma+F\gamma}{c} = \\ &= \begin{cases} 0 & ; \quad U < a, \\ -\frac{F\gamma}{c} & ; \quad U > a. \end{cases} \end{aligned}$$

By assumption $U(z) - a$ vanishes only at one point, i.e. at $z = 0$ where as a result of the discontinuity of f, the following *jump conditions* must be satisfied

$$(2.4)\qquad \begin{aligned} U(0+) - U(0-) &= 0, \\ U'(0+) - U'(0-) &= 0, \\ U''(0+) - U''(0-) &= -F. \end{aligned}$$

Solutions of equation (2.3) for $U < a$ are linear combinations of exponentials $\exp(\alpha_i z)$, $i = 1,2,3$ where α_i is a zero of the polynomial

$$(2.5)\qquad P(\alpha) = \alpha^3 - \alpha^2(c - \frac{\gamma}{c}) - \alpha(F+\gamma) - \frac{1}{c}(\sigma+F\gamma).$$

The general solution of (2.3) for $U > a$ is found by subtracting from $\hat{U}$ the general solution of (2.3) for $U < a$.

Since $\alpha_1\alpha_2\alpha_3 = (\sigma+F\gamma)/c>0$, $P(\alpha)$ has one positive real zero, α_1 say and since $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = -(F+\gamma) < 0$, two with negative real part. By the requirement of boundedness at $z = \pm\infty$, a solution U of (2.1)-(2.3) has the following expression

$$(2.6)\qquad \begin{aligned} U(z) &= ae^{\alpha_1 z}, \qquad z < 0,\\ U(z) &= A_2 e^{\alpha_2 z} + A_3 e^{\alpha_3 z} + \frac{\gamma F}{\sigma+\gamma F}, \qquad z > 0. \end{aligned}$$

Substitution of (2.6) into the jump conditions (2.4) yields

$$(2.7)\qquad \begin{aligned} a &= A_2 + A_3 + \frac{\gamma F}{\sigma+\gamma F},\\ \alpha_1 a &= \alpha_2 A_2 + \alpha_3 A_3,\\ \alpha_1^2 a - F &= \alpha_2^2 A_2 + \alpha_3^2 A_3, \end{aligned}$$

leading to

$$(2.8)\qquad \begin{aligned} A_2 &= \frac{F}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}\left[1 + \frac{\alpha_1\alpha_3\gamma}{\sigma+\gamma F}\right],\\ A_3 &= \frac{-F}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}\left[1 + \frac{\alpha_1\alpha_2\gamma}{\sigma+\gamma F}\right],\\ a &= \frac{F}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\left[1 + \frac{\alpha_2\alpha_3\gamma}{\sigma+\gamma F}\right]. \end{aligned}$$

The "zero" a of f depends on c through the numbers α_i, i = 1,2,3. Using (2.8)[3] it is much easier to calculate a as a function of c than the other way around and the result is shown in the figures 2.3a - 2.3f for several values of σ,γ and where F = .25, .5,1,2,4 (from low to high). The dotted lines correspond to the waves of type C while the solid lines correspond to the A-waves and B-waves in the way depicted in fig. 2.2 below.

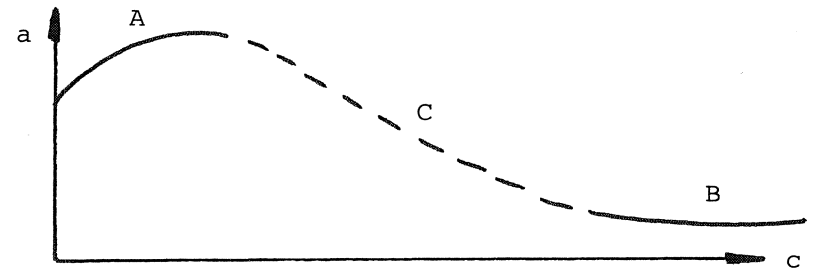


fig. 2.2-

Thus A-waves are found for smaller wave speeds than C-waves and finally B-waves.

