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EXISTENCE AND UNIQUENESS FOR A NONLINEAR DIFFUSION PROBLEM ARISING IN NEUROPHYSIOLOGY

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Existence and uniqueness for a nonlinear diffusion problem arising in neurophysiology

by

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ABSTRACT

In this paper we discuss existence and uniqueness for a nonlinear reaction-diffusion problem which arises in neurophysiology as a model for impulse propagation along nonuniform nerve axons.

KEY WORKS & PHRASES: *Existence, uniqueness, reaction-diffusion equations, neurophysiology*

1. INTRODUCTION

In our study of propagation of potential waves along nonuniform nerve axons we examined in two papers [3] [4] the qualitative behaviour of solutions of the following system of equations.

$$(1.1)\qquad \begin{aligned} u_t &= e_\varepsilon(x)u_{xx} + F(u,w), \\ w_t &= G(u,w), \qquad x \in \mathbb{R} \setminus \{0\}, \quad t > 0 \end{aligned}$$

where

$$(1.2)\qquad e_\varepsilon(x) = \begin{cases} 1 & : \quad x \le 0 \\ \varepsilon & : \quad x > 0, \end{cases}$$

where (u,w) takes on values in $\mathbb{R} \times \mathbb{R}^n$ for some $n > 0$, and F and G are Lipschitz continuous in u and w. We treated (1.1) in the framework of an initial value problem with initial conditions

$$(1.3)\qquad \begin{aligned} u(x,0) &= \chi(x), \\ w(x,0) &= \psi(x), \end{aligned} \qquad x \in \mathbb{R}.$$

for a bounded continuous function χ while ψ is bounded and Hölder continuous.

The variable u(x,t) represents the departure of the transmembrane potential from its resting value at time t and place x on the axon. The auxiliary variable w describes the process of the transport of ions (K^+,Na^+,Cl^-) across the membrane which can be regarded as the "engine" for impulse conduction. The function $e_\varepsilon(x)$ describes the nonuniformity of the nerve axon. A small value of ε means a large increase of cross-section area for increasing x.

In this report our main goal will be to verify that a unique classical solution of (1.1) and (1.3) exists. This will be done in Section 2 under the assumption of apiori boundedness, where we shall postpone the proofs of intermediate (technical) results to Appendix A. In another appendix, Appendix B, we shall verify the apiori boundedness for those examples which are of interest in [3] and [4].

2. EXISTENCE AND UNIQUENESS

We start with a few definitions and notations.

DEFINITION 2.1. Let for $m,n \in \mathbb{N}$ $D \subset \mathbb{R}^m$ and $\psi : D \to \mathbb{R}^n$ where $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$. Then we shall call ψ bounded if and only if

$$(2.1) \qquad \|\psi\|_D \equiv \sup_{x \in D} \sum_{i=1}^{n} |\psi_i(x)| < \infty.$$

NOTATION 2.1. Throughout this Section we shall use the notation $C^{k+\alpha, m+\beta}(Q \to R)$, $k,m \in \{0,1,2,\ldots\}$; $\alpha, \beta \in [0,1)$ for the set of functions $u = u(x,t)$, defined on $Q \subset \mathbb{R} \times \mathbb{R}^+$ and taking values in $R \subset \mathbb{R}^\ell$, $\ell \in \mathbb{N}$ for which $\frac{\partial^k u}{\partial t^k}$ and $\frac{\partial^m u}{\partial x^m}$ are continuous and where $\frac{\partial^k u}{\partial t^k}$ is Hölder continuous with exponent α, with respect to t if $\alpha > 0$ and $\frac{\partial^m u}{\partial x^m}$ is Hölder continuous in x with exponent β, if $\beta > 0$. $C(Q \to R)$ for the set of functions $u : Q \to R$ where $R \subset \mathbb{R}^\ell$ and $Q \subset \mathbb{R}$ or $Q \subset \mathbb{R} \times \mathbb{R}^+$, which are continuous on Q. $BC(Q \to R)$ for the set of functions $u \in C(Q \to R)$ which are bounded.

NOTATION 2.2. For a set $\mathcal{D} \subset \mathbb{R}^m$, $m \in \mathbb{N}$ we shall denote the closure of $\mathcal{D}$ by $\bar{\mathcal{D}}$.

In this section we shall investigate existence and uniqueness for the initial value problem

$$(2.2) \qquad \begin{cases} u_t = e_\varepsilon(x) u_{xx} + F(u,w) \\ \\ w_t = G(u,w), \end{cases} \qquad x \in \mathbb{R} \setminus \{0\}, \quad t > 0.$$

$$(2.3) \qquad u(x,0) = \chi(x), \quad w(x,0) = \psi(x), \quad x \in \mathbb{R}$$

as introduced in Section 1 and denoted here as *Problem* P_1.

DEFINITION 2.2. The vector function $(u,w):\mathbb{R} \times [0,T) \to \mathbb{R} \times \mathbb{R}^n$ is called a *classical solution* of P_1 on $[0,T)$ if and only if

(i) $u \in BC(\mathbb{R} \times [0,T) \to \mathbb{R})$
$u_x \in C(\mathbb{R} \times (0,T) \to \mathbb{R})$
$u_{xx}, u_t \in C(\mathbb{R}\setminus\{0\} \times (0,T) \to \mathbb{R})$
$w, w_t \in BC(\mathbb{R} \times [0,T) \to \mathbb{R}^n)$

(ii) (u,w) satisfies (2.2) and (2.3).

The plan of this section is as follows. First we reformulate P_1 as a set of integral equations, denoted as problem P_2. Using contraction arguments we shall prove local existence and uniqueness for P_2. Then we shall show by means of a regularization result due to Ladyženskaja et.al. [2], that this solution of P_2 is also a solution of P_1. Finally, using an apriori estimate for solutions of P_1 we shall extend the local existence to global existence.

In [6] Schonbek discusses among other things, questions of existence uniqueness and regularity for the FitzHugh-Nagumo equations in the quarter-plane $\{(x,t)|x > 0, t > 0\}$ with bounded Neumann boundary data at $x = 0$. However, in the present situation $u_x(0+,t)$ may behave as $t^{-\frac{1}{2}}$ for $t \downarrow 0$ (see Proposition (2.1)). We shall therefore give the proof of existence and uniqueness for P_1 in full detail, in spite of the fact that most of the techniques used, are similar to those of [6]. A more detailed application of these techniques can be found in [5]. In order not to disturb the main argument of this Section we shall give the proofs of our intermediate results, called *propositions*, in Appendix A.

