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DIPS AND SLIDINGS OF THE FORCED VAN DER POL RELAXATION OSCILLATOR

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by

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ABSTRACT

Local approximations of solutions of the periodically forced Van der Pol relaxation oscillator are constructed with singular perturbation techniques. In this report we deal exclusively with specific solutions that for some period of time follow the unstable branch of an equation being a local approximation of the oscillator. This report is meant as a supplement to Report TW 207.

KEY WORDS & PHRASES: *Van der Pol equation, relaxation oscillation, singular perturbations*

1. INTRODUCTION

In this paper we consider the Van der Pol equation with a sinusoidal forcing term

$$(1.1) \qquad \frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = (\alpha\nu+\beta)\cos t$$

for large values of the parameter ν and with $0 < \alpha < 2/3$. In a preceding report [1] we constructed asymptotic approximations of subharmonic solutions with period $T = 2\pi(2n-1)$. In order to deal with other type of solutions, as described by LEVI [2] for a modified Van der Pol oscillator, we first have to investigate a specific behaviour of the solution that may occur around the line $x=1$. Usually when the solution passes the line in a downward direction, it crosses swiftly the unstable region $|x| < 1$. However, it is observed in electronic experiments [3] and also analyzed rigorously for the modified Van der Pol oscillator [2] that the solution, instead of crossing the unstable region, may just dip and return to the region $x > 1$. Another possibility is that the solution continues in a slow motion for some time in the unstable region but then abruptly makes a sliding and approaches quickly the value $x = -2$. In this report we analyze these two cases as well as a critical case in which the solution stays over a full period of the forcing term within the unstable region.

In Figure 1 we show the characteristic regions of the x,t-plane where the solution may exhibit various types of behaviour. In the following sections we will analyze the local behaviour asymptotically.

2. ASYMPTOTIC SOLUTIONS FOR REGION A_n

The solution passes the region A_n in the time interval (t_{n-1},t_n) and is expanded as

$$(2.1) \qquad x_n(t;\nu) = x_{n0}(t) + \nu^{-1}x_{n1}(t) + \dots .$$

Substituting (2.1) into (1.1) and equating equal powers of ν we obtain a recurrent system of differential equations for $x_{nk}(t)$:

