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ONE WAY TRAFFIC OF PULSES IN A NEURON

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One way traffic of pulses in a neuron *)

by

Joop Pauwelussen

ABSTRACT

In this paper we investigate the effect of a change in geometry of a nerve axon on the propagation of potential waves along the axon. In particular we show that potential waves are stopped at a sudden large increase of cross-section area such as increase of diameter or branching. Some special examples are treated. The results do also apply to problems in population genetics and chemical reaction theory.

KEY WORDS & PHRASES: *Reaction diffusion equations, nerve conduction, blocking of travelling waves*

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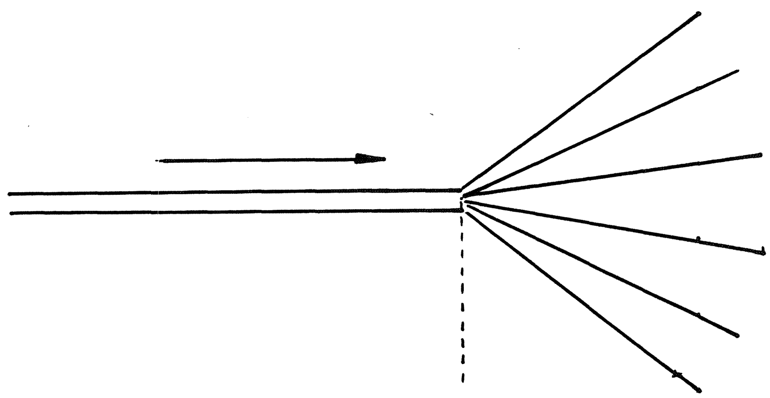
*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In theoretical studies of propagation of electrical information along an unmyelinated nerve axon, one usually assumes uniform geometric properties. It is well known that if a uniform axon is triggered at one end, and the potential across the membrane exceeds a certain threshold, the change in potential does not die out but it causes ions to move through the membrane thereby stimulating adjacent parts of the nerve. By this mechanism a wave is set up which proceeds down the axon. For a general reference see for example [3].
Whether or not one actually obtains a wave depends strongly on the shape of the specific axon. In fact, ahead of the wave, at a sudden increase of the membrane area the membrane current density falls. As a result there is a temporary decrease of action potential and if this potential falls below threshold, the wave may be stopped.

Our main goal is to study this effect by analytical means. For that reason we shall consider a tree-shaped neuron of infinite extension (schematically shown in fig. 1.1).



$x < 0$	$x = 0$	$x > 0$
radius 1		radius r
1 branch		k branches.

fig. 1.1

Let the variable x measure the distance along the neuron and t denote time. Then we shall restrict ourselves to the situation of only one branching point at $x = 0$, one branch of radius 1 for $x < 0$ and k branches of radius r for $x > 0$, and we shall mostly be concerned with waves, travelling from the region $x < 0$ towards the point $x = 0$. We shall assume that the membrane potentials on the k branches are identical for any positive time.

A general model for nerve impulse propagation along this conductor leads to the following system of equations [4]

$$
(1.1)\qquad
\begin{aligned}
u_t &= r(x)u_{xx} + F_0(u,w),\\
w_{1,t} &= G_1(u,w)\\
&\ \vdots\\
w_{n,t} &= G_n(u,w), \qquad\qquad x \in \mathbb{R}\setminus\{0\},\ t > 0
\end{aligned}
$$

where $u(x,t)$ represents the transmembrane potential while the variables $w_1,\dots,w_n$ describe the transport of ions such as K^+, Na^+ and Cl^- across the membrane.
F_0 and $G = (G_1,\dots,G_n)$ are smooth functions in u and $w = (w_1,\dots,w_n)$ (We shall assume Lipschitz continuity) and $r(x)$ is the diameter of a branch of the nerve at place x, i.e. $r(x) = 1$ for $x < 0$ and $r(x) = r$ for $x > 0$.

At $x = 0$ the following transmission conditions should be satisfied. The transmembrane potential should be continuous and so should the internal current which is proportional to the gradient of the potential times the surface area. At the branching point the surface area changes by a factor r^2k. Thus u satisfies

$$
(1.2)\qquad u_x(0-,t) = r^2ku_x(0+,t), \qquad t > 0.
$$

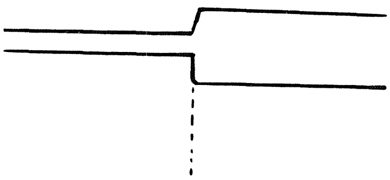
Consequently u_x is not continuous at $x = 0$. In this paper we shall make use of maximum principle techniques and for that purpose this discontinuity in u_x is not convenient. To remove it we replace x for $x > 0$ by $x/(r^2k)$. In terms of this rescaled spatial variable the equations (1.1) transform into

$$
(1.3)\qquad
\begin{aligned}
u_t &= e_\varepsilon(x)u_{xx} + F_0(u,w),\\
w_t &= G(u,w), \qquad x \in \mathbb{R}\setminus\{0\},\ t > 0
\end{aligned}
$$

where

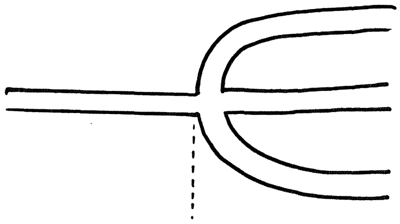
$$(1.4) \qquad e_\varepsilon(x) = \begin{cases} 1 & ; \quad x \le 0 \\ \varepsilon \equiv \dfrac{1}{r^3k^2} & ; \quad x > 0. \end{cases}$$

Two special cases are of special interest. The case that we have only an increase of diameter (i.e. $k = 1$; see fig. 1.2), and branching with constant radius (i.e. $r = 1$; see fig. 1.3).



$x = 0$

fig. 1.2. $k = 1$, $\varepsilon = r^{-3}$



$x = 0$

fig. 1.3. $r = 1$, $\varepsilon = k^{-2}$.

In the first case $\varepsilon = 1/r^3$ and thus a large r corresponds with ε being small. Similarly, in the case of only branching where $\varepsilon = 1/k^2$, a great number of daughter branches implies a small value of ε. Summarizing, in the formulation of (1.3), the change in the shape of the axon is given through the number ε and small ε corresponds with a large geometric change.

We shall treat (1.3) in the framework of an initial value problem, the initial values being

$$(1.5) \qquad \begin{aligned} u(x,0) &= \chi(x), \\ w(x,0) &= \psi(x), \qquad x \in R \end{aligned}$$

where the function χ is bounded and continuous while ψ is bounded and Hölder continuous.

The following examples are special cases of the general system (1.1):

(i) *The bistable equation* [8]

$$u_t = e_\varepsilon(x)u_{xx} + u(1-u)(u-a), \quad 0 < a < 1/2,$$

(ii) *FitzHugh - Nagumo* [12]

$$\begin{aligned} u_t &= e_\varepsilon(x)u_{xx} + u(1-u)(u-a)-w, \\ w_t &= \sigma u - \gamma w, \quad 0 < a < 1/2, \end{aligned}$$

(iii) *Goldstein - Rall* [4]

$$\begin{aligned} u_t &= e_\varepsilon(x)u_{xx} + w_1(1-u) - w_2(u+\frac{1}{10}) - u \\ w_{1,t} &= c_1u^2 + c_2u^4 - c_3w_1 - c_4w_1w_2 \\ w_{2,t} &= c_5w_1 + c_6w_1w_2 - c_7w_2 \end{aligned}$$

$$c_i > 0, \quad c_2 \gg c_1 >> c_3 > c_4, \; c_1 \gg c_7 > c_5 \gg c_6.$$

These special examples all have the property that in the uniform case, $e_\varepsilon(x) \equiv$ constant, they allow *travelling wave solutions* which in this paper, will be understood as nontrivial solutions of (1.3), depending only on the similarity variable $z = x - ct$, $c > 0$ such that $(u,w) \to (0,0)$ for $z \to \infty$.

For the second example we did some numerical experiments. Fixing $\sigma = 0.0036$, $\gamma = 0.02$ and $a = 0.2$, and initiating a wave by a method due to Muira [6] (see also [5]) we computed *) the evolution of the wave for several values of ε. Plots of the u-component of the solution for $\varepsilon = 0.162$ and 0.163 are shown in fig. 1.4 and fig. 1.5. The situation for $\varepsilon = 0.163$ is typical for all $\varepsilon \geq 0.163$, the speed of propagation changes at $x = 0$ but the wave proceeds beyond this point. The situation for $\varepsilon = 0.162$ is typical for all $\varepsilon \leq 0.162$, the waves are *blocked* at the origin. Hence the critical value ε^* of $\varepsilon = 1/(r^3k^2)$ is approximately equal to 0.162. If no branching occurs (k=1, see fig. 1.2), this corresponds to a critical value r^* of $r{:}r^* = 1.83$ while in case of only branching (r=1, see fig. 1.3) the critical value k^* of k, is $k^* = 3$. Rinzel's numerical investigations of example (ii) [12] indicated similar results.

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*) We used the algorithm M3RK of Verwer [14].