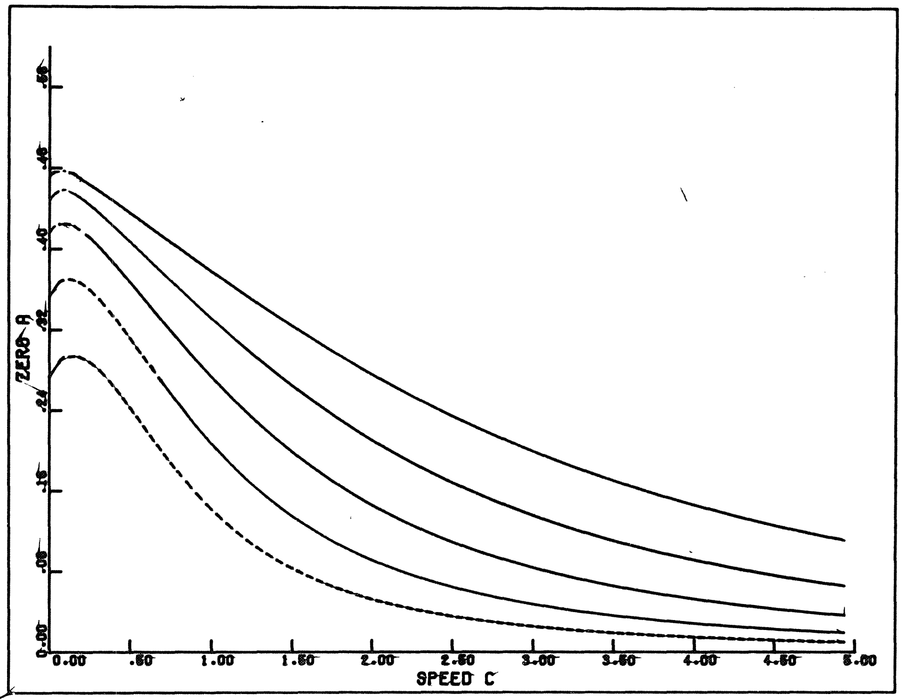


fig 2.3a.: a vs.c for σ = 0.02 and γ = 0.1.

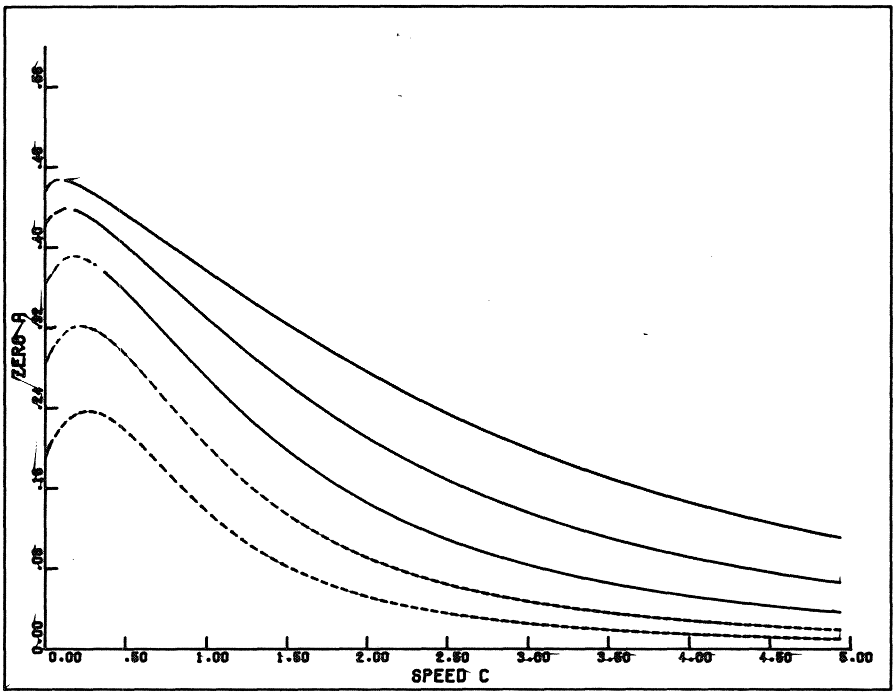


fig 2.3b.: a vs.c for σ = 0.04 and γ = 0.1.

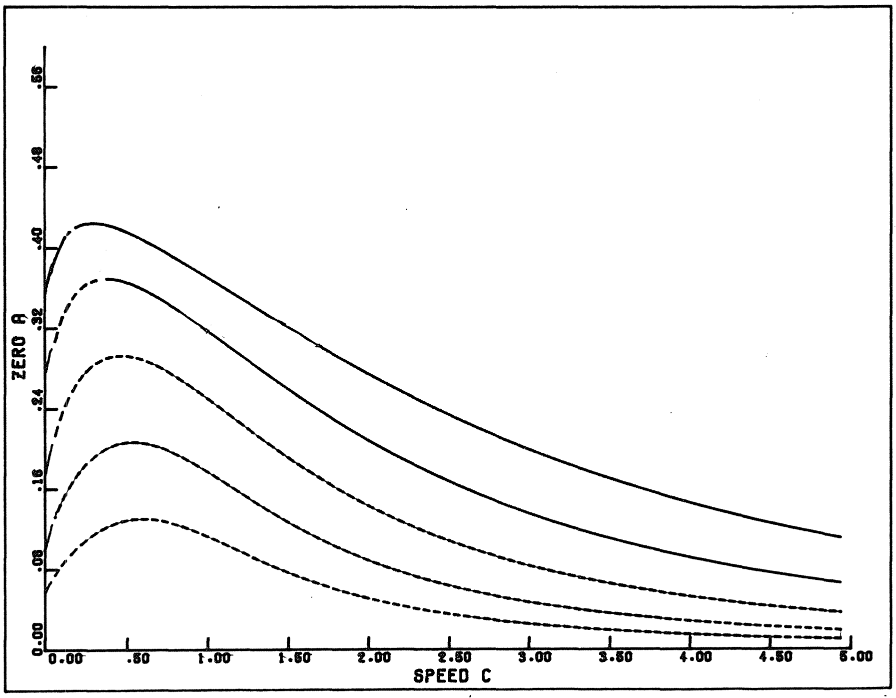


fig. 2.3c.: a vs.c for σ = 0.2 and γ = 0.1.