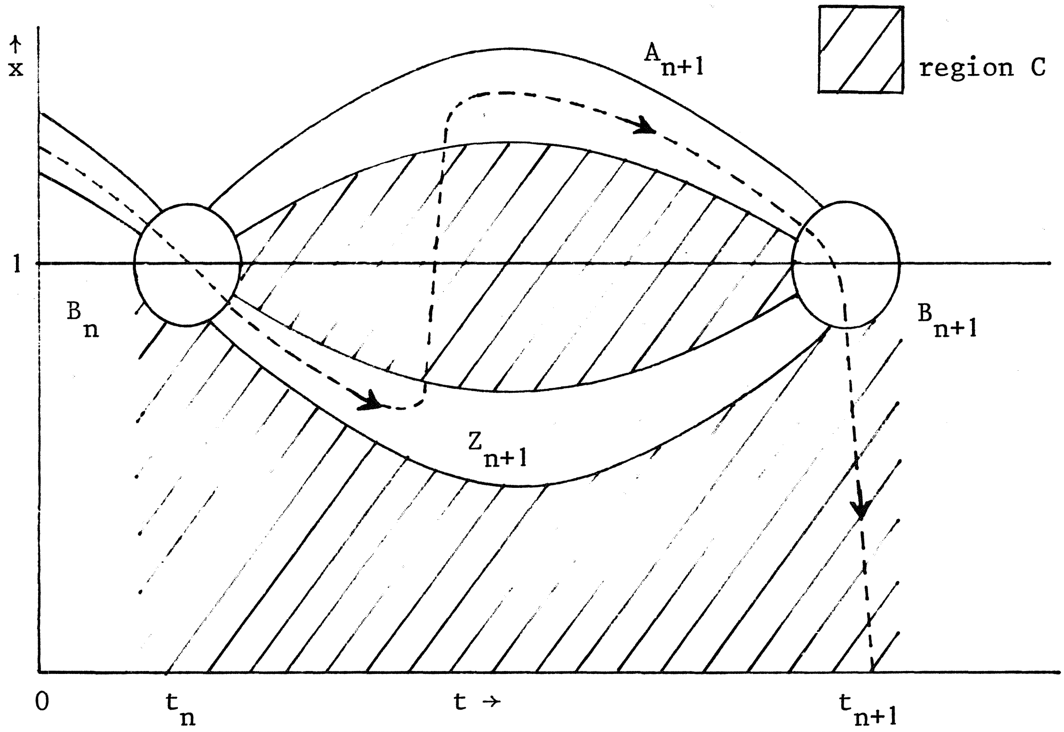

Fig. 1. Characteristic regions

(2.2a) $$(x_{n0}^2 - 1)\frac{dx_{n0}}{dt} = \alpha\cos t$$

(2.2b) $$(x_{n0}^2-1)\frac{dx_{n1}}{dt} + 2x_{n0}\frac{dx_{n0}}{dt}x_{n1} = -\frac{d^2x_{n0}}{dt^2} - x_{n0} + \beta\cos t, \dots .$$

Integration yields

(2.3a) $$\frac{1}{3}x_{n0}^3 - x_{n0} = \alpha\sin t + C_0^{(n)},$$

(2.3b) $$(x_{n0}^2-1)x_{n1} = -\frac{dx_{n0}}{dt} - \int_{t_{n-1}}^{t} x_{n0}(\bar{t})d\bar{t} + \beta\sin t + C_1^{(n)}.$$

Since for $t \downarrow t_{n-1}$ and $t \uparrow t_n$ the solution approaches the line $x = 1$, we have

(2.4) $$C_0^{(n)} = \alpha - \frac{2}{3}.$$

Consequently, the solution of (2.3a) above the line $x = 1$ reads

(2.5) $$x_{n0}(t) = 2\cos\{\tfrac{1}{3}\arccos(\tfrac{3}{2}\alpha\sin t + \tfrac{3}{2}\alpha - 1)\}.$$

As $t \uparrow t_n$ (2.1) behaves asymptotically as

(2.6a) $$x \approx 1 - \tfrac{1}{2}\sqrt{2\alpha}(t-t_n) + \nu^{-1}K_n/(t-t_n),$$

(2.6b) $$K_n = -\tfrac{1}{2} + (-c_1^{(n)} + \beta + I)/\sqrt{2\alpha},$$

(2.6c) $$I = \int_{t_{n-1}}^{t_n} x_{n0}(t)\,dt.$$

In the preceding report [1] we analyzed the case $K_n \neq 0$ and independent of ν. Now we assume that

(2.7) $$K_n(\nu) = k\exp(-a\nu)$$

with $|k|$ having an upperbound independent of ν. This choice of K_n will lead to a local behaviour which in our terminology we denote by dipping and sliding of the solution. If for a moment we take $k = 0$, the expansion (2.1) remains regular and at the point $t = t_n$ the solution will smoothly switch to a different expansion with a leading term

(2.8) $$\underline{x}_{n+1,0}(t) = 2\cos[\tfrac{1}{3}(\arccos\{\tfrac{3}{2}\alpha\sin t + \tfrac{3}{2}\alpha - 1\} + 4\pi)]$$

being the second branch of (2.3a) with $c_0^{(n)}$ given by (2.4). This solution will hold asymptotically for some region Z_{n+1} over some time interval (t_n, t^*) with $t_n \le t^* \le t_{n+1}$, where t^* depends on the value of a in (2.7).