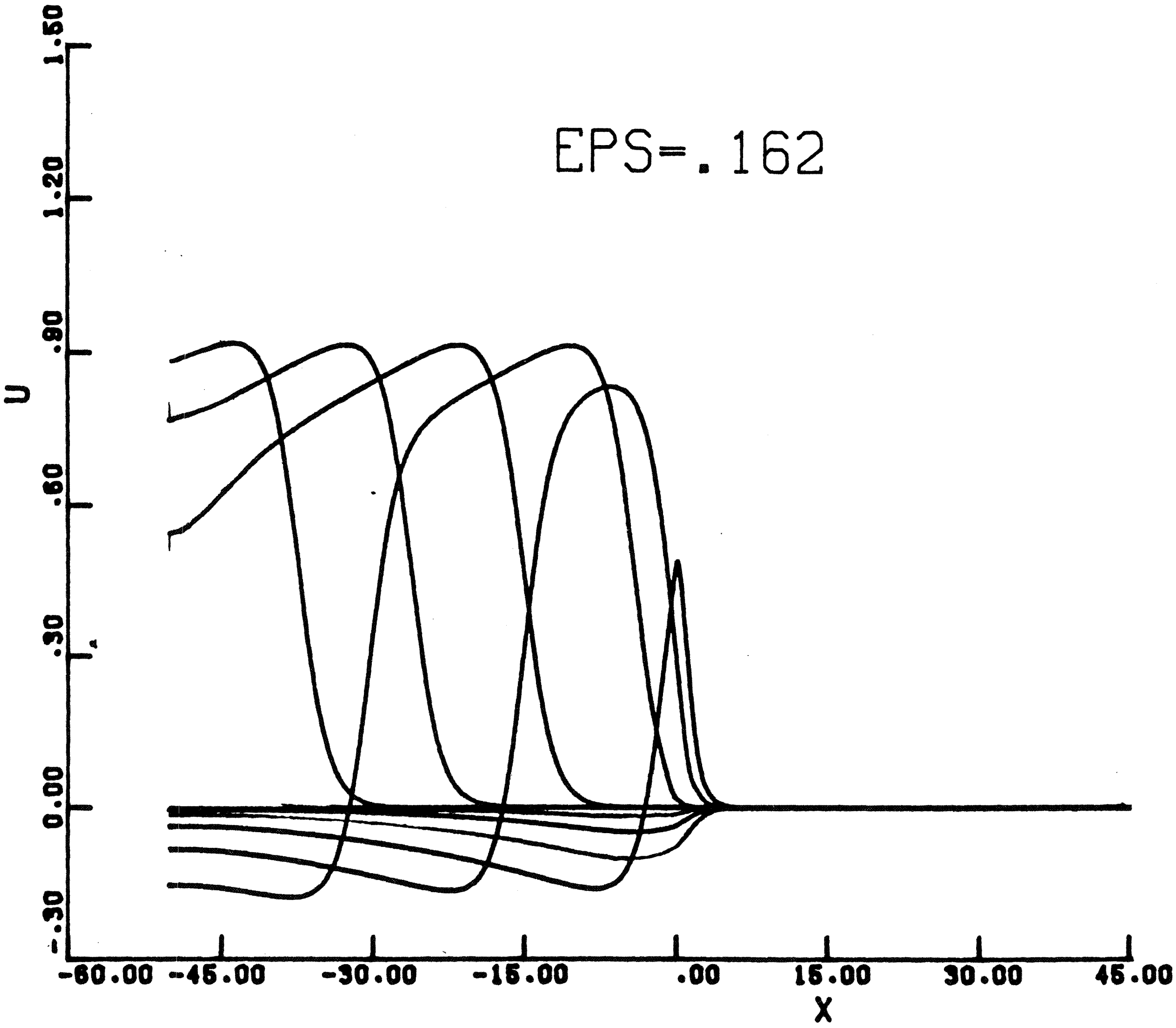

fig. 1.4.: The potential u(x,t) for example (ii) for ε = 0.162.
The time t varies from 30 to 330 with steps 30.

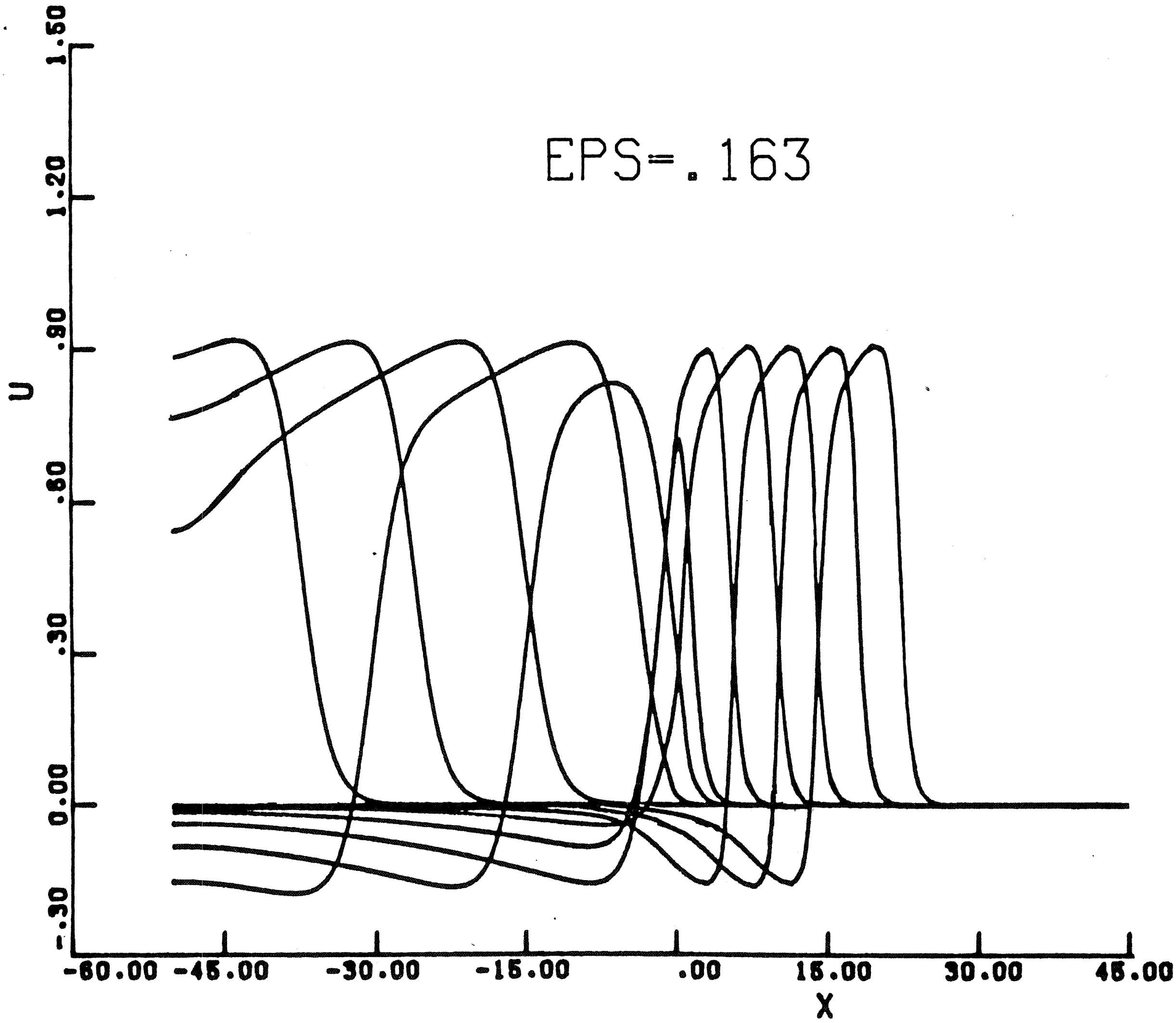

fig. 1.5: The potential u(x,t) for example (ii) for $\varepsilon = 0.163$.
The time t varies from 30 to 330 with steps 30.

Example (i) was studied in an earlier paper [8]. It was found that there exists a critical value ε^* of ε such that

(i) if $0 < \varepsilon < \varepsilon^*$ there exist exactly two stationary solutions: q_- and q_+, $q_- < q_+$. They are strictly decreasing, approach 1 as $x \to -\infty$ and 0 as $x \to +\infty$.

(ii) if $0 < \varepsilon < \varepsilon^*$ for rather general initial function χ the solution of (1.3) and (1.5) tends to q_- as $t \to \infty$ while

(iii) if $\varepsilon^* < \varepsilon < 1$ this solution tends as $t \to \infty$ to a travelling wave, corresponding to the case $e_\varepsilon(x) \equiv \varepsilon$. Similar results were obtained by Rinzel [13] for a piecewise linear approximation of f.

In [4] it is shown by means of numerical study that blocking of wave solutions for example (iii) occurs too if the increase of diameter or the amount of branching is sufficiently large. A remarkable result in this paper is that the speed of propagation does not change monotically near the branching point.