For the derivation of the integral equations in P_2, we shall make use of the Green function $U(x,\xi;t)$ for the Neumann problem in the quarter plane $\{(x,t)|x > 0, t > 0\}$ for the heat equation, given by

$$U(x,\xi;t) = K(x-\xi,t) + K(x+\xi,t)$$

where

$$K(z,t) = \frac{1}{2\sqrt{\pi t}} \exp\left[-\frac{z^2}{4t}\right].$$

Note that $U(-x,-\xi;t) = U(x,\xi;t) = U(x,-\xi;t)$ for all $x \in \mathbb{R}$, $t > 0$. In what follows we shall use the notation

$$H(u)(t) = u_x(0,t)$$
$$f(u,w)(x,t) = F(u(x,t),w(x,t)),$$
$$g(u,w)(x,t) = G(u(x,t), w(x,t)).$$

If we write

$$R_{\pm}[u,w](x,t) = \pm \int_0^t \int_0^{\pm\infty} U(x,\xi;e_\varepsilon(x)(t-\tau))f(u,w)(\xi,\tau)d\xi d\tau$$

$$L_{\pm}\, h(x,t) = \pm\, e_\varepsilon(x) \int_0^t U(x,0;e_\varepsilon(x)(t-\tau))\, h(\tau)d\tau$$

$$S_{\pm}\,(x,t) = \pm \int_0^{\pm\infty} U(x,\xi;e_\varepsilon(x)t)\chi(\xi)d\xi$$

then $u(x,t)$ may be expressed as (cf. [6])

$$u(x,t) = R_{\pm}\,[u,w](x,t) - L_{\pm}\, H(u)(x,t) + S\pm(x,t), \quad (\pm\, x > 0). \tag{2.4}$$

Integration of the second equation $(2.2)^2$ with respect to time yields

$$w(x,t) = \psi(x) + \int_0^t g(u,w)(x,\tau)d\tau. \tag{2.5}$$

Continuity of $u(x,t)$ at $x = 0$ requires (from (2.4))

$$\begin{aligned} &\tfrac{1}{2}\int_0^t \frac{H(u)(\tau)}{\sqrt{\pi(t-\tau)}}[1+\sqrt{\varepsilon}]d\tau = \\ &= \int_0^\infty K(\xi,\varepsilon t)\chi(\xi)d\xi - \int_{-\infty}^0 K(\xi,t)\chi(\xi)d\xi \; + \\ &+ \int_0^t \int_0^\infty K(\xi,\varepsilon(t-\tau))f(u,w)(\xi,\tau)d\xi d\tau - \int_0^t \int_{-\infty}^0 K(\xi,t-\tau)f(u,w)(\xi,\tau)d\xi d\tau. \end{aligned} \tag{2.6}$$

Equation (2.6) can now be solved for H(u)(t) in terms of u,w and χ. If we change the order of integration in (2.6) and calculate the Laplace transform of (2.6) we arrive at

$$\frac{(1+\sqrt{\varepsilon})\,\bar{H}(p)}{2\sqrt{p}} = \tfrac{1}{2}\int_0^\infty \chi(\xi)(p\varepsilon)^{-\frac{1}{2}}\exp[-\,\xi\sqrt{\tfrac{p}{\varepsilon}}\,]\,d\xi$$

$$(2.7)\qquad -\tfrac{1}{2}\int_{-\infty}^0 \chi(\xi)p^{-\frac{1}{2}}\exp[\xi\sqrt{p}]d\xi\ +$$

$$+\ \tfrac{1}{2}\int_0^\infty \bar{f}\,(\xi,p)(p\varepsilon)^{-\frac{1}{2}}\exp[-\,\xi\,\sqrt{\tfrac{p}{\varepsilon}}]d\xi\ -\ \tfrac{1}{2}\int_{-\infty}^0 \bar{f}(\xi,p)p^{-\frac{1}{2}}\exp[\xi\sqrt{p}]d\xi$$

where $\bar{f}$ denotes the Laplace transform of a function f (we surpressed u and w in the notation) with respect to its last variable and where we have used the identities [1]

$$(\tfrac{\pi}{p})^{\frac{1}{2}}\exp[-2(ap)^{\frac{1}{2}}] = \int_0^\infty \frac{e^{-pt}}{\sqrt{t}}\exp[-\,\tfrac{a}{t}]dt,$$

$$\exp[-ap^{\frac{1}{2}}] = \tfrac{1}{2}a\pi^{-\frac{1}{2}}\int_0^\infty t^{-3/2}e^{-pt}\exp[-\,\frac{a^2}{4t}\,]dt.$$

If we multiply (2.7) by $2\sqrt{p}$ and invert we finally find

$$2\sqrt{\pi}\,(1+\sqrt{\varepsilon})H(u)(t) =$$

$$\frac{t^{-3/2}}{\varepsilon}\int_0^\infty \xi\chi(\xi)\exp\,[-\frac{\xi^2}{4\varepsilon t}]d\xi\ +\ t^{-3/2}\int_{-\infty}^0 \xi\chi(\xi)\exp[-\,\frac{\xi^2}{4t}]d\xi\ +$$

$$(2.8)\qquad +\ \frac{1}{\varepsilon}\int_0^\infty\int_0^t \xi(t-\tau)^{-3/2}\exp[-\,\frac{\xi^2}{4\varepsilon(t-\tau)}\,]\,f(u,w)(\xi,\tau)d\tau d\xi\ +$$

$$+\int_{-\infty}^0\int_0^t \xi(t-\tau)^{-3/2}\exp[-\frac{\xi^2}{4(t-\tau)}]\,f(u,w)(\xi,\tau)d\tau d\xi.$$

For convenience we shall denote the right-hand side as $2\sqrt{\pi}(1+\sqrt{\varepsilon})\mathcal{H}[u,w](t)$.

After substitution of the expression (2.8) for H into (2.4) we arrive at the equation

$$(2.9)\qquad u = R_{\pm}[u,w] - L_{\pm}H[u,w] + S_{\pm}, \qquad \pm x > 0.$$

which, together with (2.5) will be denoted by *Problem* P_2. We define a solution of P_2 in the following way.

<u>DEFINITION 2.3</u>. The vectorfunction $(u,w):\mathbb{R} \times [0,T] \to \mathbb{R} \times \mathbb{R}^n$ is called a solution of P_2 on $[0,T]$ if and only if

(i) $(u,w) \in BC(\mathbb{R} \times [0,T] \to \mathbb{R} \times \mathbb{R}^n)$

(ii) (u,w) satisfies (2.5) and (2.9).

