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OPTIMAL LOWER BOUNDS FOR THE SPECTRUM OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH A P-INTEGRABLE COEFFICIENT

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Optimal lower bounds for the spectrum of a second order linear differential equation with a p-integrable coefficient*)

by

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ABSTRACT

In this note the differential expression $M[y] \equiv -y'' + qy$, $q \in L^p(\mathbb{R}^+)$ for some $p \geq 1$, will be considered on $[0,\infty)$ together with the boundary condition either $y(0) = 0$ or $y'(0) = 0$. Lower bounds will be given for the spectrum of the self-adjoint operators T generated by $M[\cdot]$ and these boundary conditions. The bounds depend on the L^p-norm of the coefficient q and they improve results of Everitt and Eastham. For $p > 1$ the bounds are optimal.

KEY WORDS & PHRASES: *singular second order linear differential equation, Sturm-Liouville problem, lower bound spectrum*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this note new lower bounds for the spectrum of an operator T_α acting in the Hilbert space $L^2(\mathbb{R}^+)$ with complex-valued elements will be given. The operator T_α is generated by the differential expression $M[\cdot]$

$$M[y] \equiv -y'' + qy \quad \text{on} \quad [0,\infty), \qquad ' \equiv \frac{d}{dx}, \tag{1.1}$$

where the real-valued coefficient q is element of $L^p(\mathbb{R}^+)$ for some $p \geq 1$. The domain of T_α is defined by

$$\mathcal{D}(T_\alpha) = \{f \mid f \in L^2(\mathbb{R}^+),\ f' \text{ is absolutely continuous on } [0,X] \text{ for all } X > 0,\ M[f] \in L^2(\mathbb{R}^+),\ f(0)\cos\alpha + f'(0)\sin\alpha = 0\} \tag{1.2}$$

and the operator T_α is defined by

$$T_\alpha f = M[f], \quad f \in \mathcal{D}(T_\alpha). \tag{1.3}$$

Many results concerning this type of differential operators are given in EVERITT [2]. Because $q \in L^p(\mathbb{R}^+)$, $p \geq 1$, $M[\cdot]$ is in the limit-point case at infinity. So T_α is self-adjoint. Furthermore T_α is bounded below and there is a discrete (possibly empty) spectrum below zero, while the essential spectrum is the half-line $[0,\infty)$. In [2] a lower bound for T_0 in terms of the L^p-norm of the coefficient q has been given. For $p = 1,2$ and ∞ this bound was already well-known. In this note we improve the bound for T_0 and give a similar bound for $T_{\pi/2}$. The proof is a combination of the technique of Everitt and a less known result of Sz. NAGY [5].

The idea for this improvement arose form an inequality in ROSEN [4; Appendix A]

$$\left\{\int_{-\infty}^{\infty}|u|^4\,dx\right\}^{\frac{1}{4}} \leq 3^{-\frac{1}{8}}\left\{\int_{-\infty}^{\infty}|u|^2dx\right\}^{\frac{3}{8}}\left\{\int_{-\infty}^{\infty}|u'|^2dx\right\}^{\frac{1}{8}}, \quad u \in H^1(\mathbb{R}), \tag{1.4}$$

with equality for the choice $u(x) = c_1/\cosh(c_2(x-c_3))$, c_1, $c_2 \neq 0$, c_3 arbitrary constants. The author generalized this inequality to the cases

$$(1.5)\qquad \left\{\int_{-\infty}^{\infty} |u|^r\, dx\right\}^{\frac{1}{r}} \le K(r)\left\{\int_{-\infty}^{\infty} |u|^2 dx\right\}^{\frac{r+2}{4r}}\left\{\int_{-\infty}^{\infty} |u'|^2 dx\right\}^{\frac{r-2}{4r}}, \quad u \in H^1(\mathbb{R}),\quad r\ge 2,$$

with equality for the choice

$$(1.6)\qquad u(x) = c_1\left\{\cosh(c_2\sqrt{r^2-4}\ (x-c_3))\right\}^{-\frac{2}{r-2}},$$

c_1, $c_2 \neq 0$, c_3 arbitrary. The optimal constant reads

$$(1.7)\qquad \begin{cases} K(2) = 1, \\ K(r) = 2^{\frac{6-r}{2r}}(r-2)^{\frac{r-2}{4r}}(r+2)^{\frac{r-6}{4r}}\left[\dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{r-2})}{\Gamma(\frac{r+2}{2(r-2)})}\right]^{-\frac{r-2}{2r}}, \quad r > 2. \end{cases}$$

Afterwards it appeared that Sz. NAGY [5] had already a much more general result which is given below as a theorem.