The plan of the paper is as follows. In Section 2 we shall formulate a result about existence and uniqueness of a solution of (1.3) and (1.5). Section 3 will be devoted to a conditional comparison principle which we need in Section 4 to show that under certain additional conditions, solutions of (1.3) and (1.5) remain small for $x > 0$ when ε is small enough. In particular it follows from the results that if we choose for (χ,ψ) a travelling wave solution of (1.3) corresponding to the uniform case $e_\varepsilon(x) \equiv 1$, it is always possible to shift it so far to the left that for the solution (u,w) of (1.3) and (1.5), u satisfies an inequality of the form

$$|u(x,t)| \le E\varepsilon^{\ell}, \; x \ge 0, \; t \ge 0$$

for some positive ℓ and E, not depending on ε, and for ε sufficiently small, $\varepsilon < \varepsilon^*$ say. Thus, for $\varepsilon < \varepsilon^*$ the membrane potential $u(x,t)$ is below the threshold to initiate *any* wave for $x \ge 0$, and the approaching wave is blocked. Observe that for $\varepsilon = 1$ the solution of (1.3) (1.5) is of the wave form $u(x,t) = \chi(x+ct)$, $w(x,t) = \psi(x+ct)$ for some $c \in \mathbb{R}$ and it does not slow down at $x = 0$.

REMARK. We have introduced the problem in terms of nerve conduction. However, special forms of equations (1.3) also arise in chemical reactions with space dependent diffusivities. For example if one studies target patterns in models of the Belousov-Zhabotinski reaction [2] in a narrow tube with changing diameter.

Example (i), the bistable equation arises also from population genetics [7].

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2. EXISTENCE AND UNIQUENESS

In [9] we have treated the existence-uniqueness problem for the system (1.3). In this section we shall only give the result and refer to [9] for further details. Here, and in the next Sections we shall make use of the following notations

Notation 2.1. Let for $m,n \in \mathbb{N}$ $D \subset \mathbb{R}^m$ and $\psi: D \to \mathbb{R}^n$ where $\psi = (\psi_1,\dots,\psi_n)$. Then we shall write

$$\|\psi\|_D = \sup_{x\in D} \sum_{i=1}^{n} |\psi_i(x)|. \tag{2.1}$$

We call ψ bounded if $\|\psi\|_D < \infty$.

Notation 2.2. The following sets of functions will be used.

$$C^{k+\alpha,m+\beta}(Q\to\mathbb{R}),k,m \in \{0,1,2,\dots\};\ \alpha,\beta \in [0,1)$$

for the set of functions $u = u(x,t)$, defined on $Q \subset \mathbb{R} \times \mathbb{R}^+$ and taking values in $R \subset \mathbb{R}^\ell$, $\ell \in \mathbb{N}$ for which $\frac{\partial^k u}{\partial t^k}$ and $\frac{\partial^m u}{\partial x^m}$ are continuous and where $\frac{\partial^k u}{\partial t^k}$ is Hölder-continuous with exponent α with respect to t if $\alpha > 0$, and $\frac{\partial^m u}{\partial x^m}$ is Hölder-continuous in x with exponent β if $\beta > 0$.

$C(Q \to \mathbb{R})$ for the set of functions $u: Q \to R$ where $R \subset \mathbb{R}^\ell$ and $Q \subset \mathbb{R}$ or $Q \subset \mathbb{R} \times \mathbb{R}^+$, which are continuous on Q.

$BC(Q \to \mathbb{R})$ for the set of function $u \in C(Q \to \mathbb{R})$ which are bounded.

<u>DEFINITION 2.1</u>. The vectorfunction $(u,w): \mathbb{R} \times [0,T) \to \mathbb{R} \times \mathbb{R}^n$ is called a solution of the system (1.3) on $[0,T)$, with initial conditions (1.5) if and only if

(i) $u \in BC(\mathbb{R} \times [0,T) \to \mathbb{R})$
$u_x \in C(\mathbb{R} \times (0,T) \to \mathbb{R})$
$u_{xx}, u_t \in C(\mathbb{R} \backslash \{0\} \times (0,T) \to \mathbb{R})$
$w, w_t \in BC(\mathbb{R} \times [0,T) \to \mathbb{R}^n)$

(ii) (u,w) satisfies (1.3) and (1.5).

To prove the existence of a solution of (1.3) and (1.5) we assume a priori boundedness.

H: There exists a number $K = K(\|\chi\|_{\mathbb{R}}, \|\psi\|_{\mathbb{R}})$ such that for all $T > 0$, a solution (u,w) of (1.3) and (1.5) on $[0,T)$ satisfies
$\|(u,w)\|_{\mathbb{R} \times [0,T)} < K$.

In [9] it is shown for the examples (i) - (iii), given in Section 1 that this hypothesis is true in general.

<u>THEOREM 2.1</u>. *Let* $T > 0$. *Suppose* H *is satisfied. Then problem* (1.3), (1.5) *has a unique solution on* $[0,T)$. *Moreover if* α *is a Hölder-exponent for* $\psi(x)$ *then for arbitrary* $\delta \in (0,T)$ *and any* x*-interval* J, *not including an open neighbourhood of* $x = 0$ *we have*

$$u \in C^{2+\alpha, 1+\alpha/2}(J \times (\delta,T) \to \mathbb{R}),$$

(2.2)

$$w \in C^{\alpha,1}(J \times (\delta,T) \to \mathbb{R}^n),$$

with w_t *Lipschitz continuous in* t *for* $x \in J$.

3. THE CONDITIONAL COMPARISON PRINCIPLE

In [8], where we studied the bistable equation (example (i)) we exhibited blocking of travelling solutions for small values of ε. Here this meant, generally speaking, that we found stationary upper- and lower solutions for the specific differential equation.

In the next section we want to apply similar maximum principle techniques to the reaction diffusion system (1.3) on an x-interval $I_\alpha = [\alpha,\infty)$, $\alpha \in \mathbb{R}$ (α may be negative!) in order to demonstrate blocking of travelling solutions for a certain class of initial functions. To begin with we shall choose a differential operator L for which a maximum principle holds. Then we shall construct the function $\phi(x)$ such that for all $T > 0$

$$\phi(\alpha) \le u(\alpha,t) \le \phi(\alpha) \quad \text{for all } t \in [0,T] \;,$$

and

$$(3.1) \qquad -\phi(x) \le u(x,t) \le \phi(x), \qquad x \in I_\alpha,\ t \in [0,T]$$

implies.

$$(3.2) \qquad L(-\phi) \le Lu \le L\phi, \qquad x \in I_\alpha \setminus \{0\},\ t \in (0,T].$$

Observe that in contrast to the usual comparison principle the differential inequality (3.2) need not hold in general but only for functions u lying between ϕ and $-\phi$. We shall prove in Theorem 3.1 that if (3.1) implies (3.2) on any finite time-interval this yields that (3.1) holds *for all* $t \ge 0$.

The proof of Theorem 3.1 is based on the following two unconditional comparison principles.

<u>LEMMA 3.1</u>. *Let* $\alpha,\beta \in \mathbb{R}$, $\alpha < \beta$ *where* $0 \notin (\alpha,\beta)$, *and let* $\tilde{t} \in BC([\alpha,\beta] \to (0,\infty))$. *Let*

$$\mathcal{D} = \{(x,t) \mid \alpha < x < \beta;\ 0 < t \le \tilde{t}(x)\}$$

and, let N *be a differential operator of the form*

(3.3) $$Nu = u_t - e_\varepsilon(x)u_{xx} - F(u,x)$$

where $F \in C^{1,0}(\mathbb{R}\times[\alpha,\beta] \to \mathbb{R})$. *Assume that*

$$u,v \in BC(\bar{\mathcal{D}}\to\mathbb{R}) \cap C^{2,1}(\mathcal{D}\to\mathbb{R})$$

satisfy the following conditions.

(i) $Nu \le Nv$ *on* $\mathcal{D}$.

(ii) $u(\alpha,t) \le v(\alpha,t), \quad 0 \le t \le \tilde{t}(\alpha),$

$u(\beta,t) \le v(\beta,t), \quad 0 \le t \le \tilde{t}(\beta).$

(iii) $u(x,0) \le (\not\equiv) v(x,0), \quad \alpha < x < \beta.$

Then $u(x,t) < v(x,t)$ *for all* $(x,t) \in \mathcal{D}$.

PROOF. Introduce for $\lambda > 0$ the function w by

(3.4) $$w(x,t) = e^{-\lambda t}[v(x,t)-u(x,t)].$$

If we choose λ sufficiently large (cf. [10,p.175, remark (ii)]) then we can find a bounded positive function $\tilde{F}(x,t)$ such that

$$w_t - e_\varepsilon(x)w_{xx} + \tilde{F}(x,t)w \ge 0, \quad (x,t) \in \mathcal{D}.$$

By [10; p.174, Th.7] if $w = 0$ at some point $P \in \mathcal{D}$ then $w(x,t) = 0$ at all points $(x,t) \in \mathcal{D}$ lying below P and this contradicts (iii).

If $0 \in J$ where $J \subset \mathbb{R}$ we shall write $J' = J\setminus\{0\}$.

LEMMA 3.2. *Let for* $\alpha,\beta \in \mathbb{R}$, $\alpha < 0 < \beta$ *and* $\underline{t} > 0$:

$$u,v \in BC([\alpha,\beta]\times[0,\underline{t}] \to \mathbb{R}) \cap C^{2,1}((\alpha,\beta)'\times(0,\underline{t}] \to \mathbb{R})$$

satisfy the following conditions

(i) $Nu \leq Nv$ *for* $\alpha < x < \beta$, $x \neq 0$ *and* $t \in (0,\underline{t}]$ *where* N *is given in* (3.3).