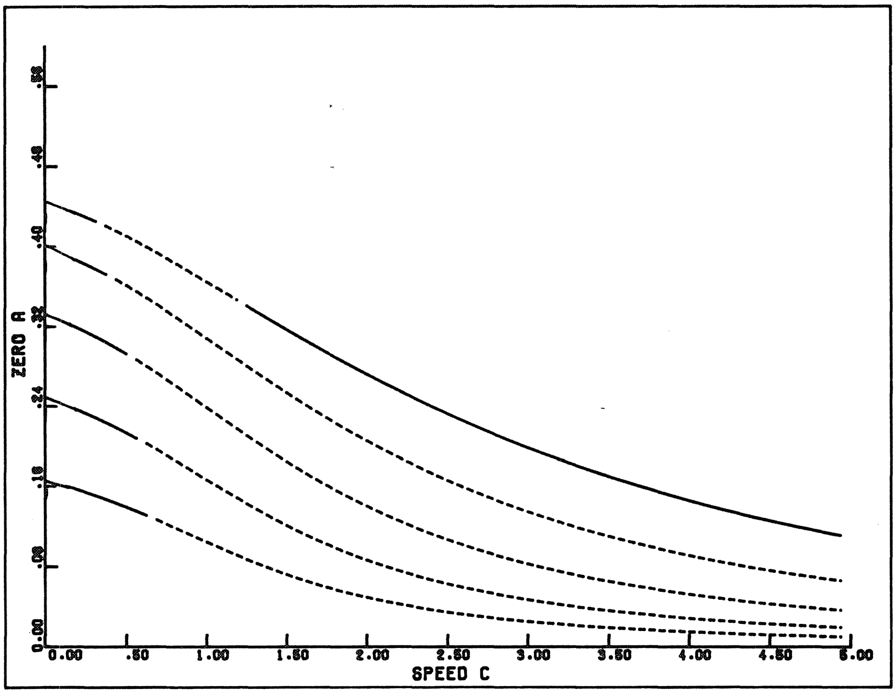


fig 2.3d.: a vs.c for σ = 0.5 and γ = 1.

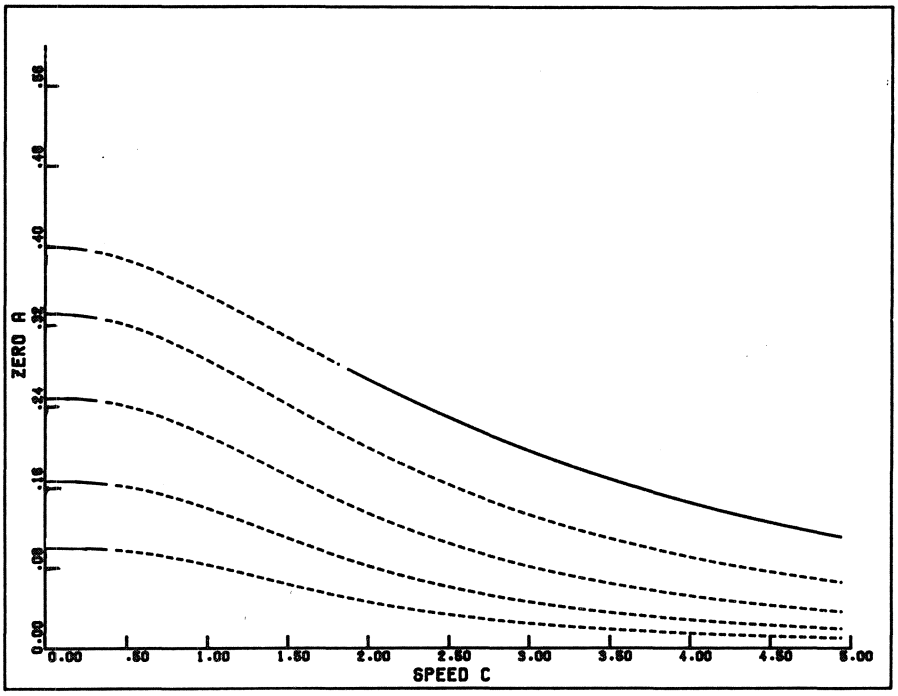


fig. 2.3e.: a vs.c for σ = 1 and γ = 1.