<u>THEOREM 1.1</u> (Sz. NAGY [5]). *Let* y *be absolutely continuous on* $\mathbb{R}$. *Let for* $q > 0$ *and* $s \ge 1$ *the following integrals be finite*

$$(1.8)\qquad \|y\|_q^q = \int_{-\infty}^{\infty} |y|^q\, dx < \infty,$$

$$(1.9)\qquad \|y'\|_s^s = \int_{-\infty}^{\infty} |y'|^s\, dx < \infty.$$

Then the following inequalities hold for $t = 1 + \frac{s-1}{s}q$, $r > q$

$$(1.10)\qquad \sup_{x\in\mathbb{R}} |y(x)| \le \left(\tfrac{t}{2}\right)^{\frac{1}{t}} \|y\|_q^{\frac{q(s-1)}{st}} \|y'\|_s^{\frac{1}{t}},$$

$$(1.11)\qquad \|y\|_r \le \left[\frac{t}{2}\, H\!\left(\frac{t}{r-q}, \frac{s-1}{s}\right)\right]^{\frac{r-q}{rt}} \|y\|_q^{\frac{q}{r}\left(1+\frac{(r-q)(s-1)}{st}\right)} \|y'\|_s^{\frac{r-q}{rt}},$$

where

$$(1.12)\qquad H(u,v) = \frac{(u+v)^{-(u+v)}\Gamma(1+u+v)}{u^{-u}\Gamma(u)v^{-v}\Gamma(v)}, \qquad H(u,0) = H(0,v) = 1.$$

Equality occurs in the following cases:

for (1.10), $s = 1$, $|y(x)|$ *monotonely increasing until* $x = x_0$,

$|y(x)|$ *monotonely decreasing from* $x = x_0$;

for (1.10), $s > 1$, $y(x) = c_1 y_{sq}(|c_2x+c_3|)$, $c_1, c_2 \neq 0$, c_3 *arbitrary constants and*

$$(1.13) \qquad q < s, \quad y_{sq}(x) = \begin{cases} (1+x)^{\frac{s}{s-q}}, & 0 \le x \le 1, \\ 0, & x > 1, \end{cases}$$

$$(1.14) \qquad q = s, \quad y_{sq}(x) = e^{-x}, \qquad x \ge 0,$$

$$(1.15) \qquad q > s, \quad y_{sq}(x) = (1+x)^{\frac{s}{s-q}}, \qquad x \ge 0;$$

for (1.11), $s = 1$, *unless* $y \equiv 0$ *there is no equality;*

for (1.11), $q \ge s > 1$, $y(x) = c_1 y_{sqr}(|c_2x+c_3|)$, $c_1, c_2 \neq 0$, c_3 *arbitrary constants and* $u = y_{sqr}(x)$ *is the inverse function of*

$$(1.16) \qquad x = \int_u^1 \frac{dv}{[v^q(1-v^{r-q})]^{1/s}}, \qquad 0 \le u \le 1;$$

for (1.11), $s > q \ge 1$, $y(x) = \max(0, y_{sqr}(x))$ *with* $y_{sqr}(x)$ *defined by* (1.16).

REMARK 1.1. For $q = s = 2$, so that $t = 2$, (1.5), (1.6) and (1.7) are found.

REMARK 1.2. The form of the exponents can easily be found by applying the scaling transformations $\tilde{u}(x) = \alpha u(\beta x)$, with α, β arbitrary constants. See also LEVINE [3] for a discussion for some results concerning estimates of the optimal constants in $\mathbb{R}^n$, $n > 1$, for this type of inequalities.

2. RESULTS

In this section we formulate and prove our results for the operators T_0 (Dirichlet boundary condition) and $T_{\pi/2}$ (Neumann boundary condition).