We will deal with a regular asymptotic solution $\hat{x}(t;\nu)$ of the form (2.1), which has two distinct representations: (2.2) - (2.5) with $c_1^{(n)}$ such that $K_n = 0$ for $t < t_n$ and for $t > t_n$ a representation given by (2.8) and an equation for $\underline{x}_{n+1,1}(t)$ of the type (2.3b) with

(2.9) $$\underline{c}_1^{(n+1)} = \beta - \tfrac{1}{2}\sqrt{2\alpha}$$

We will account for the fact that $k \neq 0$ in (2.7) by considering this as a perturbation of the regular asumptotic solution $\hat{x}(t;\nu)$ starting from a neighbourhood of $t = t_n$.

3. ASYMPTOTIC SOLUTION FOR REGION B_n

The local behaviour of the solution in a neighbourhood of $(x,t) = (1,t_n)$ is analyzed by introduction of the local variables (v,ξ):

$$t = t_n + \xi\nu^{-1/2} \tag{3.1a}$$

$$x = \hat{x}(t_n+\xi\nu^{-1/2};\nu) + v(\xi)\delta(\nu), \tag{3.1b}$$

with $\delta(\nu) = \nu^{-1/2}\exp(-a\nu)$ and with $\hat{x}$ being the regular expansion valid in a $O(1)$-neighbourhood of $t = t_n$ as described in the foregoing section. Substitution in (1.1) yields the following equation for $v(\xi)$ after equating terms of order $O(\delta(\nu)\nu)$:

$$\frac{d^2v}{d\xi^2} - \xi\sqrt{2\alpha}\,\frac{dv}{d\xi} - \sqrt{2\alpha}\,v = 0. \tag{3.2}$$

Furthermore, from (2.6a) it follows that v must satisfy

$$v(\xi) \approx k/\xi \qquad \text{for } \xi \to -\infty. \tag{3.3}$$

The function

$$v = -2^{1/2}k\sqrt[4]{\alpha/2}\,\exp(1/2\,\sqrt{\alpha/2}\,\xi^2)D_{-1}(-2^{1/2}\,\sqrt[4]{\alpha/2}\,\xi) \tag{3.4}$$

meets these requirements. In (3.4) $D_\mu(z)$ denotes the so-called parabolic cylinder function of order μ (see WHITTAKER and WATSON [4, p.347]) with

$$D_\mu(z) = \exp(-\tfrac{1}{4}\,z^2)z^\mu\{1-\tfrac{1}{2}\mu(\mu-1)z^{-2}+\ldots\} \tag{3.5}$$

for $z \to \infty$. On the other hand, as for $z \to -\infty$

$$D_\mu(z) = \exp(-\tfrac{1}{4}z^2)z^\mu\{1-\tfrac{1}{2}\mu(\mu-1)z^{-2}+\ldots\} \tag{3.6}$$

$$-\sqrt{2\pi}\,\Gamma(-\mu)^{-1}\exp(\tfrac{1}{4}z^2+\mu\pi i)z^{-\mu-1}\{1+\tfrac{1}{2}(\mu+1)(\mu+2)z^{-2}+\ldots\},$$

we find that for $\xi \to \infty$

$$v(\xi) \approx -2k\sqrt[4]{\alpha/2}\,\sqrt{\pi}\,\exp(\sqrt{\alpha/2}\,\xi^2). \tag{3.7}$$

From this asymptotic behaviour we may conclude that for $k \neq 0$ the perturbation will grow rapidly.

4. ASYMPTOTIC SOLUTION FOR REGION Z_{n+1}

As we pointed out in Section 2 we assume that the solution consists of two parts the regular part $\hat{x}$ and a perturbation due to the fact that $k \neq 0$, so

$$x = \hat{x}(t;\nu) + V(t,\nu). \tag{4.1}$$

Substitution in (1.1) yields for the leading part V_0 of $V(t;\nu)$

$$\frac{d^2V_0}{dt^2} + \nu\frac{d}{dt}\{(\underline{x}^2_{n+1,0}(t)-1)V_0\} = 0. \tag{4.2}$$