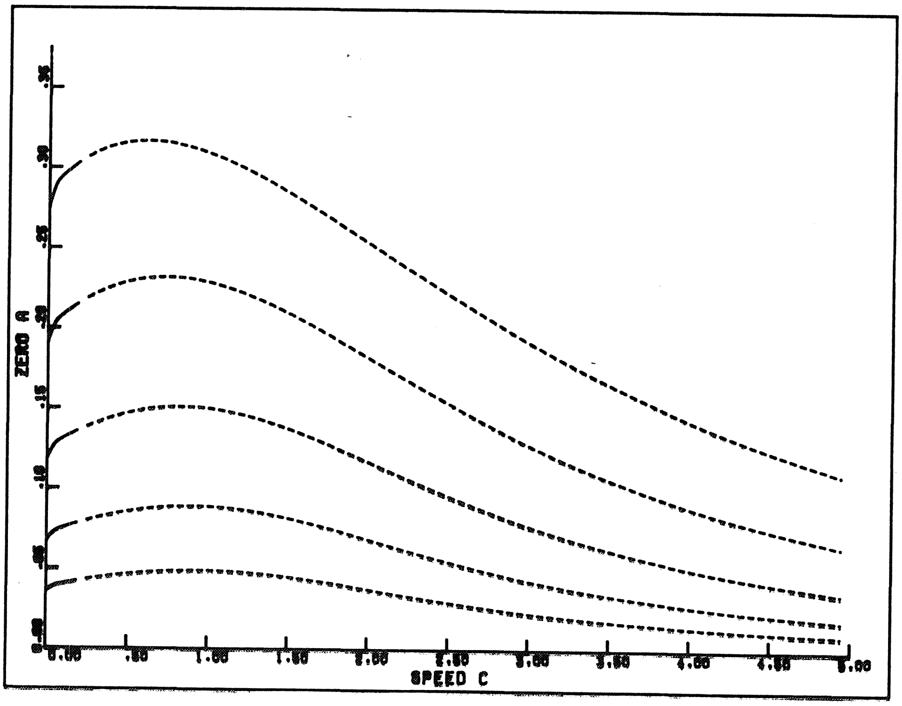


fig. 2.3f.: a vs.c for σ = 3 and γ = 1.

As indicated in the figures 2.3 the occurrence of the several types of waves depends strongly on the choice of the parameters σ,γ and F. Let us first consider the a vs.c-curves for small $c > 0$. Substitution of an expansion for α in powers of c in (2.5) leads to

$$(2.9)\qquad \begin{aligned} \alpha_1 &= \sqrt{F + \frac{\sigma}{\gamma}} + \tfrac{1}{2}c\left(1 - \frac{\sigma}{\gamma^2}\right) + O(c^2), \\ \alpha_2 &= -\sqrt{F + \frac{\sigma}{\gamma}} + \tfrac{1}{2}c\left(1 - \frac{\sigma}{\gamma^2}\right) + O(c^2), \\ \alpha_3 &= -\frac{\gamma}{c} + \frac{\sigma c}{\gamma^2} + O(c^2). \end{aligned}$$

If we insert this result into (2.8)[3] we find

$$(2.10)\qquad a = \frac{F}{2(F + \frac{\sigma}{\gamma})} + \frac{cF}{4\gamma^2 (F + \frac{\sigma}{\gamma})^{3/2}}[\sigma - \gamma^2] + O(c^2).$$

We formulate this result in a first Theorem.

<u>THEOREM 2.1.</u> *For small* $c > 0$ *the function* $a(c)$ *increases with* c *if* $\sigma > \gamma^2$ *whereas it decreases if* $\sigma < \gamma^2$. *In addition* $a(0) = \dfrac{F}{2(F + \frac{\sigma}{\gamma})}$.

This theorem is clearly demonstrated in fig. 2.3d - 2.3f. Moreover these pictures indicate the following behaviour of $a = a(c)$.

<u>ASSERTION 2.1.</u> If $\sigma \le \gamma^2$, the function $a(c)$ is strictly decreasing for all $c > 0$ whereas for $\sigma > \gamma^2$ it has a unique maximum at some point $c = c^* > 0$.

For $c = 0$ $U(z)$ satisfies the equation

$$(2.11)\qquad U'' + f(U) - \frac{\sigma}{\gamma}U = 0.$$

Since $a(0) = \dfrac{F}{2(F + \frac{\sigma}{\gamma})}$ is exactly the value of a for which

$$\int_0^{\hat{u}} [f(v) - \frac{\sigma}{\gamma}v]\,dv = 0,$$

there exists an increasing solution of (2.11) which satisfies the boundary condition (2.1) (see [13]). Hence the wave is of type A. This result can be extended to the following theorem

THEOREM 2.2. *For small positive* c *any heteroclinic wave is necessarily of type* A.

PROOF. The discriminant $\mathcal{D}$ of a third order polynomial

$$P(x) = x^3 + a_2 x^2 + a_1 x + a_0$$

with real coefficients $a_0, a_1 . a_2$ is defined by

$$\mathcal{D} = Q^3 + R^2$$

where

$$Q = \frac{1}{3}a_1 - \frac{1}{9}a_2^2,$$
$$R = \frac{1}{6}(a_1 a_2 - 3a_0) - \frac{1}{27}a_2^3,$$

see Abramowitz & Stegun [1]. If $\mathcal{D} < 0$ then P(x) has three real zeroes whereas for positive $\mathcal{D}$ two of the zeroes are complex conjugates. For the polynomial given in (2.5) we find for small $c > 0$ that

$$\mathcal{D} = -\frac{\gamma^3}{27c^4}[F\gamma + \sigma] + 0(\frac{1}{c^2}).$$

Hence $\alpha_2, \alpha_3 \in \mathbb{R}$. Moreover by substitution of the expressions (2.9) into (2.8) it follows that A_2 and A_3 are of the same sign for $c \downarrow 0$. Consequently U(z) given by (2.6) cannot have a maximum for $z = 0$ which proves this Theorem.