Let us first list some properties of $H[u,w](t)$. We split H in a part $H_0(t)$ which does not depend on u and w, and a remaining part $H_1[u,w](t)$:

$$(2.10)\qquad \begin{aligned} H_0(t) &= t^{-3/2}[2\sqrt{\pi}(1+\sqrt{\varepsilon})]^{-1}. \\ &[\frac{1}{\varepsilon}\int_0^\infty \xi\chi(\xi)\exp(-\frac{\xi^2}{4\varepsilon t})d\xi + \int_{-\infty}^0 \xi\chi(\xi)\exp(-\frac{\xi^2}{4t})d\xi], \end{aligned}$$

$$(2.11)\qquad H_1[u,w](t) = H[u,w](t) - H_0(t).$$

<u>PROPOSITION 2.1</u>. *Let* $h_0 = \dfrac{2}{\sqrt{\pi}(1+\sqrt{\varepsilon})}$. *Then the following properties hold.*

(i) $|H_0(t)| \le h_0 \dfrac{\|\chi\|_{\mathbb{R}}}{\sqrt{t}}$, $\qquad t > 0$

(ii) $|H_1[u,w](t)| \le 2h_0\sqrt{t}\,\|f\|_{\mathbb{R}\times[0,t]}$, $\qquad t > 0$

(iii) *Let* $\delta > 0$. *For* $T > \delta > 0$ *and* $\alpha \in (0,\frac{1}{2})$ *there exists a number* $K = K(\delta,T,\alpha) > 0$ *such that for all* $t_1 \in (0,T)$ *and* $t_2 \in (\delta,T)$

$$(2.12)\qquad |H_0(t_2) - H_0(t_1)| \le \frac{K}{\sqrt{t_1}}|t_2 - t_1|,$$

$$(2.13)\qquad |H_1[u,w](t_2) - H_1[u,w](t_1)| \le K|t_2 - t_1|^\alpha.$$

Observe that by the Lipschitz continuity of F there exists for each constant $M > 0$ a constant $L_M > 0$ such that for $t > 0$ and for any pair of vector functions (u_1,w_1), (u_2,w_2) satisfying

$$\|(u_i,w_i)\|_{\mathbb{R} \times [0,t]} \le M, \qquad i = 1,2, \tag{2.14}$$

we have

$$\|f(u_1,w_1) - f(u_2,w_2)\|_{\mathbb{R} \times [0,t]} \le L_M \|(u_1,w_1) - (u_2,w_2)\|_{\mathbb{R} \times [0,t]}.$$

The following proposition is a consequence of this observation and (2.8)

PROPOSITION 2.2. *Under the condition* (2.14) *we have*

$$|H_1[u_1,w_1](t) - H_1[u_2,w_2](t)| \le$$

$$\le 2\, h_0\, L_M \sqrt{t}\, \|(u_1,w_1) - (u_2,w_2)\|_{\mathbb{R} \times [0,t]}.$$

We need the above estimates in the proof of local solvability of P_2. As a preparation for this proof we isolate from the right-hand sides of (2.5) and (2.9) the parts which do not depend on u and w:

$$(2.15) \qquad \begin{aligned} u_0(x,t) &= - L_{\pm} H_0(x,t) + S_{\pm}(x,t), && \pm x > 0,\\ w_0(x,t) &= \psi(x), && x \in \mathbb{R}. \end{aligned}$$

Define

$$\Phi = 2\|\chi\|_{\mathbb{R}} + \|\psi\|_{\mathbb{R}} \tag{2.16}$$

We shall operate in the following function space, defined for $t_0 > 0$

$$F_{t_0} = \{(u,w) \in BC(\mathbb{R} \times [0,t_0] \to \mathbb{R} \times \mathbb{R}^n) \mid$$

$$\|(u,w)-(u_0,w_0)\|_{\mathbb{R} \times [0,t_0]} \le \Phi\}$$

which is a complete metric space with respect to the norm $\|\cdot\|_{\mathbb{R} \times [0,t_0]}$. On F_{t_0} we consider the following operator Γ.

$$\Gamma[u,w]_1 = R_{\pm}[u,w] - L_{\pm} H_1[u,w] + u_0, \qquad \pm x > 0,$$

$$\Gamma[u,w]_{j+1} = w_{0j} + \int_0^t g(u,w_j)d\tau, \qquad j = 1,2,\dots,n.$$

Obviously, fixed points of Γ in F_{t_0} are solutions of Problem P_2. Now the following proposition holds.

PROPOSITION 2.3.

(i) *There exists a time* $t_0 = t_0(\Phi)$ *such that* Γ *as a mapping from* F_{t_0} *into* F_{t_0} *is well defined.*

(ii) *There exists a number* $N > 0$, $N = N(\Phi)$ *such that for any pair* $(u_1,w_1),(u_2,w_2) \in F_{t_0}$.

$$(2.17) \qquad \|\Gamma[u_1,w_1] - \Gamma[u_2,w_2]\|_{\mathbb{R} \times [0,t_0]} \le Nt_0 \|(u_1,w_1) - (u_2,w_2)\|_{\mathbb{R} \times [0,t_0]}$$

LEMMA 2.1. *There exists a time* $t_0 = t_0(\Phi)$ *such that Problem* P_2 *has a unique solution on* $[0,t_0]$.

PROOF. By Proposition 2.3 there exists a number $t_0 > 0$ such that Γ is a contraction on F_{t_0}. Hence Γ has a unique fixed point in F_{t_0} which is a solution of P_2 on $[0,t_0]$.

We shall show that this solution of P_2 is smooth enough to be a solution of P_1, in two steps. First we shall show that it is a generalized solution of P_1 as specified below. Then it follows in a standard way that it is also a classical solution of P_1.

Let I be a bounded open interval of $\mathbb{R}$ with $0 \notin \bar{I}$ and let $Q_T = I \times (0,T)$ for $T > 0$. Consider the differential equation

$$(2.18) \qquad Lu \equiv u_t - e_\varepsilon(x)u_{xx} = h(x,t)$$

where e_ε is given in (1.2) and $h \in C(\bar{Q}_T)$.

DEFINITION 2.4. By a *generalized solution of the equation* $Lu = h$ *on* Q_T we mean a function $u:Q_T \to \mathbb{R}$ with the properties

(i) $u, u_x \in C(\bar{Q}_T)$

(ii) For all $\eta \in C^{1,1}(\bar{Q}_T \to \mathbb{R})$ with $\eta(x,t)$ vanishing on $\bar{I}\backslash I \times (0,T)$ and $\eta(x,0) = 0$ on I we have

$$\int_I u(x,t)\eta(x,t)dx - \int_0^t \int_I u(x,\tau)\eta_t(x,\tau)dxd\tau +$$
$$+ \int_0^t \int_I [e_\varepsilon(x)u_x(x,\tau)\eta_x(x,\tau) - h(x,\tau)\eta(x,\tau)]dxd\tau = 0.$$

Let us first verify that u is C^1-smooth.

PROPOSITION 2.4. *Let* (u,w) *be the solution of* P_2 on $[0,t_0]$. *Then* $u_x \in C(\mathbb{R} \times (0,t_0] \to \mathbb{R})$ *and* $u_x(0\pm,t) = H[u,w](t)$ *for* $t > 0$.

THEOREM 2.1. *Let* (u,w) *be the solution of* P_2 *on* $[0,t_0]$. *Let* $\tilde{t} \in (0,t_0)$ *and* $T = t_0-\tilde{t}$. *Then for* $h = f_0(u,w)$, $u(x,t+\tilde{t})$ *is a generalized solution of the equation.* $Lu = h$ *on* Q_T.

PROOF. It is quite standard to prove that u, given by (2.4) satisfies (2.19) for $I \subset (-\infty,0)$ or $I \subset (0,\infty)$, $t \in (\tilde{t},t_0)$. Part (i) in Definition 2.4 for $u(x,t+\tilde{t})$ follows from Definition 2.3(i) and Proposition 2.4.