Integration gives

$$\frac{dV_0}{dt} + \nu\{\underline{x}^2_{n+1,0}(t)-1\}V_0 = -2k\sqrt{\frac{\alpha}{2}}\,\nu^{-1/2}\delta(\nu), \tag{4.3}$$

where the right-hand side follows from matching conditions between $V_0(t)$ and $v(\xi)$ given by (3.4) and (3.7). From the conditions we also derive the value of the integration constant in the solution

$$V_0(t) = \exp\{-\nu A(t)\}[C - 2k\sqrt{\frac{\alpha}{2}}\,\nu^{1/2}\delta\int_{\bar{t}=t_n}^{\bar{t}=t}\exp\{\nu A(\bar{t})\}d\bar{t}],$$

$$A(t) = \int_{t_n}^{t}\{\underline{x}^2_{n+1,0}(\bar{t})-1\}d\bar{t}. \tag{4.4}$$

It is easily verified that we must have

$$C = -k \sqrt[4]{\frac{\alpha}{2}}\, \nu^{1/2} \delta \sqrt{\pi}. \tag{4.5}$$

Let the constant a of (2.7) be such that for some $t = t^*$

$$-A(t^*) = a, \qquad t_n < t^* < t_{n+1}. \tag{4.6}$$

Then, as t approaches t^*, the asymptotic solution (4.1) looses its validity and the solution enters the boundary layer region C.

5. ASYMPTOTIC SOLUTION FOR REGION C

We introduce the local coordinate

$$\eta = (t-t^*)\nu \tag{5.1}$$

and assume that the solution can be expanded locally as

$$x = W_0(\eta) + \nu^{-1}W_1(\eta) + \nu^{-2}W_2(\eta) + \dots . \tag{5.2}$$

Applied to equation (1.1) this yields the recurrent system

$$\frac{d^2W_0}{d\eta^2} + (W_0^2-1)\frac{dW_0}{d\eta} = 0, \tag{5.3a}$$

$$\frac{d^2W_1}{d\eta^2} + (W_0^2-1)\frac{dW_1}{d\eta} + 2W_0W_1\frac{dW_0}{d\eta} = \alpha\cos t^*, \dots . \tag{5.3b}$$

From (4.1) it follows that for $\xi \to -\infty$, W_i have to satisfy the matching conditions

$$W_0 \approx \underline{x}_0^* - k\sqrt{\pi\alpha/2}\,\exp\{-((\underline{x}_0^*)^2-1)\eta\}, \tag{5.4a}$$

$$W_1 \approx \frac{\alpha\cos t^*}{(\underline{x}_0^*)^2-1} + \frac{1}{(\underline{x}_0^*)^2-1}\left\{\frac{-\alpha\cos t^*}{(\underline{x}_0^*)^2-1} - \int_{t_n}^{t^*} \underline{x}_{n+1,0}^2(t)\,dt + \beta\sin t^* + \underline{c}_1^{(n+1)}\right\}, \tag{5.4b}$$

where $\underline{x}_0^* = \underline{x}_{n+1,0}(t^*)$ given by (2.8). Using (5.4a) we obtain, integrating (5.3a) once,

$$(5.5) \qquad \frac{dW_0}{d\eta} + \frac{1}{3}(W_0-\underline{x}_0^*)(W_0-y_0^*)(W_0-\bar{x}_0^*) = 0,$$

where $y_0^* < -1$ and $\bar{x}_0^* > 1$ are the two other roots of the algebraic equation

$$(5.6) \qquad \frac{1}{3}W_0^3 - W_0 = \frac{1}{3}(\underline{x}_0^*)^3 - \underline{x}_0^*,$$

Carrying out the integration of (5.5), while using (5.4a), we obtain

$$(5.7) \qquad \frac{\ln|W_0-\underline{x}_0^*|}{(\underline{x}_0^*)^2-1} + \frac{\ln|W_0-y_0^*|}{(y_0^*)^2-1} + \frac{\ln|W_0-\bar{x}_0^*|}{(\bar{x}_0^*)^2-1} = -\eta + \frac{\ln|-k\sqrt{\pi\alpha/2}|}{(\underline{x}_0^*)^2-1}$$