THEOREM 2.1. *Let* T_0 *be defined by* (1.1), (1.2), (1.3). *Let* $q \in L^p(\mathbb{R}^+)$ *for some* $p \ge 1$, *and let* q *be real-valued. Then the following inequality holds*

$$(2.1)\qquad (T_0f,f) \geq -\ \ell(p)\ \|q\|_p^{\frac{2p}{2p-1}}\ \|f\|_2^2, \qquad f \in \mathcal{D}(T_0),$$

where (,) *denotes the inner-product in* $L^2(\mathbb{R}^+)$, $\|q\|_p = \{\int_0^\infty |q|^p\, dx\}^{1/p}$, *and*

$$(2.2)\qquad \begin{cases} \ell(1) = 1/4, \\ \ell(p) = 2^{-\frac{2p}{2p-1}}\, p^{-\frac{2p}{2p-1}}(2p-1)^{\frac{2}{2p-1}}(p-1)^{\frac{2p-4}{2p-1}}\left[\dfrac{\Gamma(\frac12)\Gamma(p-1)}{\Gamma(p-\frac12)}\right]^{-\frac{2}{2p-1}} \\ \quad = 2^{-2}p^{-\frac{2p}{2p-1}}(2p-1)^{\frac{3}{2p-1}}(p-1)^{\frac{2p-2}{2p-1}}[(p-1)!]^{-\frac{4}{2p-1}}[(2p-2)!]^{\frac{2}{2p-1}}, \quad p > 1. \end{cases}$$

PROOF. Integration by parts and application of the boundary condition yields

$$(2.3)\qquad (-f'',f) = \|f'\|_2^2.$$

The estimate for $p > 1$ is obtained as follows. Using Hölder's inequality we find

$$(2.4)\qquad (T_0f,f) = \|f'\|_2^2 + \int_0^\infty q|f|^2 dx \geq \|f'\|_2^2 - \int_0^\infty |q||f|^2\, dx$$
$$\geq \|f'\|_2^2 - \left\{\int_0^\infty |q|^p\, dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty |f|^{2s}\, dx\right\}^{\frac{1}{s}},$$

where s is the conjugate index of p: $s = p/(p-1)$. Application of (1.5) with $r = 2s$ and with $f(x) \equiv 0$ for $x < 0$ gives

$$(2.5)\qquad (T_0f,f) \geq \|f'\|_2^2 - \|q\|_p\, K^2(2s)\|f\|_2^{\frac{s+1}{s}}\, \|f'\|_2^{\frac{s-1}{s}}.$$

By application of the inequality

$$(2.6)\qquad ab \leq a^P/P + b^Q/Q, \quad a,b > 0, \quad 1 < P < \infty, \quad P^{-1} + Q^{-1} = 1,$$

with the choices

$$(2.7)\qquad P = 2s/(s-1) = 2p; \quad Q = 2s/(s+1) = 2p/(2p-1),$$

$$(2.8)\qquad a = P^{\frac{1}{P}}\|f'\|_2^{\frac{2}{P}} = (2p)^{\frac{1}{2p}}\|f'\|_2^{\frac{1}{p}},$$

$$(2.9)\qquad b = P^{-\frac{1}{P}} K^2(2s)\|q\|_p\|f\|_2^{\frac{2}{Q}} = (2p)^{\frac{1}{2p}} K^2(\tfrac{2p}{p-1})\|q\|_p\|f\|_2^{\frac{2p-1}{p}},$$

the following bound can be found:

$$(2.10)\qquad (T_0f,f) \geq -b^Q/Q$$

$$= -\left\{K(2s)\right\}^{\frac{4s}{s+1}}\|q\|_p^{\frac{2s}{s+1}}\left(\frac{2s}{s-1}\right)^{-\frac{s-1}{s+1}}\left(\frac{s+1}{2s}\right)\|f\|_2^2$$

$$= -\left\{K(\tfrac{2p}{p-1})\right\}^{\frac{4p}{2p-1}}(2p)^{-\frac{2p}{2p-1}}(2p-1)\|q\|^{\frac{2p}{2p-1}}\|f\|_2^2.$$

Inserting (1.7) with $r = 2p/(p-1)$ gives the result (2.2). For $p = 1$ the proof follows the same lines. We now use (1.10) with $q = s = t = 2$, i.e. $K(\infty) = 1$ and (2.6) with $P = Q = 2$. □