Now let u be a generalized solution of $Lu = h$ in $Q_T = I \times (0,T)$, where $0 \notin I$. It is well known [2;p.224] that if for some $\alpha > 0$

$$h \in C^{\alpha,\alpha/2}(Q_T)$$

then

$$u \in C^{2+\alpha,1+\alpha/2}(Q_T) \tag{2.20}$$

PROPOSITION 2.5. *Let* $\alpha > 0$ *be a Hölder exponent for the initial function* $\psi(x) = w(x,0)$ *and let* $\tilde{t} \in (0,t_0)$. *Let* I *be a bounded open interval such that*

$0 \notin \bar{I}$. *Then*

(i) $w(\cdot,t)$ *is Hölder continuous with exponent* α *on* $\mathbb{R}$ *for all* $t \in [0,t_0)$.

(ii) *For all* $\beta \in (0,1)$, $u(x,\cdot)$ *is Hölder continuous with exponent* β *on* $(\tilde{t},t_0)$ *for all* $x \in I$.

(iii) $w_t \in C(\mathbb{R} \times [0,t_0] \to \mathbb{R}^n)$.

By this proposition and the Lipschitz continuity of F we have that $f(u,w) \in$ $\in C^{\alpha,\alpha/2}(I \times (\tilde{t},t_0)$ for any $\tilde{t} \in (0,t_0)$ and I as above. This implies that the u-component of the solution of P_2 is in $C^{2+\alpha,1+\alpha/2}(I \times (\tilde{t},t_0))$.

We shall now state a local existence result for Problem P_1.

<u>THEOREM 2.2</u>. *The unique solution* (u,w) *of* P_2 *on* $[0,t_0]$ *is also a solution of* P_1. *Moreover if* α *is a Hölder exponent for* $\psi(x)$ *then for* $\tilde{t} \in (0,t_0)$ *and any x-interval* I, $0 \notin \bar{I}$ *we have*

$$(2.21)\qquad \begin{aligned} &u \in C^{2+\alpha,1+\alpha/2}(I \times (\tilde{t},t_0) \to \mathbb{R}),\\ &w \in C^{\alpha,1}(I \times (\tilde{t},t_0) \to \mathbb{R}^n), \end{aligned}$$

with w_t *Lipschitz continuous in* t *for* $x \in I$.

<u>PROOF</u>. By (2.20) and the corollary to Proposition 2.4, u satisfies (2.21)[1] for all $\tilde{t} \in (0,t_0)$ and $I \subset \mathbb{R}$, when $0 \notin \bar{I}$. Since $\tilde{t}$ and I are arbitrary, the smoothness of u_{xx} and u_t, required in Definition 2.2 follows. The smoothness of u_x follows from Proposition 2.4. Since (u,w) solves P_2, u and w are continuous on $\mathbb{R} \times [0,t_0]$ and by (2.5), the same results hold for w_t. By the Lipschitz continuity of G and the fact that both u_t and w_t exist we have that $g(u,w)$ is Lipschitz continuous in t. Hence $w_t(x,\cdot)$ is Lipschitz continuous for $x \in I$. The verification of (2.3) is standard.

To conclude this section we shall extend the local existence to a global one. Thereby we shall make the following hypothesis.

H: There exists a number $K = K(\|\chi\|_{\mathbb{R}}, \|\psi\|_{\mathbb{R}})$ such that for all

$T > 0$, a solution (u,w) of P_1 on $[0,T)$ satisfies
$\| (u,w) \|_{\mathbb{R} \times [0,T)} < K$.

Thus we assume an a priori bound for solutions of P_1. In Appendix B it is shown that this hypothesis is true for three typical examples.

<u>THEOREM 2.3</u>. *Suppose* H *is satisfied. Then for every* $T > 0$ *Problem* P_1 *has a unique classical solution on* $[0,T)$ *with regularity properties as stated in Theorem 2.1 where* t_0 *is replaced by* T.

<u>PROOF</u>. A Corollary to Theorem 2.2 is that the solution (u,w) of P_1 for fixed $t \in (0,t_0)$ has the same regularity as the pair of initial functions (χ,ψ) (i.e. bounded continuous with $\psi \in C^{\alpha}(\mathbb{R} \to \mathbb{R}^n)$) . If we replace t in (2.2) by $t' = t-t_1$ for $t_1 \in [0,t_0)$ and put $u'(x,t') \equiv u(x,t_1+t')$, $w'(x,t') \equiv$ $\equiv w(x,t_1+t')$ then the corresponding problem $P_1{}'$ (i.e. with initial functions $\chi'(x) = u(x,t_1)$, $\psi'(x) = w(x,t_1)$) has a classical solution (u',w') for $t' \in [0,t_0')$ for some $t_0{}'$, only depending on K, introduced in H. This result follows by the same arguments as used above to prove Theorem 2.2. Hence by repeated application of this theorem the local solution of P_1 is extended to a solution on $[0,t_1+mt_0{}')$ for any $m \in \mathbb{N}$ and global existence follows.

APPENDIX A: The proofs of the propositions in Section 2

PROOF OF PROPOSITION 2.1.

(i) $H_0(t)$ is majorized by

$$|H_0(t)| \le \tfrac{1}{4} t^{-3/2} h_0 [\frac{\|\chi\|_{[0,\infty)}}{\varepsilon} \int_0^\infty \xi \exp[-\frac{\xi^2}{4\varepsilon t}] d\xi +$$

$$+ \|\chi\|_{(-\infty,0]} \int_{-\infty}^0 \xi \exp[-\frac{\xi^2}{4t}] d\xi] \le t^{-\frac{1}{2}} h_0 \|\chi\|_{\mathbb{R}}$$

(ii) An explicit expression for $H_1(t)$ follows by subtraction of $H_0(t)$ from the expression for H(u)(t) given in (2.8). From this expression we find for $H_1[u,w]$

$$|H_1[u,w](t)| \le \tfrac{1}{4} h_0 \|f\|_{\mathbb{R} \times [0,t]} [\frac{1}{\varepsilon} \int_0^\infty \int_0^t \xi (t-\tau)^{-3/2} \exp[-\frac{\xi^2}{4\varepsilon(t-\tau)}] d\tau d\xi +$$

$$+ \int_{-\infty}^0 \int_0^t \xi (t-\tau)^{-3/2} \exp[-\frac{\xi^2}{4(t-\tau)}] d\tau d\xi$$

$$\le 2 h_0 \sqrt{t} \|f\|_{\mathbb{R} \times [0,t]}, \qquad t > 0.$$

(iii) Assume $t_2 \ge t_1$ and split the integrals in the expression for $H_0(t_2) - H_0(t_1)$ in the following manner (we take $\varepsilon = 1$, for simplicity)

$$t_2^{-3/2} \int_0^\infty \xi \chi(\xi) \exp[-\frac{\xi^2}{4t_2}] d\xi - t_1^{-3/2} \int_0^\infty \xi \chi(\xi) \exp[-\frac{\xi^2}{4t_1}] d\xi$$

$$\text{(A1)} \qquad = t_2^{-1} \int_0^\infty [\frac{1}{\sqrt{t_2}} \exp[-\frac{\xi^2}{4t_2}] - \frac{1}{\sqrt{t_1}} \exp[-\frac{\xi^2}{4t_1}]] \xi \chi(\xi) d\xi$$

$$+ \int_0^\infty \frac{t_1 - t_2}{t_1 t_2} \frac{1}{\sqrt{t_1}} \xi \chi(\xi) \exp[-\frac{\xi^2}{4t_1}] d\xi.$$