Using (5.4b) we may replace (5.3b) by

$$(5.8) \qquad \frac{dW_1}{d\eta} + (W_0^2-1)W_1 = \alpha\eta\cos t^* - \int_{t_n}^{t^*} \underline{x}_{n+1,0}(t)dt + \beta\sin t^* + \underline{C}_1^{(n+1)}.$$

Consequently, for $k > 0$ and $\eta \to \infty$ the coefficients of (5.2) behave as

$$(5.9a) \qquad W_0 \approx y_0^* + \exp[-\{(y_0^*)^2-1\}\eta + \frac{(\underline{x}_0^*)^2-1}{(y_0^*)^2-1}\ln|-k\sqrt{\pi\alpha/2}|],$$

$$(5.9b) \qquad W_1 \approx \frac{\alpha\cos t^*}{(y_0^*)^2-1}\eta + \frac{1}{(y_0^*)^2-1}\{\frac{-\alpha\cos t^*}{(y_0^*)^2-1} - \int_{t_n}^{t^*} \underline{x}_{n+1,0}(t)dt + \beta\sin t^* + \underline{C}_1^{(n+1)}\}.$$

In this way the system arrives at a region A, as described in report [1], where a two-variable expansion for the solution can be made. The asymptotic behaviour (5.9) is such that the present boundary layer solution for region C matches this two variable expansion.

For $k < 0$ and $\eta \to \infty$ the coefficients of (5.2) behave as (5.9) with y_0^* replaced by $\bar{x}_0^*$.