(ii) $u(\alpha,t) \leq v(\alpha,t)$,

$u(\beta,t) \leq v(\beta,t)$, $\quad 0 \leq t \leq \underline{t}$

(iii) $u(x,0) \leq (\neq) v(x,0)$, $\quad \alpha < x < \beta$

(iv) $u_x(0+,t)-u_x(0-,t) \geq v_x(0+,t)-v_x(0-,t)$, $\quad 0 < t \leq \underline{t}$.

Then $u(x,t) < v(x,t)$ *for all* $x \in (\alpha,\beta)$, $t \in (0,\underline{t}]$.

<u>PROOF</u>. If at some point $(x_0,t_0) \in \mathcal{D}_{\underline{t}} = \{(x,t)\,|\,x \in (\alpha,0)\cup(0,\beta),\ t \in (0,\underline{t}]\}$ $u(x_0,t_0) = v(x_0,t_0)$ then $u(x,t) = v(x,t)$ at all points $(x,t) \in \mathcal{D}_{\underline{t}}$, lying below (x_0,t_0) and with sign x = sign x_0. This follows in the same way as in the proof of the preceding Lemma. If $u(0,t_0) = v(0,t_0)$ where $t_0 \in (0,\underline{t})$ and $u(x,t) < v(x,t)$ for $x \neq 0$, $t \in (0,\underline{t})$ then application of [10, p.174, Theorem 7] to the function w introduced in (3.4), in $\mathcal{D}^+_{\underline{t}} = \{(x,t) \in \mathcal{D}_{\underline{t}}\,|\,x > 0\}$ as well as in $\mathcal{D}^-_{\underline{t}} = \{(x,t) \in \mathcal{D}_{\underline{t}}\,|\,x < 0\}$ yields that $v_x(0+,t_0)-u_x(0+,t_0) > 0$ and $v_x(0-,t_0)-u_x(0-,t_0) < 0$, respectively. This contradicts (iv).
This same conclusion holds if $t_0 = \underline{t}$, as pointed out in the proof of [10; Theorem 4.1]. If for some $(x_0,t_0) \in \mathcal{D}^+_{\underline{t}}(\mathcal{D}^-_{\underline{t}})$ $u(x_0,t_0) = v(x_0,t_0)$ and $u(x,t) <$ $< v(x,t)$ for all $(x,t) \in \mathcal{D}^-_{\underline{t}}(\mathcal{D}^+_{\underline{t}})$ then for all $t < t_0$

$$v_x(0-,t)-u_x(0-,t) < 0 = v_x(0+,t)-u_x(0+,t)$$

and this contradicts (iv) again.

<u>THEOREM 3.1</u>. (*The conditional comparison principle*).
Let for $\alpha \in \mathbb{R}^-$

$$\phi,u,\psi \in BC([\alpha,\infty)\times[0,\infty) \to \mathbb{R}) \cap C^{2,1}(I'_\alpha \times (0,\infty) \to \mathbb{R})$$

satisfy for all $T > 0$ *and* $x \in I'_\alpha$

(3.5) $\qquad \phi \leq u \leq \psi$ *on* $[0,T] \Rightarrow N\phi \leq Nu \leq N\psi$ *for* $t \in (0,T]$,

where N *is the differential operator, introduced in* (3.3), *with* $[\alpha,\beta]$ *replaced by* $[\alpha,\infty)$. *Moreover let* ϕ, u *and* ψ *satisfy*

$$(3.6)\qquad \phi_x(0+,t)-\phi_x(0-,t)\geq u_x(0+,t)-u_x(0-,t) \geq \psi_x(0+,t)-\psi_x(0-,t),$$

$$(3.7)\qquad \phi(\alpha,t) \leq u(\alpha,t) \leq \psi(\alpha,t), \quad t > 0,$$

$$(3.8)\qquad \phi(x,0) < u(x,0) < \psi(x,0), \quad x > \alpha.$$

Then

$$(3.9)\qquad \phi(x,t) < u(x,t) < \psi(x,t), \quad x \in (\alpha,\infty), \quad t \geq 0.$$

PROOF. Suppose the following set is nonempty

$$V = \{(x,t) \in [\alpha,\infty)x[0,\infty) \mid (u(x,t)-\phi(x,t))(u(x,t)-\psi(x,t)) = 0,$$
$$\phi(x,\tau) < u(x,\tau) < \psi(x,\tau) \textit{ for all } \tau \in [0,t)\},$$

and let $(x_2,t_2) \in V$. Take any point (x_1,t_1), $x_1 < x_2$ such that $\phi(x_1,\tau) < u(x_1,\tau) < \psi(x_1,t)$ for $\tau \in [0,t_1)$. We shall write $m = (t_2-t_1)/(x_2-x_1)$ and we shall first show that, if $x_1 > 0$, below the line $\ell = \{(x,t) | t-t_1 = m(x-x_1)\}$ through the points (x_i,t_i), there are no points of V for $x_1 < x < x_2$. Then we shall show that this implies that $t_1 \leq t_2$. By this fact it will be sufficient to consider ϕ, u and ψ only on a bounded domain and application of Lemma 3.2 will complete the proof.

Suppose there exist points of V, lying below ℓ for $x_1 < x < x_2$. We define for $q \in \mathbb{R}$

$$\ell_q = \{(x,t) \mid t = mx + q\},$$

$$\Omega_q = \{(x,t) \mid t \in [0,mx+q), x_1 < x < x_2\}$$

and

$$q_0 = \inf\{q \mid \Omega_q \cap V \neq \emptyset\}.$$

By (3.8), q_0 is well defined and ℓ_{q_0} lies below ℓ. Hence there exist points

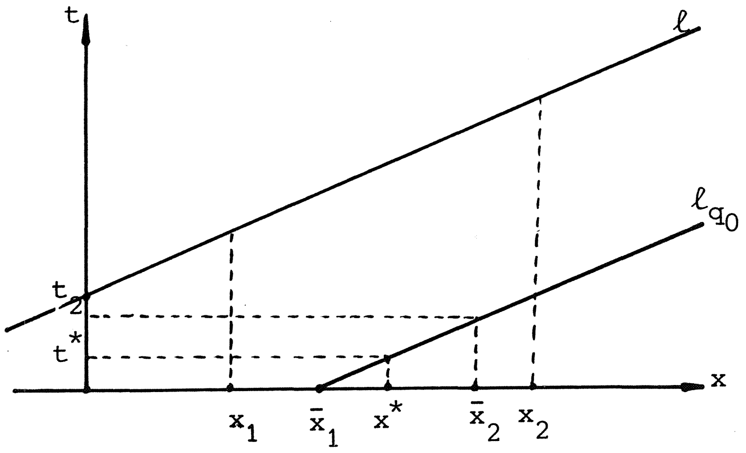

fig. 3.1.

(x^*,t^*), $(\bar{x}_1,\bar{t}_1)$ and $(\bar{x}_2,\bar{t}_2)$ on ℓ_{q_0} where $x_1 \leq \bar{x}_1 < x^* < \bar{x}_2 \leq x_2$ and such that $(x^*,t^*) \in V$, $\phi \leq u \leq \psi$ in the points $(\bar{x}_1,\bar{t}_1)$ and $(\bar{x}_2,\bar{t}_2)$, and $\phi(x,t) < u(x,t) < \psi(x,t)$ below ℓ_{q_0} for $\bar{x}_1 \leq x \leq \bar{x}_2$ (see fig. 3.1, where we assumed $\bar{t}_1 = 0$). By Lemma 3.1, applied on $[\bar{x}_1,\bar{x}_2]$ where $\tilde{t}(x) = mx + q_0$, we have a contradiction in case $u = \psi$ as well as if $u = \phi$ at (x^*,t^*).

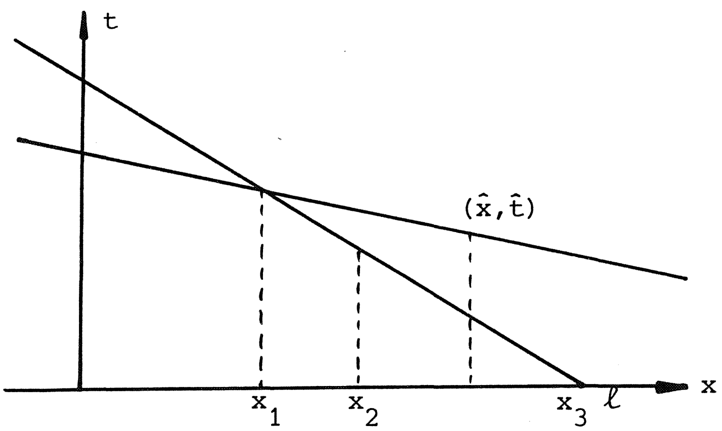

fig. 3.2.