In the integrand of the first part we apply the mean value theorem;

$$\frac{1}{\sqrt{t_2}} \exp[-\frac{\xi^2}{4t_2}] - \frac{1}{\sqrt{t_1}} \exp[-\frac{\xi^2}{4t_1}] =$$

$$(t_2-t_1)[-\tfrac{1}{2}\tau + \frac{\xi^2}{4}]\tau^{-5/2}\exp[-\frac{\xi^2}{4\tau}], \tag{A2}$$

for some τ between t_1 and t_2. This gives us the factor $|t_2-t_1|$ times an integral J and in view of (2.12) we must show that $J.\sqrt{t_1}$ is bounded for $t_1 \in (0,T)$ and $t_2 \in (\delta,T)$ for $T > 0$, $\delta \in (0,T)$. The integral J is composed of integrals bounded by

$$J^i = \|\chi\|_{\mathbb{R}} \ \tau^{-3/2-i} \int_0^\infty \xi^{2i+1} \exp[-\frac{\xi^2}{4\tau}]d\xi, \qquad i \in \{0,1\}.$$

We split J^i in integrals J_1 over $[0,2((\frac{3}{2}+i)t_1)^{\frac{1}{2}})$, J_2 over $[2((\frac{3}{2}+i)t_1)^{\frac{1}{2}}, 2((\frac{3}{2}+i)t_2)^{\frac{1}{2}}]$ and J_3 over $(2((\frac{3}{2}+i)t_2)^{\frac{1}{2}},\infty)$. Then J_1 increases if we replace τ by t_1 and J_3 increases if we replace τ by t_2. Calculation of the resulting integrals yields that $J_i\sqrt{t_1}$ is bounded, $i = 1,3$. In J_2 we use that

$$\tau^{-3/2-i} \ \xi^{2i+1}\exp[-\frac{\xi^2}{4\tau}] \le (6+4i)^{-3/2-i}\xi^{-2}\exp[-\frac{3}{2}-i]$$

and again we find after calculation that the resulting integral is of order $O(t_1^{-\frac{1}{2}})$. The second term in (A1) is easily estimated by majorizing χ by $\|\chi\|_{\mathbb{R}}$.

Finally we note that if $t_1 > t_2$ the proof of (2.12) is easy since in this case t_1 is bounded away from zero.

To prove (2.13) we split the integrals in the expression $H_1[u,w](t_2) - H_1[u,w](t_1)$ in the following way

$$J \equiv \int_0^\infty \int_0^{t_2} \xi(t_2-s)^{-3/2}\exp[-\frac{\xi^2}{4(t_2-s)}] \ f(u,w)(\xi,s)dsd\xi$$

$$- \int_0^\infty \int_0^{t_1} \xi(t_1-s)^{-3/2}\exp[-\frac{\xi^2}{4(t_1-s)}] \ f(u,w)(\xi,s)dsd\xi \tag{A3}$$

$$\equiv J_1 + J_2$$

where

$$J_2 = \int_0^\infty \int_{t_1}^{t_2} \xi(t_2-s)^{-3/2} \exp[-\frac{\xi^2}{4(t_2-s)}] \, f(u,w)(\xi,s)ds d\xi$$

Evaluation of J_2 with f replaced by $\|f\|_{\mathbb{R}\times[0,T]}$ yields

$$|J_2| \le 4\|f\|_{\mathbb{R}\times[0,T]} \cdot (t_2-t_1)^{\frac{1}{2}}.$$

In the integrand of J_1, the difference $B(t_2-s) - B(t_1-s)$ occurs where $B(t) = t^{-3/2}\exp[-\xi^2/4t]$. We shall use the relation

$$\text{(A4)} \qquad B(t_2-s) - B(t_1-s) \le (t_2-t_1)^\alpha |-3\tau + \tfrac{1}{2}\xi^2| \frac{\tau^{-\frac{5}{2}-\alpha}}{2\alpha} \exp[-\frac{\xi^2}{4\tau}]$$

for some τ between t_1-s and t_2-s where $\alpha \in (0,\frac{1}{2})$. This relation arises from application of the mean value theorem with respect to the variable $|t-s|^\alpha$ to $B(t_2-s) - B(t_1-s)$ together with the inequality

$$||t_2-s|^\alpha - |t_1-s|^\alpha| \le |t_2-t_1|^\alpha.$$

Then a further estimation of J_1 can be given along the same lines as in the estimation of J^i above, where the absence of a factor $t_1^{-\frac{1}{2}}$ is due to the extra integration with respect to time.

PROOF OF PROPOSITION 2.2. This proof is entirely analogous to the proof of Proposition 2.1 (ii).

PROOF OF PROPOSITION 2.3.

(i) To begin with we shall estimate $\|(u_0,w_0)\|_{\mathbb{R}\times[0,t_0]}$. For $x > 0$, the expression (2.15) for u_0 consists of an integral in terms of H_0 and an integral, of which the value depends on χ. For the first integral we use Proposition 2.1:

$$|\varepsilon \int_0^t U(x,0:\varepsilon(t-\tau))H_0(\tau)d\tau| \le \frac{\sqrt{\varepsilon}\, h_0}{\sqrt{\pi}} \|\chi\|_{\mathbb{R}} \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}}$$

$$\le h_0 \sqrt{\pi\varepsilon} \; \|\chi\|_{\mathbb{R}}.$$

The second term in (2.15) is majorized by

$$\|\chi\|_{\mathbb{R}} \frac{1}{2\sqrt{\pi\varepsilon t}} \int_0^\infty \exp[-\frac{(x-\xi)^2}{4\varepsilon t}] + \exp[-\frac{(x+\xi)^2}{4\varepsilon t}]d\xi$$

$$= \frac{\|\chi\|_{\mathbb{R}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \|\chi\|_{\mathbb{R}}.$$

Similar inequalities hold if $x < 0$ and, by the definition h_0 we find

$$\|u_0\|_{\mathbb{R} \times [0,t_0]} \leq 2\|\chi\|_{\mathbb{R}}.$$

Obviously $\|w_0\|_{\mathbb{R}\times [0,t_0]} = \|\psi\|_{\mathbb{R}}$ and hence

$$\|(u_0,w_0)\|_{\mathbb{R}\times [0,t_0]} \leq \Phi$$

and as a corollary

$$(u,w) \in F_{t_0} \Rightarrow \|(u,w)\|_{\mathbb{R}\times [0,t_0]} \leq 2\Phi.$$

To prove that Γ is well defined we must verify that $(u,w) \in F_{t_0}$ implies that $\|\Gamma(u,w) - (u_0,w_0\|_{\mathbb{R}\times [0,t_0]} \leq \Phi$.