Next we turn to the situation for large c. Then we find for α_1 the asymptotic expansion

$$\alpha_1 = c + \frac{F}{c} + 0(\frac{1}{c^2}) \qquad (2.12)$$

and for α_2 and α_3 the estimate

$$(2.13)\qquad \begin{pmatrix}\alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\frac{1}{c} + O(\frac{1}{c^2})$$

where $\mu_i, i = 1,2$ is a solution of

$$(2.14)\qquad \mu^2 + (F + \gamma)\mu + (\sigma + F\gamma) = 0.$$

Substitution of (2.12) and (2.13) into (2.8) yields that A_2 and A_3 are of different sign for large c and real α_2 and α_3. As a result we state the following Theorem.

THEOREM 2.3. *For large* c *the heteroclinic waves are of the following type*

(i) B *if* $F \le \gamma - 2\sqrt{\sigma}$ *or* $F \ge \gamma + 2\sqrt{\sigma}$,

(ii) C *if* $\gamma - 2\sqrt{\sigma} < F < \gamma + 2\sqrt{\sigma}$.

PROOF. The discriminant $\tilde{\mathcal{D}}$ of (2.14) is

$$\tilde{\mathcal{D}} = (F - \gamma)^2 - 4\sigma$$

which yields part (ii). In case that $\sigma \le \frac{(F-\gamma)^2}{4}$ the waves have a maximum for $z > 0$ because of the fact that $A_2A_3 < 0$ for large c. Consequently, these waves are of type B.

In fig. 2.3a we had chosen $\sigma = 0.02$ and $\gamma = 0.1$ whence for large c the waves are of type B if $F \ge 0.383$. Therefore except for the lowest one the curves in the picture are solid lines for large c. In fig. 2.3b the waves are of type B if $F \ge 0.5$ and therefore again the lowest curve will remain a dotted line for all $c > 0$. The critical value for F in fig. 2.3c is 0.994.

Finally we note that by substitution of (2.12) and (2.13) into the expression $(2.8)^3$ for a it follows that $a(c) \to 0$ if $c \to \infty$ as indicated in fig. 2.3a-f.

This observation together with theorem 2.1 yield the following result.

THEOREM 2.4. *Let* $\sigma > \gamma^2$. *Then there exists an* $\varepsilon > 0$ *such that for all* $a \in (a(0), a(0) + \varepsilon)$ *there exist two heteroclinic wave solutions to the system (1.1) which satisfy (2.1) and (2.2).*

3. STABILITY

We introduce in (1.1) the moving coordinates (z,t), z = x + ct. Then (1.1) transforms into

$$(3.1)\qquad \begin{aligned} u_t &= u_{zz} - cu_z + f(u) - w, \\ w_t &= -cw_z + \sigma u - \gamma w. \end{aligned}$$

The wave solution (U,W) in a stationary (i.e. time-independent) solution of (3.1). The formal linearization of the system (3.1) about (U,W) is given by

$$(3.2)\qquad \begin{aligned} u_t &= u_{zz} - cu_z + F[-1 + \frac{\delta(z)}{U'(0)}]\, u - w, \\ w_t &= -cw_z + \sigma u - \gamma w, \end{aligned}$$

where δ is the Dirac delta function and where we have used that $\delta(z) = U'(0)\delta(U(z)-a)$.

In a series of papers [4] - [7] Evans shows for smooth f that the stability of a homoclinic travelling wave, defined in a certain sense is ensured if the linearization (3.2) about this solution of (3.1) has no solution of the form

$$(3.3)\qquad (u(z,t),w(z,t)) = e^{\lambda t}(u_0(z),w_0(z))$$

with (u_0,w_0) bounded and for Re $\lambda > 0$.

If a solution (u(z,t),w(z,t)) of the form (3.3) exists for Re $\lambda > 0$ then the wave solution (U,W) is unstable. A review of these papers can be found in [8]. Exploiting Evans' methods but for f given by (1.4), Feroe [10] determined the stability character of the homoclinic waves. The object of this section is to use techniques similar to those of Evans and Feroe to study the stability of the heteroclinic solutions. In correspondence with Evans' results we say that a wave solution is *stable* if the system (3.2) has no solution of the form (3.3) with Re $\lambda > 0$ and (u_0, w_0) bounded, whereas a wave solution is called *unstable* if such a solution of (3.2) does exist. In the latter case we shall call λ an *eigenvalue*.