Next we suppose that $t_2 < t_1$. Then ℓ intersects the positive x-axis at some point $(x_3,0)$, $x_3 > x_2$ and for all $x \in (x_2,x_3)$ there exists a time $t \in [0,mx+t_1-mx_1]$ such that $(x,t) \in V$. For if not, then $\phi < u < \psi$ at $(\hat{x},t)$ for some $\hat{x} \in (x_2,x_3)$ and all $t \in [0,\hat{t}]$ with $(\hat{x},\hat{t})$ lying above ℓ. Using the same arguments as above, with (x_2,t_2) replaced by $(\hat{x},\hat{t})$ we find that below the straight line segment connecting (x_1,t_1) and $(\hat{x},\hat{t})$ there are no points of V. However, (x_2,t_2) lies below this line segment (see fig.3.2) and we obtain a contradiction. Thus arbitrarily close to the point $(x_3,0)$ there are points of V. This contradicts (3.8). Hence $t_2 \geq t_1$. Summarizing, if the inequality (3.9) holds for $x = x_1$, and $t \in [0,t_1)$ where $x_1 > 0$, $t_1 > 0$ then $\phi(x,t) < u(x,t) < \psi(x,t)$ for all $x \geq x_1$ and $t < t_1$. Hence

$$\underline{t} \equiv \inf\{t > 0 \mid (x,t) \in V \quad \text{for some } x > \alpha\}$$

exists and is positive. Moreover it follows that if this infimum occurs at a point $(\underline{x},\underline{t})$ where $\underline{x} > 0$ then it must also occur at the point $(0,\underline{t})$. Now, by application of Lemma 3.2 to any set of the form $\{(x,t) \mid \alpha < x < \beta,$ $0 < t \leq \underline{t}\}$ where $\beta > 0$, we arrive at a contradiction because of (3.6).

4. STANDSTILL OF SOLUTIONS

In this Section we shall demonstrate that for small ε solutions of the reaction diffusion system (1.3), such as travelling wave solutions are blocked at the point $x = 0$, and we shall first say what we mean by "blocking".

<u>DEFINITION 4.1</u>. Let V be a class of admissible initial functions and let $\phi_1,\phi_2\colon \mathbb{R} \to \mathbb{R}$ be two functions with the property

$$(\chi,\psi) \in V \;\Rightarrow\; \phi_1 \leq u(\cdot,t) \leq \phi_2 \quad \text{on } \mathbb{R} \text{ for all } t \geq 0.$$

Such a pair of functions (ϕ_1,ϕ_2) we shall call a *trap* for V.

<u>DEFINITION 4.2</u>. For a class V of initial functions (χ,ψ) we shall say that a solution of (1.3) is *blocked* if there exists a trap (ϕ_1,ϕ_2) such that

$$\|\phi_i\|_{\mathbb{R}^+} \leq \rho(\varepsilon), \qquad i = 1,2$$

and $\rho(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$.

To begin with we shall consider for $\delta_0 > 0$, $K_0 > 0$ the class V_0 of initial functions (χ,ψ) which satisfy

(i) $\chi \in BC(\mathbb{R}\to\mathbb{R})$, $\psi \in BC(\mathbb{R}\to\mathbb{R}^n)$ and ψ is Hölder continuous,

(ii) $(\chi(x),\psi(x)) = 0$, $x \geq -\delta_0$,

(iii) $\|(\chi,\psi)\|_{\mathbb{R}} \leq K_0$.

As we shall see, for small ε solutions of (1.3) with $(\chi,\psi) \in V_0$ are blocked by a trap $(-\phi,\phi)$ where, for some E, $\ell > 0$, $\rho(\varepsilon)=E\varepsilon^{\ell}$. A consequence is that travelling solutions of (1.3) approaching. $x = 0$ from the left do not proceed beyond $x = 0$.

We shall make the following assumptions on F_0 and G:

HF_1: F_0 and G are continuously differentiable,

HF_2: $F_0(0,0) = 0$, $-A \equiv F_{0,u}(0,0) < 0$,
$G(0,0) = 0$,

HF_3: For $n \geq 1$ the spectrum $\Sigma(J_0)$ of $J_0 \equiv G_w(0,0)$ is contained in $\{z \in \mathbb{C} | \operatorname{Re} z < 0\}$.

We shall write

$$F(u,w) = F_0(u,w) + Au. \tag{4.1}$$

For the several examples in Section 1 we have

(i) *Bistable equation*: $A = a$

(ii) *FitzHugh-Nagumo* : $A = a$,
$J_0 = -\gamma < 0$.

(iii) *Goldstein-Rall* : $A = 1$,

$$J_0 = \begin{pmatrix} -k_3 & 0 \\ k_5 & -k_7 \end{pmatrix}$$

The plan of this Section is as follows. First we shall construct a trap of the form $(-\phi,\phi)$, where the function ϕ has the shape shown in figure 4.1 below.

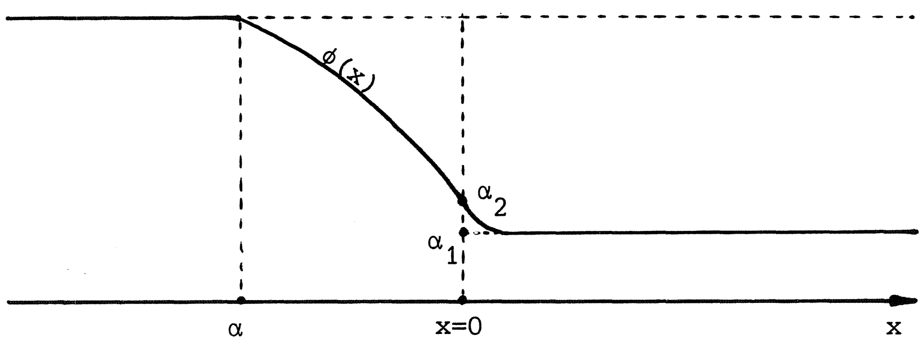

fig.4.1.

Here, K is a bound for u where (u,w) is a solution of (1.3) (cf. assumption H) and α, α_1 and α_2 are all of order of some positive power of ε. To begin with we shall show for example (ii) above, the nonuniform FitzHugh-Nagumo equations, that α_1, α_2 and α can be chosen in such a way that there exists an ε^* so that when $\varepsilon \in (0,\varepsilon^*)$, $(-\phi,\phi)$ is a trap for (1.3) and V_0. All the ingredients of the proof for the more general equation where F_0 and G satisfy HF_1 - HF_3 are already present in this special situation and existence of a trap of the form $(-\phi,\phi)$ is proved along the same lines. Finally we shall extend the result to a more general class V_0^* of initial functions.

4.1. Standstill of solutions of the non-uniform FitzHugh-Nagumo equations

Recall that the non-uniform FitzHugh-Nagumo equations are given by

$$(4.2) \qquad \begin{aligned} u_t &= e_\varepsilon(x) u_{xx} + f(u) - w. \\ w_t &= \sigma u - \gamma w. \end{aligned}$$

We shall construct $\phi(x)$, shown in figure 4.1. such that for the solution (u,w) of (4.2) which satisfies (1.3) and $-\phi(\alpha) < u(\alpha,t) < \phi(\alpha)$ for all $t \geq 0$, we have: if for any $T > 0$

$$(4.3) \qquad -\phi(x) \leq u(x,t) \leq \phi(x), \qquad t \in [0,T], \qquad x \in I'_\alpha$$

then

$$(4.4) \qquad L(-\phi) \le Lu \le L\phi, \quad t \in (0,T],\ x \in I'_\alpha$$

where

$$(4.5) \qquad Lu \equiv u_t - e_\varepsilon(x)u_{xx} + au.$$

Application of the conditional comparison principle Theorem 3.1 then yields that if $-\phi(x) < \chi(x) < \phi(x)$ for $x \ge \alpha$, u must remain between $-\phi$ and ϕ for $x \ge \alpha$ and for all time.

For solutions of (4.2) it follows that

$$(4.6) \qquad Lu = f(u) + au - w.$$

Since $f(u) + au$ is of order u^2 as $u \downarrow 0$ we find that for large x, if $|u| < \phi$, Lu is close to $-w$. If we solve $(4.2)^2$ for $w(x,0) = 0$ we obtain an expression for w from which it is easily deduced that on $[0,t] \subset [0,T]$

$$(4.7) \qquad |w(x,t)| \le \frac{\sigma}{\gamma} \|u(x,\cdot)\|_{[0,t]} \le \frac{\sigma}{\gamma}\phi(x).$$

On the other hand, for large x we have $|\phi''| \ll 1$ and since ϕ is bounded away from zero (cf. fig.4.1) we obtain that $L\phi$ is approximately equal to $a\phi$. Hence, in order to derive (4.4) from (4.3), the condition

$$(4.8) \qquad \frac{\sigma}{\gamma} < a$$

seems to be needed. This is the same condition as appeared in [11] where stability properties of the zero-solution of the uniform FitzHugh-Nagumo equations were discussed by means of contracting rectangles which are a special type of positively invariant regions [1].

Let us now treat the first part of our program: the construction of $\phi(x)$. Since this construction is rathe technical we shall here only state the result and give the proof at the end of this section. For large x where u is small the implication (4.3) $\Rightarrow$ (4.4) is a consequence of (4.8) Since K is a bound for u it is clear that $u(x,t)$ stays between $\phi(x)$ and $-\phi(x)$ for all $t \geq 0$. On the remaining interval (α,β) where ϕ is neither small nor constant and equal to K, the construction of ϕ will involve a differential inequality of the form $e_\varepsilon(x)\phi'' - a\phi \leq -B\varepsilon^{-\mu}$; $B,\mu > 0$. Hence, since u and w are bounded it is possible to satisfy (4.4) by selecting ε sufficiently small. Recall that α_1,α_2 and α in figure 4.1 are of the order of some positive power of ε.