Let K be such that $|F(u,w)|$, $|G_j(u,w)| \leq K$ for all $(u,w) \in F_{t_0}$. For $x > 0$ and $t \leq t_0$ we have for $\Gamma[u,w]_1 - u_0$

$$|\Gamma[u,w]_1(x,t) - u_0(x,t)| \leq K \int_0^t \int_0^\infty |U(x,\xi;\varepsilon(t-\tau))|d\xi d\tau +$$

$$+ \varepsilon \int_0^t |U(x,0;\varepsilon(t-\tau))\ H_1[u,w](\tau)|d\tau$$

$$\leq K\ t_0 + 2K\ t_0$$

where we have used Proposition 2.1(ii) and the definition of h_0. This inequality also holds if $x < 0$. For $j \in \{1,\ldots,n\}$ we have

$$|\Gamma[u,w]_{j+1}(x,t) - w_{0j}(x,t)| \leq K\ t_0.$$

and therefore, for t_0 sufficiently small ($t_0 < \frac{\Phi}{K(3+n)}$), $\Gamma(u,w] \in F_{t_0}$. The other conditions in the definition of F_{t_0}, for $\Gamma(u,w)$ are easily verified.

(ii) If we apply, in the expression for $\Gamma[u_1,w_1]_1 - \Gamma[u_2,w_2]_1$ the Lipschitz continuity of F and Proposition 2.2 we arrive at a relation of the form

$$|\Gamma[u_1,w_1]_1(x,t) - \Gamma[u_2\, ,\, w_2]_1(x,t)| \le M(I_1 + I_2)$$

for some M > 0, where I_1 and I_2 are integrals which can be estimated using the arguments in the proof of (i), leading to (2.17).

PROOF OF PROPOSITION 2.4. The first derivative u_x of u for x > 0 is found by differentiation of (2.4)

$$\begin{aligned} u_x(x,t) = &- \frac{1}{4\sqrt{\pi}} \int_0^t \int_0^\infty [\varepsilon(t-\tau)]^{-3/2} [(x-\xi)\exp[-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}] + \\ &+ (x+\xi)\exp[-\frac{(x+\xi)^2}{4\varepsilon(t-\tau)}]].\ f(u,w)(\xi,\tau)d\tau + \\ &+ \frac{x\varepsilon}{2\sqrt{\pi}} \int_0^t [\varepsilon(t-\tau)]^{-3/2}\exp[-\frac{x^2}{4\varepsilon(t-\tau)}]H(u)(\tau)d\tau + \\ &- \frac{1}{4\sqrt{\pi}}(\varepsilon t)^{-3/2} \int_0^\infty [(x-\varepsilon)\exp[-\frac{(x-\xi)^2}{4\varepsilon t}] + (x+\xi)\exp[-\frac{(x+\xi)^2}{4\varepsilon t}]]\chi(\xi)d\xi, \end{aligned} \tag{A5}$$

provided that this expression exists. For $t \le t_0$, f is bounded and the first term in (A5) can be majorized by

$$\begin{aligned} &\frac{\|f\|_{\mathbb{R}\times[0,t_0]}}{4\sqrt{\pi}} \int_0^t \int_0^\infty (\varepsilon(t-\tau))^{-3/2} |(x-\xi)\exp[-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}] + \\ &+ (x+\xi)\exp[-\frac{(x+\xi)^2}{4\varepsilon(t-\tau)}]|d\xi d\tau \\ &\le \frac{2\|f\|_{\mathbb{R}\times[0,t_0]}}{\sqrt{\pi}} \int_0^{t_0} \frac{d\tau}{\sqrt{\varepsilon\tau}} = \frac{4\|f\|_{\mathbb{R}\times[0,t_0]}}{\sqrt{\pi\varepsilon}}.\sqrt{t_0} \end{aligned} \tag{A6}$$

In a similar fashion, the third term in (A5) is bounded by $K\|\chi\|_{\mathbb{R}}/\sqrt{t}$ for some $K > 0$. The second term is convergent for $x > 0$, by Proposition 2.1 and if we let $x \downarrow 0$ we arrive at

$$\frac{x\varepsilon}{2\sqrt{\pi}} \int_0^t [\varepsilon(t-\tau)]^{-3/2} \exp[-\frac{x^2}{4\varepsilon(t-\tau)}] H(u)(\tau)d\tau =$$

$$= \frac{1}{\sqrt{\pi}} H(t) \int_{\frac{x^2}{4\varepsilon t}}^{\infty} \frac{e^{-\eta}}{\sqrt{\eta}} d\eta + \frac{1}{\sqrt{\pi}} \int_{\frac{x^2}{4\varepsilon t}}^{\infty} \frac{e^{-\eta}}{\sqrt{\eta}} [H(u)(t - \frac{x^2}{4\varepsilon\eta}) - H(u)(t)]d\eta$$

$$\to H(u)(t), \quad x \downarrow 0,$$

by Proposition 2.1 (iii), for $t > 0$. The first and the third term in (A5) tend to zero as $x \downarrow 0$. For $x < 0$ we find similar results, including $u_x(x,t) \to H(u)(t)$ as $x \uparrow 0$ for $t > 0$. The derivative u_x is continuous for $x \neq 0$ and $t \in (0,t_0]$ and by the above observations, also at $x = 0$ if $t > 0$.

PROOF OF PROPOSITION 2.5.

(i) Let $x_1, x_2 \in \mathbb{R}$. By the Lipschitz continuity of G we find from (2.5) for any $j \in \{1,2,...,n\}$ and $t \in (0,t_0]$

$$|w_j(x_1,t) - w_j(x_2,t)| \leq |\psi_j(x_1) - \psi_j(x_2)| +$$

$$+ t[L_u \|u(x_1,\cdot) - u(x_2,\cdot)\|_{[0,t_0]} + L_w \|w_j(x_1,\cdot) - w_j(x_2,\cdot)\|_{[0,t_0]}]$$

for some constants L_u and L_w. Hence, w is Hölder continuous in x with exponent α for small t, $t \leq t_1$ say. However, we can repeat the arguments on $[t_1,2t_1]$ and so forth. Thus w is Hölder continuous in $x \in \mathbb{R}$ for all $t \in [0,t_0]$. As a consequence, $f(u,w)(x,t)$ is Hölder continuous in x with exponent α.