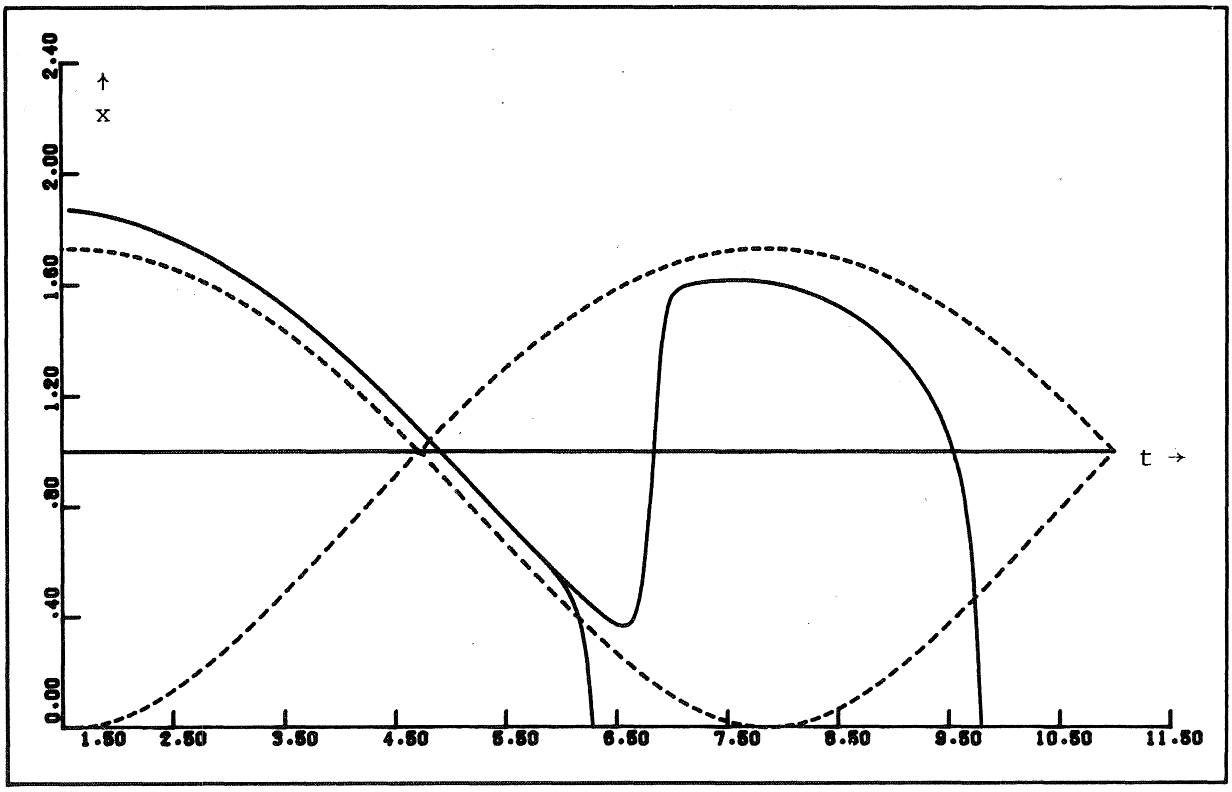

Fig. 2. Two trajectories for $\alpha = 1/3$, $\beta = 0$, $\nu = 15$:

$x(\pi/2) = 1.8711914$, $x'(\pi/2) = -0.0521795$ (dipping)

$x(\pi/2) = 1.8711901$, $x'(\pi/2) = -0.0521795$ (sliding)

6. ASYMPTOTIC SOLUTION FOR REGION A_{n+1}

The asymptotic expansion for this region takes the form (2.1) with leading term $\bar{x}_{n+1,0}(t)$ identical to (2.5) and with a second term satisfying

$$(\bar{x}^2_{n+1,0}-1)\bar{x}_{n+1,1} = -\frac{d\bar{x}_{n+1,0}}{dt} - \int_{t^*}^{t} \bar{x}_{n+1,0}(\bar{t})d\bar{t} + \beta \sin t + \bar{C}_1^{(n+1)}. \tag{6.1}$$

Consequently, this asymptotic solution matches the solution of region C, if

$$\bar{C}_1^{(n+1)} = \underline{C}_1^{(n+1)} - \int_{t_n}^{t^*} \bar{x}_{n+1,0}(\bar{t})d\bar{t}. \tag{6.2}$$

Using (2.9), (6.1) and (6.2) we deduce that for $t \uparrow t_{n+1}$ the expansion for region (6.1) behaves as

$$x \approx 1 - \frac{1}{2}\sqrt{2\alpha}(t-t_{n+1}) + \nu^{-1}K_{n+1}/(t-t_{n+1}) \tag{6.3}$$

with

$$K_{n+1} = \{\int_{t_n}^{t^*} \underline{x}_{n+1,0}(t)dt + \int_{t^*}^{t_{n+1}} \bar{x}_{n+1,0}(t)dt\}/\sqrt{2\alpha}. \tag{6.4}$$

Clearly, the solution now enters the region B_{n+1} in a regular way as K_{n+1} is positive and bounded away from zero. This case is analyzed in [1, Sections 4 and 5]: Starting from B_{n+1} the solution crosses the unstable interval $-1 < x < 1$ and will arrive at the point $x = -2$, where it takes up with the two-variable asymptotic solution of region A described in [1, Section 2].

7. THE CRITICAL CASE

Let us now consider the case

$$K_n(\nu)\exp\{-\nu A(t_{n+1})\} \to 0 \qquad \text{as } \nu \to \infty. \tag{7.1}$$

Then the solution follows the branch (2.8) within the region Z_{n+1} until it arrives in a neighbourhood of the point $(x,t) = (1,t_{n+1})$, where

$$x \approx 1 + \frac{1}{2}\sqrt{2\alpha}(t-t_{n+1}) - \frac{(1+H/\sqrt{2\alpha})\nu^{-1}}{(t-t_{n+1})}, \tag{7.2a}$$

$$H = \int_{t_n}^{t_{n+1}} \underline{x}_{n+1,0}(t)dt. \tag{7.2b}$$

For a region B_{n+1} of order $O(\nu^{-1/2})$ in a neighbourhood of $(x,t) = (1,t_{n+1})$ we introduce local variables v and ξ:

$$(7.3ab) \qquad x = 1 + v(\xi)\nu^{-1/2}, \qquad t = t_{n+1} + \xi\nu^{-1/2}.$$

Substituting (7.3ab) into (1.1) and multiplying this equation with $\nu^{-1/2}$ we obtain, after taking the limit $\nu \to \infty$:

$$(7.4) \qquad \frac{d^2v_0}{d\xi^2} + 2v_0\frac{dv_0}{d\xi} = \alpha\xi.$$

On the other hand, because of (7.2), we have the matching condition

$$(7.5) \qquad v_0(\xi) \approx \frac{1}{2}\,\xi\sqrt{2\alpha} - (1+H/\sqrt{2\alpha})\xi^{-1}.$$