<u>LEMMA 4.1</u> (Construction $\phi(x)$).
Let K, δ, $a > 0$. *Then there exists a positive number* ε^*, *positive constants* B_1,B_2,ℓ,μ_1,μ_2 *and a function* $\alpha:(0,\varepsilon^*) \to \mathbb{R}^-$ *such that* $\alpha(\varepsilon) = O(\varepsilon^{\mu_2})$, *and a function* $\phi\colon \mathbb{R} \to \mathbb{R}$, *depending on* ε, *with the properties*

(i) $\phi \in C^1(\mathbb{R}\setminus\{\alpha\}) \cap C^2(\mathbb{R}\setminus\{\alpha,0\})$

(ii) $\phi(x) = K$ *for* $x \leq \alpha$, ϕ *is decreasing for* $x > \alpha$ *and* $\phi(\infty) = B_1(1+\tfrac{1}{2}\delta)\varepsilon^\ell/a$,

(iii) *for some* $\beta \in (\alpha,0)$ *we have*

(4.9) (a) *on* $(\beta,\infty)'$: $e_\varepsilon(x)\phi'' - a\phi \leq -B_1\varepsilon^\ell$
(b) *on* (α,β): $\phi'' - a\phi \leq -\delta B_2\varepsilon^{-\mu_1}$

Moreover $\phi(0) = B_1(1+\delta)\varepsilon^\ell/a$ *and* $\phi(\beta) = B_1(1+2\delta)\varepsilon^\ell/a$.

Thus in terms of the parameters, introduced in this lemma we have $\alpha_1 = B_1(1+\delta/2)\varepsilon^\ell/a$ and $\alpha_2 = B_1(1+\delta)\varepsilon^\ell/a$.

<u>THEOREM 4.1</u>. *Suppose* $\sigma/\gamma < a$. *Then there exist numbers* ε^*, E *and* $\ell > 0$ *such that for any* $\varepsilon \in (0,\varepsilon^*)$

(4.10) $$|u(x,t)| \leq E\varepsilon^\ell, \qquad x \geq 0,\ t \geq 0.$$

<u>PROOF</u>. By Hypothesis H there exists a number $K \geq K_0$ such that for any $(\chi,\psi) \in V_0$, for the solution (u,w) of (1.3), (1.5)

$$\|u\|_{\mathbb{R}\times\mathbb{R}^+},\ \|w\|_{\mathbb{R}\times\mathbb{R}^+} < K.$$

Choose $m_1 \in (\frac{\sigma}{\gamma},a)$. Then by (4.7) there exists a $\delta_1 > 0$ such that for any $t > 0$

$$(4.11) \qquad |(f(u)+au-w)(x,t)| \le m_1 \|u(x,\cdot)\|_{[0,t]},$$

$$\|u(x,\cdot)\|_{[0,t]} \le \delta_1, x \ge -\delta_0.$$

Choose

$$(4.12) \qquad \delta \in (0, \tfrac{1}{2}(\frac{a}{m_1} - 1)).$$

By Lemma 4.1 there exists an $\varepsilon^* > 0$ such that a function $\phi(x)$ can be constructed as described in that Lemma for some positive constants, introduced in that Lemma and which will be used here too. In particular, ϕ satisfies the inequalities (4.9) and is differentiable on (α,∞) where $\alpha = \alpha(\varepsilon) = O(\varepsilon^{\mu_2})$ on $(0,\varepsilon^*)$ for some $\mu_2 > 0$. We assume ε^* to be so small that

$$(4.13) \qquad -\delta_0 < \bar{\alpha} \equiv \inf\{\alpha(\varepsilon) \mid 0 \le \varepsilon \le \varepsilon^*\}$$

and

$$(4.14) \qquad \phi(\beta) = \frac{B_1}{a}(1+2\delta)(\varepsilon^*)^{\ell} < \delta_1.$$

By (4.13) and the definition of V_0 we have that $u(x,0) = 0$ for $x \ge \bar{\alpha}$. It follows from (4.14) that (4.11) holds on (β,∞) as long as u remains between $-\phi$ and ϕ on (β,∞). We want to apply the conditional comparison principle of the previous Section and we shall verify that $-\phi$, u and ϕ satisfy in that order condition (3.5) of Theorem 3.1, for differential operator L.

Let $T > 0$ and assume $|u(x,t)| \le \phi(x)$ on $[0,T]$. Then we have for

(i) $\underline{x > 0}$: $u_t - \varepsilon u_{xx} + au = f(u) + au - w \le \frac{m_1 B_1}{a}(1+\delta)\varepsilon^{\ell}$ by (4.11)

$\le -\varepsilon\phi'' + a\phi + (\frac{m_1}{a}(1+\delta)-1)B_1\varepsilon^{\ell}$ by (4.9)

$\le -\varepsilon\phi'' + a\phi$ by (4.12),

(ii) $\underline{\beta < x < 0}$: $u_t - u_{xx} + au = f(u) + au - w \le \frac{m_1 B_1}{a}(1+2\delta)\varepsilon^{\ell}$ by (4.11)

$\le -\phi'' + a\phi + (\frac{m_1}{a}(1+2\delta)-1)B_1\varepsilon^{\ell}$ by (4.9)

$\le -\phi'' + a\phi$ by (4.12),

(iii) $\underline{\alpha < x < \beta}$: $u_t - u_{xx} + au = f(u) + au - w \le \tilde{K}$ for some $\tilde{K} > 0$

$$\le -\phi'' + a\phi + \tilde{K} - \delta B_2(\varepsilon)^{-\mu_1} \text{ by (4.9)}$$

$$\le -\phi'' + a\phi \text{ if}$$

(4.15) $$\varepsilon^* \le [\frac{\delta B_2}{\tilde{K}}]^{1/\mu_1} .$$

The conditions of Theorem 3.1 are satisfied for $\varepsilon < \varepsilon^*$, where ε^* satisfies (4.13) - (4.15) and as a consequence it follows that $(-\phi,\phi)$ is a trap for (4.2) and class of initial functions V_0. Since $\|\phi\|_{\mathbb{R}^+} = B_1(1+\delta)\varepsilon^{\ell}/a$, (4.10) follows with $E = B_1(1+\delta)/a$.

4.2. <u>The general system (1.3)</u>

From HF_1 - HF_3 it follows that

(i) for all $K > 0$ there exists a number K_G such that $|u| < K$ together with $\sum_{j=1}^{n} |w_j| < K$ implies

(4.16) $$|F(u,w)| < o(|u|) + K_G \sum_{j=1}^{n} |w_j|, \quad (|u| \to 0).$$

(ii) $G(u,w)$ can be written as

(4.17) $$G(u,w) = G_u(0,0)u + J_0 w + R(u,w).$$

where for all $j = 1,\dots,n$

(4.18) $$R_j(u,w) = o(|u| + \sum_{k=1}^{n} |w_k|), \quad (|u| + \sum_{k=1}^{n} |w_k| \to 0).$$

We shall use the differential operator L, given by (4.5) and the function $\phi(x)$, constructed in Lemma 4.1, with a replaced by A. We shall first verify that an inequality of the form (4.7) can be found in this general situation.

It is well known that for $\mu \in (0,-M(J_0))$ where $M(J_0) = \max\{\text{Re } \lambda \mid \lambda \in \Sigma(J_0)\}(<0)$ there exists a number J_μ such that

(4.19) $$\|e^{J_0 t}\|_M \le J_\mu \exp[-\mu t], \quad t \ge 0$$

where for a matrix $B = \{B_{jk}\}$

$$\|B\|_M \equiv \max_j \sum_k |B_{jk}|. \tag{4.20}$$

The next lemma which is the extension of (4.7) shows that in the general case, the rôle of $\frac{\sigma}{\gamma}$ in (4.7) is taken over by

$$c_\mu = \frac{nJ_\mu}{\mu} \max_i |G_{i,u}(0,0)|, \tag{4.21}$$

where $\mu \in (0,-M(J_0))$. To be precise we shall prove.

<u>LEMMA 4.2</u>. *Let* F_0 *and* G *satisfy* $HF_1 - HF_3$ *and let* $\mu \in (0,-M(J_0))$. *If* $w(x,0) = 0$ *then for* $t \geq 0$

$$\limsup_{\|u(x,\cdot)\|_{[0,t]} \longrightarrow 0} \frac{\|w(x,\cdot)\|_{[0,t]}}{\|u(x,\cdot)\|_{[0,t]}} \leq c_\mu.$$

<u>PROOF</u>. If we solve equation $(1.3)^2$ for w and use the representation (4.17) for $G(u,w)$ we arrive at

$$\begin{aligned} w(x,t) &= \int_0^t u(x,\tau)e^{J_0(t-\tau)} G_u(0,0)d\tau + \\ &+ \int_0^t e^{J_0(t-\tau)} R(u(x,\tau),w(x,\tau))d\tau \equiv w^1(x,t) + w^2(x,t). \end{aligned} \tag{4.22}$$

By (4.17) if follows for the entries w_j^1 of w^1 that

$$|w_j^1(x,t)| \leq \frac{1}{\mu} J_\mu \|u(x,\cdot)\|_{[0,t]} \max_i |G_{i,u}(0,0)| \tag{4.23}$$

By (4.18) there exists a function $\rho(\delta)$, vanishing at $\delta = 0$ such that for all $j = 1,2,\dots,n$

$$\|R_j(u(x,\cdot),w(x,\cdot)\|_{[0,t]} \leq \rho(\delta)[\|u(x,\cdot)\|_{[0,t]} + \|w(x,\cdot)\|_{[0,t]}], \tag{4.24}$$

where $\|u(x,\cdot)\|_{[0,t]} + \|w(x,\cdot)\|_{[0,t]} \leq \delta$.