REMARK. Essential for the work of Evans is the value of f' near the resting points of which there was only one in his case. In the present paper, with f given by (1.4) there are two resting points. Because f' has the same value at both points, Evans' results are also applicable to these heteroclinic waves.

For the system (3.2) to have a solution of the form (3.3), (u_0, u_0', w_0) must satisfy the equation

$$(3.4) \qquad \frac{d}{dz}\begin{pmatrix} u_0 \\ u_0' \\ w_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \lambda+F & c & 1 \\ \frac{\sigma}{c} & 0 & -\frac{\lambda+\gamma}{c} \end{pmatrix}\begin{pmatrix} u_0 \\ u_0' \\ w_0 \end{pmatrix}$$

for $z \neq 0$, together with the jumpconditions

$$(3.5) \qquad \begin{aligned} u_0(0+) &= u_0(0-), \\ u_0'(0+) &= u_0'(0-) = -\frac{F}{\alpha_1 a}\, u_0(0), \\ w_0(0+) &= w_0(0-). \end{aligned}$$

The condition $(3.5)^2$ is obtained if we integrate $(u_0'' - cu_0')\exp(-cz)$ over an interval $(-\varepsilon_1, \varepsilon_2)$, $\varepsilon_i > 0$, and let $\varepsilon_i \downarrow 0$ where we use equation (3.4). Since the equation (3.4) is homogeneous we might as well put $u_0(0) = 1$. Observe that for $\lambda = 0$, (3.4) and (3.5) are satisfied by (U', U'', W').

Solutions of equation (3.4) are linear combinations of the exponentials $X_i \exp(\beta_i z)$ where β_i, $i = 1,2,3$ are the zeroes of the third order polynomial

$$(3.6) \qquad \begin{aligned} P(\beta;\lambda) \equiv \beta^3 &+ \frac{1}{c}(\gamma+\lambda-c^2)\beta^2 - (F+\gamma+2\lambda)\beta \\ &- \frac{1}{c}(\lambda^2+\lambda(F+\gamma)+\sigma+F\gamma), \end{aligned}$$

and X_i is given by

$$(3.7) \qquad X_i = \begin{pmatrix} \beta_i \\ 1 \\ \dfrac{\sigma}{\beta_i c+\lambda+\gamma} \end{pmatrix}.$$

It was shown in [5] that for Re $\lambda \geq 0$ $P(\beta;\lambda)$ has exactly one zero with positive real part to be denoted by β_1, and two with negative real part.

The remaining part of this section will have the following structure. First we search for a solution of (3.4) which is bounded on $(-\infty,0)$. Then we extend this solution over the point $z = 0$ and investigate for what λ it remains bounded in $(0,\infty)$. Thus we search for solutions of which the u_0-component is of the following type

$$(3.8) \qquad u_0(z) = \begin{cases} e^{\beta_1 z}, & z < 0 \\ B_1 e^{\beta_1 z} + B_2 e^{\beta_2 z} + B_3 e^{\beta_3 z}, & z > 0 \end{cases}$$

and we look for zeroes of $B_1 = B_1(\lambda)$. Substitution of (u_0,u_0',w_0) into the jump conditions (3.5) leads after a great deal of calculations to

$$(3.9) \qquad B_1(\lambda) = 1 - \frac{F(c-\beta_2-\beta_3)}{\alpha_1 a(\beta_1-\beta_2)(\beta_1-\beta_3)}.$$

In [7] Evans introduces a function $D(\lambda)$ which in our terminology is given by

$$(3.10) \qquad D(\lambda) = [2\beta_1 - c + \frac{\sigma c}{(\gamma+\lambda+\beta_1 c)^2}]\, B_1(\lambda)$$

and which satisfies the following properties:

D_1. $D(\lambda)$ is analytic for Re $\lambda \geq 0$.

D_2. $D(\lambda) = 0 \Longleftrightarrow B_1(\lambda) = 0$.

D_3. $\operatorname{Re} D(\lambda) \to \infty$ as $|\lambda| \to \infty$, $\operatorname{Re}\lambda \geq 0$.

D_4. $D(\bar{\lambda}) = \overline{D(\lambda)}$.

Since $\lambda = 0$ is an eigenvalue we have $D(0) = 0$.

From the equation

$$P(\beta;\lambda) = 0,$$

the following expansions for β_1, β_2 and β_3 follow

$$\beta_1 = \sqrt{\lambda} + \tfrac{1}{2}c - \frac{F + \frac{1}{4}c^2}{2\sqrt{\lambda}} + o(1),$$

$$(3.11) \qquad \beta_2 = -\sqrt{\lambda} + \tfrac{1}{2}c + \frac{F + \frac{1}{4}c^2}{2\sqrt{\lambda}} + o(1),$$

$$\beta_3 = -\frac{\lambda}{c} - \frac{\gamma}{c} + o(|\lambda|^{-1}), \quad (|\lambda| \to \infty, \operatorname{Re}\lambda \geq 0)$$

where we choose $|\arg \lambda| < \pi$. Substitution of (3.11) into (3.9) and (3.10) yields

$$(3.12) \qquad D(\lambda) = 2\sqrt{\lambda} - \frac{F}{\alpha_1 a} + o(|\lambda|^{-\frac{1}{2}})$$

from which D_3 immediately follows. In addition we find

$$D_5. \quad \frac{\operatorname{Im} D(\lambda)}{|\lambda|^{\frac{1}{2}}} = o(1), \qquad (\operatorname{Re}\lambda \geq 0).$$