REMARK 2.1. EVERITT [2] used instead of the optimal constant $K(2p/(p-1))$ the value $2^{\frac{s-1}{2s}} = 2^{\frac{1}{2p}}$ and his result reads

$$(2.11)\qquad (T_0f,f) \geq -k(p)\ \|q\|_p^{\frac{2p}{2p-1}}\ \|f\|_2^2,$$

where

$$(2.12)\qquad k(p) = 2^{-\frac{2p-2}{2p-1}}\ p^{-\frac{2p}{2p-1}}(2p-1).$$

In the following table we compare the constants

p	s	r = 2s	$2^{\frac{1}{2p}}$	K(2p/(p-1)) = K(r)	k(p)	$\ell(p)$
1	∞	∞	$2^{\frac{1}{2}} = 1.414$	1	1	$2^{-2} = 0.250$
2	2	4	$2^{\frac{1}{4}} = 1.189$	$3^{-\frac{1}{8}} = 0.872$	$2^{-2}3 = 0.750$	$2^{-\frac{8}{3}}3^{\frac{2}{3}} = 0.328$
3	3/2	3	$2^{\frac{1}{6}} = 1.122$	$2^{\frac{1}{6}}3^{\frac{1}{6}}5^{-\frac{1}{4}} = 0.901$	$2^{-\frac{4}{5}}3^{-\frac{6}{5}}5 = 0.768$	$2^{-\frac{4}{5}}3^{-\frac{4}{5}}5^{\frac{2}{5}} = 0.454$
∞	1	2	1	1	1	1

THEOREM 2.2. *Let* $T_{\pi/2}$ *be defined by* (1.1), (1.2), (1.3). *Let* $q \in L^p(\mathbb{R}^+)$ *for some* $p \geq 1$, *and let* q *be real-valued. Then the following inequality holds*

$$(2.13)\qquad (T_{\pi/2}f,f) \geq -\ \ell^*(p)\ \|q\|^{\frac{2p}{2p-1}}\ \|f\|_2^2,$$

where

$$(2.14)\qquad \ell^*(p) = 2^{\frac{2}{2p-1}}\ \ell(p).$$

PROOF. For T_0 as well as for $T_{\pi/2}$ equation (2.3) is valid. Instead of an application of (1.5) with $f(x) \equiv 0$, $x < 0$, we extend f as $f(-x) = f(x)$, $x \geq 0$. For symmetric functions f (1.5) gives, $r \geq 2$,

$$(2.15)\qquad \left\{\int_0^\infty |f|^r dx\right\}^{\frac{1}{r}} \leq K^*(r)\left\{\int_0^\infty |f|^2 dx\right\}^{\frac{r+2}{4r}} \left\{\int_0^\infty |f'|^2 dx\right\}^{\frac{r-2}{4r}},$$

where

$$(2.16)\qquad K^*(r) = 2^{-\frac{1}{r}+\frac{r+2}{4r}+\frac{r-2}{4r}}\ K(r) = 2^{\frac{r-2}{2r}}K(r).$$

For the choice $r = 2p/(p-1)$ $K^*(2p/(p-1)) = 2^{\frac{1}{2p}}K(2p/(p-1))$, which has to be raised to the power $4p/(2p-1)$ (see (2.10)). This gives the result (2.14). □

REMARK 2.2. EASTHAM [1] has noticed that the estimates for T_0 given by EVERITT [2] are also applicable for the operator $T_{\pi/2}$. In view of the fact that

$$(2.17)\qquad K(2p/(p-1)) = \{H(p-1,\tfrac{1}{2})\}^{\frac{1}{2p}} < 1, \qquad p > 1,$$

(see (1.12) for $H(u,v)$), because $H(u,v)$ is a decreasing function in both variables, we find that $K^*(2p/(p-1)) < 2^{1/(2p)}$. But then Remark 2.1 implies that $\ell^*(p) < k(p)$.