(ii) Suppose $t_1 < t_2$ and $x < 0$. By (2.4) we may write for $u(x,t_1) - u(x,t_2)$

$$u(x,t_1) - u(x,t_2) =$$

$$
\begin{aligned}
&= \int_0^{t_1}\int_{-\infty}^{0} [U(x,\xi;t_1-\tau) - U(x,\xi;t_2-\tau)]f(u,w)(\xi,\tau)d\xi d\tau \\
&- \int_{t_1}^{t_2}\int_{-\infty}^{0} U(x,\xi;t_2-\tau)f(u,w)(\xi,\tau)d\xi d\tau \\
\text{(A7)}\qquad &+ \int_0^{t_1} [U(x,0:t_1-\tau) - U(x,0:t_2-\tau)]H(u)(\tau)d\tau \\
&- \int_{t_1}^{t_2} U(x,0:t_2-\tau)H(u)(\tau)d\tau \\
&+ \int_{-\infty}^{0} [U(x,\xi:t_1) - U(x,\xi:t_2)]\chi(\xi)d\xi \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

If we replace in I_2, f by $\|f\|_{\mathbb{R}\times[0,t_0]}$ and evaluate the resulting integral we find that $I_2 = O(t_2-t_1)$. Similarly, using Proposition 2.1, we can majorize I_4 by

$$K\int_{t_1}^{t_2} \frac{d\tau}{1\times 1\sqrt{\tau}} \le \frac{K}{1\times 1\sqrt{t_1}}(t_2-t_1)$$

for some $K > 0$. In the integrand of I_1 we apply the mean value theorem with respect to the variable $|t-\tau|^{\beta}$, $\beta \in (0,1)$ to $U(x,\xi;t_2-\tau) - U(x,\xi;t_1-\tau)$. Since $\beta < 1$, the resulting integral is convergent. We have applied the same technique in the proof of (2.13) in Proposition 2.1 and similar to that proof it follows for I_1 that $I_1 = O(|t_2-t_1|^{\beta})$.

In the integrand of I_3 we apply the mean value theorem with respect to the variable $t-\tau$ to $U(x,0:t_1-\tau) - U(x,0:t_2-\tau)$. The resulting integral does converge since $x \neq 0$ and as a result $I_3 = O(t_2-t_1)$. Finally, using the mean value theorem in I_5 in a similar fashion as in I_3, it follows, by the boundedness of χ that $I_3 = O(t_2-t_1)$.

(iii) This statement follows from the continuity of u,w and G, and equation (1.3).

APPENDIX B: A priori bounds for solutions of Problem P_1

In this appendix we shall show that the assumption of boundedness of solutions of problem P_1, as formulated in hypothesis H, holds for the following three examples:

(i) The bistable equation:

$$F(u) = u(1-u)(u-a), \qquad 0 < a < \tfrac{1}{2},$$

(ii) FitzHugh-Nagumo equations

$$F(u,w) = u(1-u)(u-a)-w, \qquad 0 < a < \tfrac{1}{2},$$

$$G(u,w) = \sigma u - \gamma w, \qquad \sigma,\gamma > 0$$

(iii) Goldstein-Rall equations

$$F(u,w) = w_1(1-u) - w_2(u + \frac{1}{10}) - u,$$

$$G(u,w) = \begin{pmatrix} k_1u^2 + k_2u^4 - k_3w_1 - k_4w_1w_2 \\ k_5w_1 + k_6w_1w_2 - k_7w_2 \end{pmatrix}$$

$$k_i > 0,\ k_2 >> k_1 >> k_3 > k_4,\ k_1 >> k_7 > k_5 >> k_6.$$

which were of interest in [3] and [4].

Recall that the initial functions χ and ψ are bounded.

We shall make use of a conditional comparison principle which is a modification of Theorem 3.1 in [4]

THEOREM B1. *Let*

$$\phi,u,\psi \in BC(\mathbb{R} \times [0,\infty) \to \mathbb{R}) \cap C^{2,1}(\mathbb{R} \setminus \{0\} \times (0,\infty) \to \mathbb{R})$$

satisfy for all $T > 0$ *and* $x \in \mathbb{R} \setminus \{0\}$

$$\phi \le u \le \psi \text{ on } [0,T] \Rightarrow N\phi \le Nu \le N\psi, \qquad t \in (0,T]$$

where N *is a differential operator of the form*

$$Nu \equiv u_t - e_\varepsilon(x)u_{xx} - F_0(u;x,t)$$

where $F_0 \in C^{1,0,0}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R})$. *Moreover let* ϕ, u *and* ψ *satisfy*

$$\phi_x(0+,t) - \phi_x(0-,t) > u_x(0+,t) - u_x(0-,t) \ge \psi_x(0+,t) - \psi_x(0-,t) \qquad t > 0$$

$$\phi(x,0) < u(x,0) < \psi(x,0), \qquad x \in \mathbb{R} .$$

Then

$$\phi(x,t) < u(x,t) < \psi(x,t), \qquad x \in \mathbb{R},\ t \ge 0.$$

Note: This theorem differs from Theorem 3.1 in [4] in the sense that here we have $\mathbb{R}$ as x-domain whereas in [4] we selected x from an interval $[\alpha,\infty)$. Also F_0 may depend on t.

PROOF. Following the proof of Theorem 3.1 on an interval $[-\gamma,\infty)$ and on an interval $(-\infty,\gamma]$ for $\gamma > 0$ we find that $(\phi(x,t) - u(x,t))(\psi(x,t) - u(x,t))$ vanishes at some point if and only if $(\phi(0,t) - u(0,t))(\psi(0,t) - u(0,t)) = 0$ for some $t > 0$. Then if $\underline{t}$ is the smallest time for which $(\phi(0,t) - u(0,t))(\psi(0,t) - u(0,t))$ vanishes, application of the ordinary unconditional comparison principle (Lemma 3.1 in [4]) to any set of the form $\{(x,t) | \alpha < x < \beta,\ 0 < t \le \underline{t}\}$ will yield a contradiction as in the proof of Theorem 3.1..

LEMMA B_1. (*Example* (ii)). *For example* (ii), *there exists a number* M *such that for every solution* (u,w) *of* P_1 *we have*

$$(B_1) \qquad \begin{aligned} &|u(x,t)| \le M, \\ &|w(x,t)| \le \|\psi\|_{\mathbb{R}} + \frac{\sigma}{\gamma}M, \qquad x \in \mathbb{R}, \qquad t \ge 0. \end{aligned}$$

PROOF. For M we choose a number such that

$$M \geq \max\{1, \|\chi\|_{\mathbb{R}}, \|\psi\|_{\mathbb{R}}\}$$

(B2)

$$M + \frac{\sigma}{\gamma} M \pm f(\pm M) \leq 0.$$

This is possible because $f(u) \sim -u^3$, $(|u| \to \infty)$.

Then, as long as $|u(x,t)| < M$ we have

(B3)
$$w(x,t) = \psi(x,t)e^{-\gamma t} + \sigma \int_0^t u(x,\tau)e^{-\gamma(t-\tau)}d\tau$$
$$\geq -M - \frac{\sigma}{\gamma} M,$$

which yields

$$u_t - e_\varepsilon(x)u_{xx} - f(u)$$
$$\leq u_t - e_\varepsilon(x)u_{xx} - f(u) + w + M + \frac{\sigma}{\gamma} M$$
$$\leq -f(M).$$

by (B2), and similarly

$$u_t - e_\varepsilon(x)u_{xx} - f(u) \geq -f(-M).$$

By Theorem B1 $|u(x,t)| < M$ and together with (B3) this implies $|w(x,t)| \leq$ $\leq \|\psi\|_{\mathbb{R}} + \frac{\sigma}{\gamma} M$.