We introduce a new function $E(\lambda)$ by

$$(3.13) \qquad E(\lambda) = \frac{D(\lambda)}{\lambda}(\lambda+1)$$

and we shall search for its zeroes in the open right half plane $\operatorname{Re}\lambda > 0$. This function is analytic for $\operatorname{Re}\lambda \geq 0$, the zeroes of $E(\lambda)$ and $D(\lambda)$ coincide for $\lambda \neq 0$, we have $E(\bar{\lambda}) = \overline{E(\lambda)}$ and it follows by D_5 that $\operatorname{Re} E(\lambda) \to \infty$ as $|\lambda| \to \infty$, $\operatorname{Re}\lambda \geq 0$.

We plotted the curve $\{E(i\mu) | \mu \in \mathbb{R}\}$ for several values of F, c, σ and γ.

Three examples are shown in fig. 3.1a - fig. 3.1c. It appears that in general these curves do not intersect the origin (and thus $\lambda = 0$ is a zero of $D(\lambda)$ with multiplicity one). The number of zeroes of $D(\lambda)$ with Re $\lambda > 0$ is now given by the winding number around the origin of the image under E of the imaginary axis. The curves in fig. 3.1a and fig. 3.1b wind once which reflects the existence of an unstable mode.

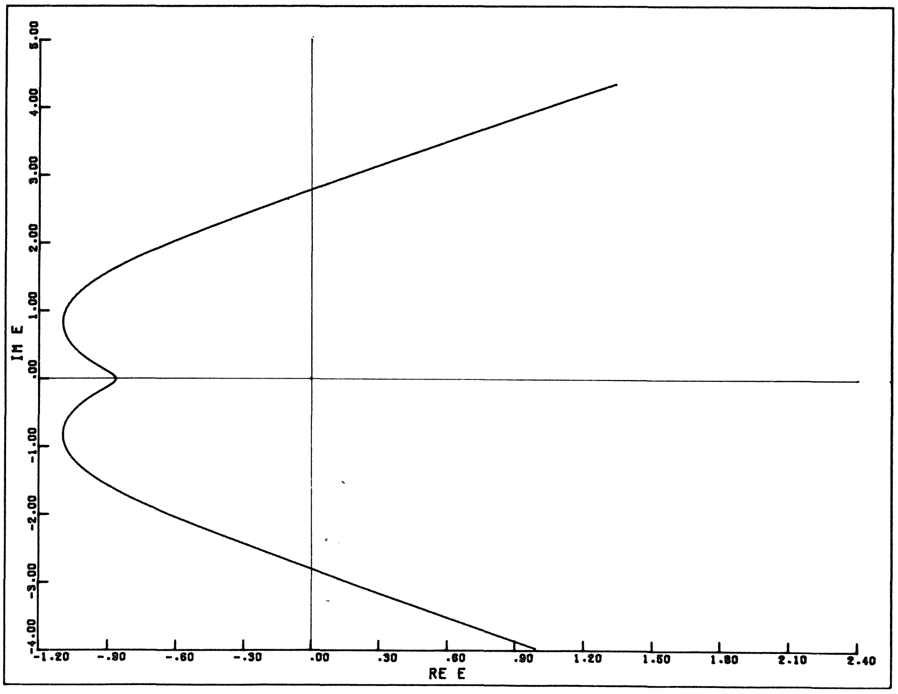


fig. 3.1a: Plot of $E(i\mu)$, $\mu \in \mathbb{R}$ for $F = 1$, $\sigma = 3$, $\gamma = 1$ and $c = 0.125$.