REMARK 2.3. It is clear from the proof of Theorem 2.1 and 2.2 that the inequalities in (2.1) and (2.13) can only be equalities if all three estimates (2.4), (2.5) and (2.6) are equalities. In (2.4) this is so if $q = -|q|$ and

$$(2.18)\qquad q = -c_1|f|^{\frac{2}{p-1}}, \qquad p > 1,$$

where c_1 is an arbitrary positive constant. Equality in (2.5) is assured if the eigenfunction belonging to the first eigenvalue of the operator T_0 or $T_{\pi/2}$ equals

$$(2.19)\qquad f(x) = \{\cosh(c_2(x-c_3))\}^{-(p-1)}, \qquad p > 1,$$

with $c_2 \neq 0$ and c_3 arbitrary constants (see (1.6)). Equality in (2.6) is assured if $a^P = b^Q$ or

$$(2.20)\qquad (2p)\|f'\|_2^2 = \left\{K\left(\frac{2p}{2p-1}\right)\right\}^{\frac{4p}{2p-1}}(2p)^{-\frac{1}{2p-1}}(2p-1)\|q\|_p^{\frac{2p}{2p-1}}\|f\|_2^2 .$$

It can be proved that (2.18) and (2.19) imply (2.20). From (2.19) it is clear that the estimate for T_0 can never be an equality, because $f(0) \neq 0$, but since for $c_3 = 0$, $f'(0) = 0$, the estimates for $T_{\pi/2}$ offer perspectives. Below we shall demonstrate that it is possible to achieve equality for $T_{\pi/2}$, $p > 1$, and that the estimates for T_0, $p > 1$, are optimal in the sense that the bounds for some special choices of q are arbitrarily close to the first eigenvalue.

Using (2.18), (2.19) we find

$$(2.21)\qquad M[f] = -(p-1)^2 c_2^{\,2} f - p(1-p)c_2^{\,2} f^{\frac{p+1}{p-1}} - c_1 f^{\frac{p+1}{p-1}} .$$

For $c_1 = p(p-1)$, $c_2 = 1$, $c_3 = 0$ f is an eigenfunction of the operator $T_{\pi/2}$. Since f does not possess any zero, it belongs to the lowest eigenvalue $\lambda_1 = -(p-1)^2$. Making use of the identity

$$(2.22)\qquad \int_0^\infty \cosh^{-\alpha}(x)dx = \frac{1}{2}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})}, \qquad \alpha > 0,$$

we find

$$(2.23)\qquad -\ell^*(p)\left\| -p(p-1)\left\{\cosh(.)\right\}^{-2}\right\|_p^{\frac{2p}{2p-1}} = -(p-1)^2.$$

It means that for $T = T_{\pi/2}$, $p > 1$, the bound in this note can give equality.

For $c_1 = p(p-1)$, $c_2 = 1$, $c_3 = A$ and q defined by (2.18) the estimate (2.1) for T_0 gives for all $\tilde{f} \in \mathcal{D}(T_0)$

$$(2.24)\qquad (T_0\tilde{f},\tilde{f})/(\tilde{f},\tilde{f}) \geq \ell(p)\left\| -p(p-1)\left\{\cosh(.-A)\right\}^{-2}\right\|^{\frac{2p}{2p-1}} =$$

$$= -(p-1)^2\left(1-2^{2p-1}p^{-1}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(p)}e^{-2pA} + O(e^{-4pA})\right),$$

$$A \to \infty.$$

It is also possible to give an upper bound for the first eigenvalue by evaluating the Rayleigh quotient $(T_0\tilde{f},\tilde{f})/(\tilde{f},\tilde{f})$ with the choice

$$(2.25)\qquad \tilde{f}(x) = \{\cosh(x-A)\}^{1-p} - \{\cosh(A)\}^{1-p}e^{-(p-1)x}.$$

So $\tilde{f} \in \mathcal{D}(T_0)$. We now find

$$(2.26)\qquad (T_0\tilde{f},\tilde{f})/(\tilde{f},\tilde{f}) \le -(p-1)^2(1-K_p e^{-(p-1)A} + O(e^{-2(p-1)A})),$$

$$A \to \infty,$$

where K_p is a positive constant, which depends on p only. So for $T = T_0$, $p > 1$, it follows from (2.24) and (2.26) that the bound (2.1) comes arbitrarily close to the first eigenvalue for the special choices of q by letting $A \to \infty$. In this sense the given bounds for $T = T_0$, $p > 1$, are optimal.

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