Estimation of $w(x,t)$, using (4.23) and (4.24) yields

$$\|w(x,\cdot)\|_{[0,t]} \le \frac{n}{\mu} J_\mu \|u(x,\cdot)\|_{[0,t]}[\max_i |G_{i,u}(0,0)|+\rho(\delta)] +$$

$$+ \frac{n\rho(\delta)}{\mu} J_\mu \|w(x,\cdot)\|_{[0,t]}.$$

Select δ so small such that $n\rho(\delta)J_\mu/\mu \in (0,1)$. Then if

(4.25) $\quad \|u(x,\cdot)\|_{[0,t]} + \|w(x,\cdot)\|_{[0,t]} \le \delta$

if follows that

$$\|w(x,\cdot)\|_{[0,t]} \le \frac{n}{\mu - nJ_\mu\rho(\delta)} J_\mu \|u(x,\cdot)\|_{[0,t]}.$$

(4.26)

$$\cdot[\max_j |G_{i,u}(0,0)| + \rho(\delta)] \equiv Q(\delta) \|u(x,\cdot)\|_{[0,t]}.$$

In figure 4.2 if $\|u(x,\cdot)\|_{[0,t]} < \delta_0$ where $\delta_0 < \frac{\delta}{1+Q(\delta)}$, this means that the points $(\|u(x,\cdot)\|_{[0,t]}, \|w(x,\cdot)\|_{[0,t]})$ lie either in the set A or in B, indicated in this figure.

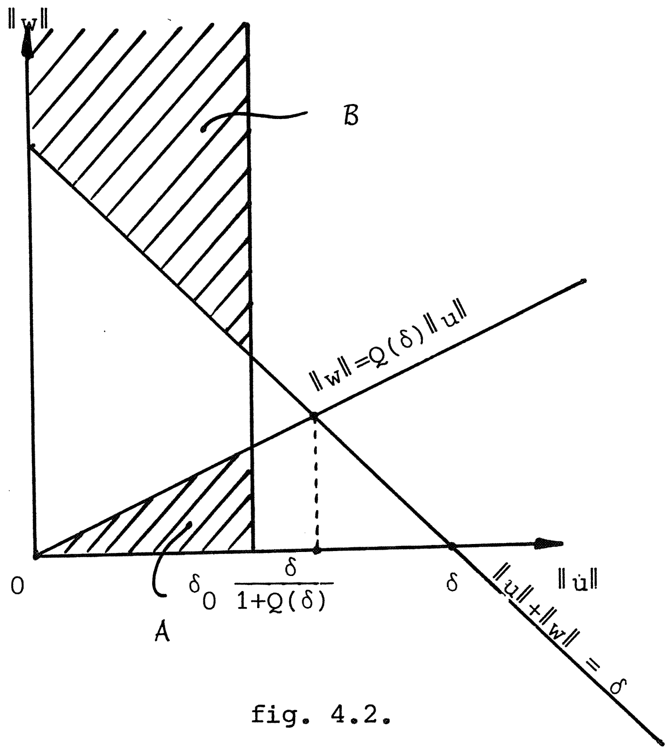

fig. 4.2.

Since $w(x,0) = 0$, these points lie in $\hat{A}$ for small t and, by the continuity of w in t, they do so for all time. Hence (4.26) holds for all $t > 0$ if $\|u(x,\cdot)\|_{[0,t]} \le \delta_0$. If we let $\delta \to 0$, the result follows.

<u>COROLLARY 4.1</u>. *Let* K *be such that* $\|(u,w)\|_{\mathbb{R}\times\mathbb{R}^+} < K$. *Let* $\mu \in (0,-M(J_0))$. *If* $w(x,0) = 0$ *then for every* $\varepsilon_1 > 0$ *there exists a* $\delta_1 > 0$ *such that* $\|u(x,\cdot)\|_{[0,t]} \le \delta_1$ *implies*

$$\|F(u(x,\cdot),w(x,\cdot))\|_{[0,t]} \le K_G\, c_\mu(1+\varepsilon_1)\|u(x,\cdot)\|_{[0,t]}. \tag{4.27}$$

where K_G *was introduced in* (4.16).

<u>PROOF</u>. The inequality follows easily from (4.16), using Lemma 4.2.

<u>REMARK 4.1</u>. For the FitzHugh-Nagumo equations we have $n = 1$, $\Sigma(J_0) = \{-\gamma\}$ and $G_u(0,0) = \sigma$ so that we may take $J_\mu = 1$ where $\mu \in (0,\gamma)$. Hence for this example

$$\inf\{c_\mu \,|\, \mu \in (0,-M(J_0|)\} = \frac{\sigma}{\gamma}\,.$$

<u>REMARK 4.2</u>. For the Goldstein-Rall equations $G_{i,u}$ is of order $O(|u|)$ as $|u| \to 0$ and therefore, by (4.21), (4.27) may be replaced by

$$\|F(u(x,\cdot),w(x,\cdot)\|_{[0,t]} = o(\|u(x,\cdot)\|_{[0,t]}),$$
$$(\|u(x,\cdot)\|_{[0,t]} \to 0).$$

If we not turn to the proof of Theorem 4.1 then, except for the appearance of a instead of A, only in (4.10) it is apparent that we restrict ourselves to the specific example (ii) instead of treating the general equation (1.3). However, by Corollary 4.1, the counterpart of (4.11) for the general equations is the inequality

$$|F(u(x,t),w(x,t))| \le m_1\, \|u(x,\cdot)\|_{[0,t]} \tag{4.28}$$

where $m_1 \in (K_G c_\mu, A)$, at least if $A > K_G c_\mu$. The remaining part of the proof can be extended immediately leading to

THEOREM 4.2. *Suppose that for the general equations* (1.3)

$$(4.29)\qquad K_G \inf\{c_\mu \mid 0 < \mu < -M(J_0)\} < A.$$

Then there exist numbers ε^*, E *and* $\ell > 0$ *such that for any* $(\chi,\gamma) \in V_0$ *and* $\varepsilon \in (0,\varepsilon^*)$ *the solution of* (1.3), (1.5) *satisfies*

$$(4.30)\qquad |u(x,t)| \le E\,\varepsilon^{\ell}, \qquad x \ge 0,\ t \ge 0.$$

REMARK 4.3. For the Goldstein-Rall equations K_G may be replaced by an $O(\|u\|_{\mathbb{R}^+\times[0,t]})$-term (cf. Remark 4.2) and thus, if we start at t = 0 with small u there is no condition needed for (4.30) to be satisfied.

Finally we shall point out in what way the proofs of the above theorems and lemma's can be adjusted so that the initial functions need not be equal to zero for $x \ge -\delta_0$. If we proceed as in the proof of Lemma 4.2 with $w(x,0) = 0$ replaced by $w(x,0) = \psi(x)$, then, instead of (4.26) we arrive at

$$\|w(x,\cdot)\|_{[0,t]} \le \frac{n}{\mu - nJ_\mu \rho(\delta)}\, J_\mu\, \|u(x,\cdot)\|_{[0,t]}.$$

$$(4.31)\qquad [\max_i |G_{i,u}(0,0)| + \rho(\delta)] + \frac{\mu|\psi(x)|}{\mu - n\rho(\delta)},$$

if $\|u(x,\cdot)\|_{[0,t]}$ and $|\psi_j(x)|$, $1 \le j \le n$ are small. In the proof of Theorem 4.1 we established the existence of numbers $\tilde{E}$, ε^* and $\ell > 0$ such that for some $\beta < 0$, $|u(x,t)| < \tilde{E}\,\varepsilon^{\ell}$ for $t \ge 0$, $x \ge \beta$ and $\varepsilon < \varepsilon^*$. If we bound $\|\psi\|_{[-\delta_0,\infty)}$ by a constant which we take much smaller than $\tilde{E}\,\varepsilon^{\ell}$ (so that we do not need to adjust condition (4.29)) for example by $\tilde{E}\,\varepsilon^{\ell+\varepsilon_\ell}$ for some $\varepsilon_\ell > 0$ then for ε^* small enough using (4.31)

$$|F(u(x,t),w(x,t))| \le m_1\,\tilde{E}\,\varepsilon^{\ell}$$

where $m_1 \in (K_G c_\mu, A)$ for $x \ge \beta$ if $|u(x,t)| < \tilde{E}\,\varepsilon^{\ell}$, $\varepsilon < \varepsilon^*$.

If, instead of (4.11) we use this inequality in the parts (i) and (ii) of the proof of Theorem 4.1, the differential inequality

$$L_u \le L\phi, \qquad x > \beta, \qquad x \neq 0, \qquad t \in (0,T]$$

with a replaced by A, holds if $|u(x,t)| \le \phi(x)$ on $[0,T]$. The proof of part (iii) in this case is given, using the same arguments as in the proof of Theorem 4.1. Putting the pieces together we may now state an extension of Theorem 4.2 for the class $V_0^*(E,\ell,\varepsilon)$ of initial functions (χ,ψ) which satisfy for some δ_0, $K_0 > 0$

(i) $\chi \in BC(\mathbb{R} \to \mathbb{R})$, $\psi \in BC(\mathbb{R} \to \mathbb{R}^n)$ and ψ is Hōlder continuous.