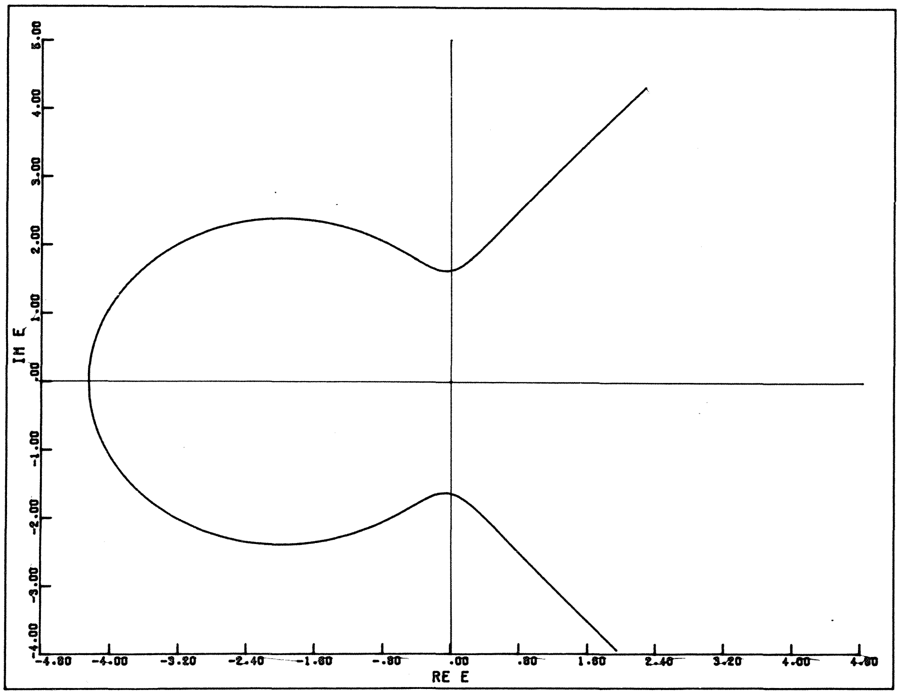


fig. 3.1b: Plot of $E(i\mu)$, $\mu \in \mathbb{R}$ for $F = 1$, $\sigma = 0.2$, $\gamma = 0.1$ and $c = 0.1$.

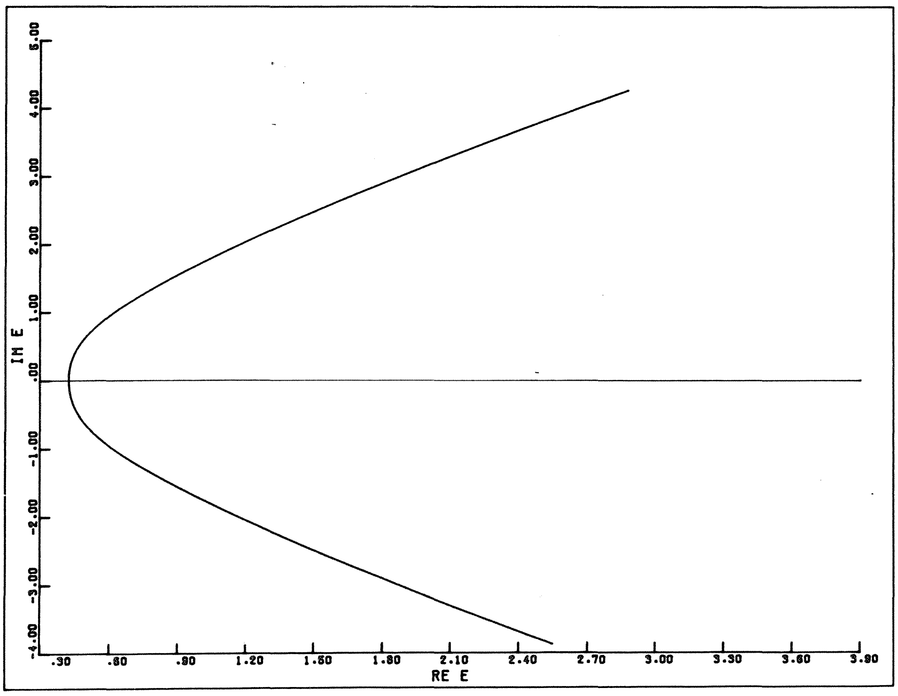


fig. 3.1c: Plot of $E(i\mu)$, $\mu \in \mathbb{R}$ for $F = 1$, $\sigma = 0.5$, $\gamma = 1$ and $c = 0.1$.

The curve in fig. 3.1c does not wind which indicates that $D(\lambda)$ has no zeroes with positive real part and thus, that the wave is stable.

Observe in fig. 2.3f that the parameter values in case of fig. 3.1a are such that the function $a = a(c)$ is increasing at this value of c while in case of fig. 3.1c the function $a = a(c)$ is decreasing at the value of c (see fig. 2.3d). This result is not coincidental. As with the homoclinic waves [19], the numerical results indicate that the slope of $a = a(c)$ is crucial in the determination of stability of the heteroclinic waves and they make the following conjecture plausible.

ASSERTION 3.1. Let $c_0 \geq 0$.
(i) If the function $a(c)$ is decreasing at $c = c_0$ then the corresponding heteroclinic wave is stable.
(ii) If the function $a(c)$ is increasing at $c = c_0$ then the corresponding wave is unstable.

REMARK. If we let c grow towards the value $c = c^*$ where the slope of $a(c)$ is horizontal (the knee of the curve as Rinzel and Keller call it) the zero of $D(\lambda)$ with positive real part and $\lambda = 0$ coincide. The heteroclinic wave is then called *neutrally stable*.

COROLLARY. *If* $\sigma \leq \gamma^2$ *then all heteroclinic wave solutions with positive speed are stable. If* $\sigma > \gamma^2$ *then there exists a number* $c^* > 0$ *such that a heteroclinic wave is stable if* $c > c^*$ *and it is unstable of* $0 < c < c^*$.

DEMONSTRATION. From Assertion 2.1 and Assertion 3.1.

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