If, in the proof of Lemma B1 one assumes $\sigma = 0$ and $\psi(x) \equiv 0$ then one arrives at the following Lemma.

LEMMA B2. *Let*

(B4) $$F(u,w) = u(1-u)(u-a) \equiv f(u), \qquad 0 < a < \tfrac{1}{2}$$

Then there exists a number $M \geq 1$ *such that for every solution* (u,w) *of* P_1 *we have*

(B5) $|u(x,t)| \leq M.$

In the situation of example (iii), the Goldstein-Rall equations, we shall verify hypothesis H under the additional condition that the negative values of $\psi_1(x)$ and $\psi_2(x)$ are not too large in absolute value. This is the case one is usually interested in.

LEMMA B3. *Suppose for the Goldstein-Rall equations that the initial function* $\psi(x) = (\psi_1(x),\psi_2(x))^T$ *satisfies*

(B6) $$\psi_i(x) \geq -\tfrac{1}{2} + \rho, \qquad i = 1,2$$

for some $\rho \in (0,\tfrac{1}{2})$. *Then there exists a number* M *such that*

(B7) $$|u(x,t)| \leq M, \qquad |w_i(x,t)| \leq M, \qquad i = 1,2, \quad x \in \mathbb{R}, \quad t \geq 0.$$

PROOF. We shall select numbers $\mu > 0$, $U^{\pm}$ and $W_i^{\pm}$ such that

(B8) $$U^- < \chi < U^+, \qquad W_i^- < \psi_i < W_i^+$$

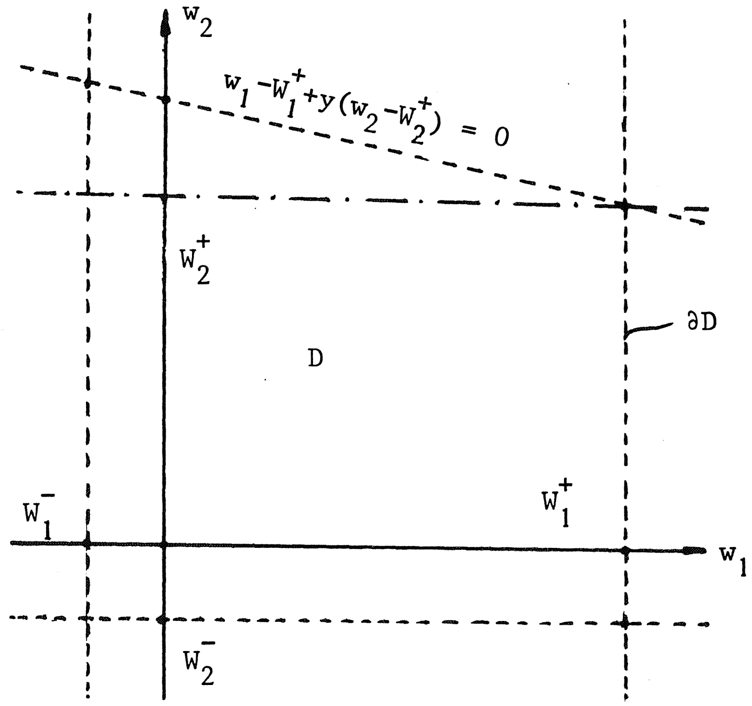
and

(B9) $$\pm F(U^{\pm},w) \leq 0, \qquad w_i \geq W_i^-$$

and on the boundary ∂D of a trapezium shaped region D in the (w_1,w_2) plane as sketched in figure B1 the condition

(B10) $$\frac{\partial G}{\partial \nu} < 0$$

holds for all $w \in \partial D$, $U^- \leq U \leq U^+$ and where $\frac{\partial}{\partial \nu}$ denotes any directional derivative in an outward direction at w.



By (B10), $w(x,t)$ cannot take on values on ∂D and therefore remains inside D. Then it follows by (B9), in a similar way as in Lemma B1 that $U(x,t)$ remains between U^- and U^+ for all $x \in \mathbb{R}$, $t \geq 0$.

Let $|\chi|$ and $|\psi_i|$ for $i = 1,2$, be bounded by K. We shall select the numbers $U^\pm$ and W_i^- such that they satisfy

(B11) $$U^+ > \max\{1,K\},\ U^- < \min\{K,-\frac{1}{10}\}$$

and

(B12) $$W_i^- \in (-\tfrac{1}{2} + \tfrac{1}{2}\rho_0, -\tfrac{1}{2} + \rho_0), \qquad i = 1,2,$$

for some $\rho_0 \in (0,\rho)$. Then $F(U^+,w) \leq 0$ for $w_i \geq W_i^-$ and $F(U^-,w) \geq -\rho_0 U^- + O(1)$ for $w_i \geq W_i^-$. Hence, by choosing $-U^-$ sufficiently large, (B11) can be satisfied.

Let L be a number satisfying

(B13) $$L \geq k_1\, U^{+2} + k_2 U^{+4}.$$

Then, if we choose $W_1^+ > K$ such that

(B14) $(\frac{1}{2}k_4 - k_3)W_1^+ + L < 0$

then for $U^- \le u < U^+$ and $w_2 \ge W_2^-$.

(B15) $G_1(u,W_1^+,w_2) \le L - k_3W_1^+ + \frac{1}{2}k_4W_1^+ < 0.$

By (B6) it follows that

(B16) $G_1(u,W_1^-,w_2) \ge - k_3W_1^- + \frac{1}{2}k_4W_1^- > 0.$

By (B6) the term $k_6\, w_1W_2^-$ in $G_2(w_1,W_2^-)$ is positive or small in absolute value, compared to $- k_7W_2^-$. Since $k_5 < k_7$, if we choose ρ_0 small enough (and therefore W_1^- close to W_2^-) then

(B17) $G_2(w_1,W_2^-) > 0, \quad W_1^- \le w_1 \le W_1^+.$

It remains to be shown that along the part of ∂D given by a line segment

(B18) $w_1 - W_1^+ + \gamma(w_2-W_2^+) = 0, \quad W_1^- \le w_1 \le W_1^+, \qquad \gamma > 0$

the inequality

$$G_1 + \gamma G_2 < 0, \qquad U^- \le u \le U^+$$

holds for suitable choice of γ and W_2^+.

We have

$$G_1(u,w_1,w_2) + \gamma G_2(w_1,w_2) =$$
$$= - w_2[(k_4 - \gamma k_6)w_1 + \gamma k_7] + R(u,w_1)$$

where R is bounded for u and w_1 bounded.

If we take $\gamma < k_4/k_6$ then for W_2^+ sufficiently large we have that $G_1 + \gamma G_2 < 0$ along the line segment given by (B18).

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