A solution of (7.4) satisfying (7.5) exists and has the form

$$(7.6) \qquad v_0(\xi) = \hat{a}D_b'(\hat{a}\xi)/D_b(\hat{a}\xi), \qquad \hat{a} = \sqrt[4]{2\alpha} \quad \text{and} \quad b = H/\hat{a}^2.$$

Since b is positive the parabolic cylinder function $D_b(\hat{a}\xi)$ will have at least one zero. Let $\xi = \xi_0$ be the point where the smallest zero arises, then as $\xi \uparrow \xi_0$ the local solution behaves as

$$(7.7) \qquad v(\xi) = (\xi-\xi_0)^{-1} + \frac{1}{3}\,a^2(\frac{1}{4}\,a^2\xi_0^2 - b - \frac{1}{2})(\xi-\xi_0).$$

From this result we conclude that the solution leaves the region B_{n+1} in a way identical to the regular case as described in [1, Section 4]. Thus, we have completed our analysis of the critical case, as the solution passes a well-known boundary layer region on its path to the value x = -2 in exactly the same way as in [1].

8. THE TRANSITIONAL CASE

Finally, we consider the case where

$$(8.1) \qquad A(t_{n+1}+\xi\nu^{-1/2}) \approx -a + \frac{1}{2}\nu^{-1}\ell n\nu + \{\frac{1}{2}\sqrt{2\alpha}\;\xi^2 - 2(1+H/\sqrt{2\alpha})\ell n\xi\}\nu^{-1}.$$

This forms the transition from the cases where $t^* < t_{n+1}$ to the critical case. We analyze the local behaviour of the solution near $(x,t) = (1,t_{n+1})$ by introducing the local variables

$$(8.2ab) \qquad x = 1 + v(\xi;\nu)\nu^{-1/2}, \qquad t = t_{n+1} + \xi\nu^{-1/2}.$$

The following matching condition holds for this local solution

$$(8.3) \qquad v \approx \frac{1}{2}\hat{a}\xi - k\sqrt[4]{\alpha/2}\,\sqrt{\pi}\,\xi^{2+2b}\exp(-\frac{1}{2}\hat{a}\xi^2) - (1+b)\xi^{-1}$$

with $\hat{a}$ and b satisfying (7.6). The limit function $v_0(\xi) = \lim_{\nu\to\infty} v(\xi,\nu)$, satisfying equation (7.4) and matching relation (8.3), becomes a transitional expression:

$$(8.4) \qquad v_0 = \hat{a}\,\frac{D_b'(\hat{a}\xi) - CD_b'(-\hat{a}\xi)}{D_b(\hat{a}\xi) + CD_b(-\hat{a}\xi)}$$

with

$$(8.5) \qquad C = -k\pi\sqrt[4]{2\alpha}/\{\hat{a}^{2b+3}\Gamma(-b)\}.$$

This solution is singular for $\xi = \xi_0$ satisfying

$$(8.6) \qquad D_b(\hat{a}\xi_0) + CD_b(-\hat{a}\xi_0) = 0.$$

Then for $\xi \uparrow \xi_0$ we have

$$(8.7) \qquad v_0 \approx (\xi-\xi_0)^{-1} + \frac{1}{3}\hat{a}^2(\frac{1}{4}\hat{a}^2\xi_0^2 - b - \frac{1}{2})(\xi-\xi_0).$$

Consequently the solution arrives in the boundary layer region in a similar manner as for the critical case and the regular case with K_{n+1} positive and independent of ν.

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