(ii) $\|\chi\|_{[-\delta_0,\infty)} \le E\,\varepsilon^{\ell}$, $\|\psi\|_{[-\delta_0,\infty)} = o(E\varepsilon^{\ell})$, $(\varepsilon \downarrow 0)$.

(iii) $\|(\chi,\psi)\|_{\mathbb{R}} \le K_0$.

THEOREM 4.3. *Suppose for the general equations* (1.3)

$$K_G \inf\{c_\mu \mid 0 < \mu < -M(J_0)\} < A.$$

Then there exist numbers ε^*, E *and* $\ell > 0$ *such that for* $\varepsilon \in (0,\varepsilon^*)$ *and any* $(\chi,\psi) \in V_0^*(E,\ell,\varepsilon)$ *the solution of* (1.3), (1.5) *satisfies*

$$|u(x,t)| \le E\,\varepsilon^{\ell}, \qquad x \ge 0,\ t \ge 0.$$

COROLLARY 4.2. *Travelling wave solutions of* (1.3) *for* $\varepsilon = 1$, *which are shifted far enough to the left, are blocked for small* $\varepsilon > 0$ *under the condition* (4.29).

PROOF OF LEMMA 4.1. Choose numbers ℓ and n such that

$$(4.32) \qquad 0 < 2\ell < n < 1 - 3\ell < \frac{2}{3}.$$

Introduce for $B_1 \in (0, \frac{Ka}{1+2\delta})$ the numbers $\alpha_i = B_1(1+\delta 2^{-2+i})\varepsilon^{\ell}/a$ for $i = 1,2,3$. Observe that $\alpha_1 < \alpha_2 < \alpha_3 < K$. Let $M > K$ then we introduce the cubic

(4.33) $\quad g(u) = (u-\alpha_1)(u-\alpha_2)(M-u).$

One may choose ε^* so small such that for $0 < \varepsilon < \varepsilon^*$

(4.34) $\quad g(u) + au \geq B_1\varepsilon^{\ell}, \qquad \alpha_1 \leq u \leq \alpha_3.$

Moreover there exists a number $B_2 > 0$ such that

(4.35) $\quad g(u) \geq \delta B_2\varepsilon^{2\ell}, \qquad \alpha_3 \leq u \leq K.$

The building bricks for $\phi(x)$ are solutions $\phi_1(x), \phi_2(x)$ of the equations

$$\begin{aligned} &\varepsilon\phi_1'' + g(\phi_1) = 0, \\ &\varepsilon^n\phi_2'' + g(\phi_2) = 0, \end{aligned} \tag{4.36}$$

respectively. For small ε their phase portraits are sketched in fig. 4.3 below.

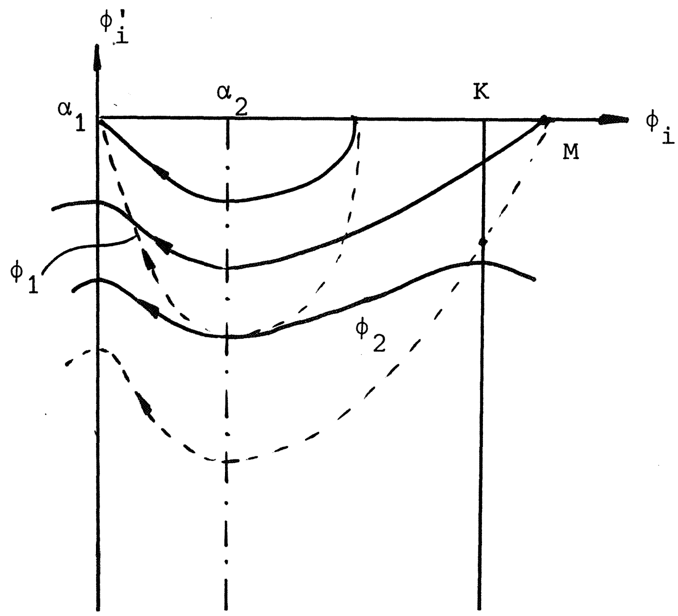

fig. 4.3.

The dotted lines belong to ϕ_1 while the solid lines correspond to the equation $(4.36)^2$ (cf. [8] where we derived the phase portrait for similar equations, but for n = 0). The functions ϕ_1 and ϕ_2 are chosen such that

$$\phi_1(0) = \alpha_2, \qquad \phi_1'(0) < 0, \qquad \phi_1(\infty) = \alpha_1$$

$$\phi_2(0) = \alpha_2, \qquad \phi_2'(0) = \phi_1'(0).$$

A sufficient condition for this to be possible is

$$\varepsilon^{1-n} < -\frac{\int_{\alpha_1}^{\alpha_2} g(u)du}{\int_{\alpha_2}^{M} g(u)du} = O(\varepsilon^{3\ell}), \qquad (\varepsilon\downarrow 0)$$

which is true for small ε, by (4.32). Next we shall estimate the values of $\phi_2'(x)$ while $\alpha_2 \le \phi_2 \le K$. Since we consider the functions ϕ_i, $i = 1,2$ only when they are strictly decreasing we may introduce $\phi = \phi_i$ as independent variable, leading to

$$(4.37) \qquad \begin{aligned} &P_1P_{1,\phi} + \varepsilon^{-1}g(\phi) = 0, \\ &P_2P_{2,\phi} + \varepsilon^{-n}g(\phi) = 0, \end{aligned}$$

where $P_i = \phi_i'$. Integration of $(4.37)^1$ over (α_1,α_2) if $P_1(\alpha_1) = 0$ yields

$$(4.38) \qquad \tfrac{1}{2} P_1^2(\alpha_2) = -\varepsilon^{-1}\int_{\alpha_1}^{\alpha_2} g(\phi)d\phi \ge K_1\varepsilon^{3\ell-1}$$

for some $K_1 > 0$. Integration of $(4.37)^2$ over (α_2,ϕ), $\alpha_2 < \phi < K$ under the condition that $P_1(\alpha_2) = P_2(\alpha_2)$ yields

$$\begin{aligned} \tfrac{1}{2}P_2^2(\phi) &= \tfrac{1}{2} P_1^2(\alpha_2) - \varepsilon^{-n}\int_{\alpha_2}^{\phi} g(\zeta)d\zeta = \tfrac{1}{2} P_1^2(\alpha_2) + O(\varepsilon^{-n}) \\ &\ge K_2\,\varepsilon^{3\ell-1}, \qquad \alpha_2 < \phi < K, \end{aligned}$$

for some $K_2 > 0$, not depending on ε, since by (4.32) $n < 1 - 3\ell$.
Define

$$(4.40)\qquad \phi(x) = \begin{cases} \phi_1(x)\ , & x > 0 \\ \phi_2(x)\ , & x < 0 \text{ while } \phi_2 \le K. \\ K\ , & \text{elsewhere.} \end{cases}$$

We shall exploit the fact that ϕ_1'' and ϕ_2'' have different but constant sign:

$$(4.4)\qquad \begin{aligned} &\phi_1''(x) \ge 0, && x > 0 \\ &\phi_2''(x) \le 0, && x < 0 \text{ while } \phi_2 \le K, \end{aligned}$$

in order to verify the inequalities (4.9). Let us first estimate the place $x = \alpha(\varepsilon)$ where $\phi_2(x) = K$. From

$$K = \phi_2(\alpha) = \phi_2(0) - \int_{\beta_1}^{0} \phi_2'(x)\,dx$$

it follows by (4.39) that

$$-R\varepsilon^{\frac{1}{2}(1-3\ell)} \le \alpha \le 0$$

for some $R > 0$, independent of ε. Thus choose

$$\mu_2 = \tfrac{1}{2}(1-3\ell)$$

and with β from $\phi(\beta) = \frac{B_1}{a}(1+2\delta)\varepsilon^{\ell}$ we shall now verify the inequalities (4.9)

(i) <u>on $(\beta,\infty)\setminus\{0\}$:</u>

$$e_\varepsilon(x)\phi'' - a\phi = \begin{cases} -g(\phi_1) - a\phi_1, & x > 0 \\ (1-\varepsilon^n)\phi_2'' - g(\phi_2) - a\phi_2, & x \in (\beta,0) \end{cases}$$

$$\le -B_1\varepsilon^{\ell} \text{ by (4.34) and (4.41)}^2$$

(ii) <u>on (α,β):</u>

$$\phi'' - a\phi = (1-\varepsilon^n)\phi_2'' - g(\phi_2) - a\phi_2$$

$$\le -\frac{1-\varepsilon^n}{\varepsilon^n}\, g(\phi_2) \qquad \text{by } (4.36)^2$$

$$\le (1-\varepsilon^{-n})\,\delta B_2 \varepsilon^{2\ell} \qquad \text{by } (4.35)^2$$

$$\le -\,\delta B_2\, \varepsilon^{-\mu_1}$$

if $\mu_1 \in (0,n-2\ell)$ is chosen such that $\varepsilon^{2\ell} - \varepsilon^{2\ell-n} \le -\varepsilon^{-\mu_1}$. For example if $\mu_1 \le n - 2\ell - \kappa$ and $(\varepsilon^*)^{-\kappa}(1-\varepsilon^{*n}) \ge 1$.

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