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VARIATIONAL ANALYSIS OF A PERTURBED FREE BOUNDARY PROBLEM

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Variational analysis of a perturbed free boundary problem*)

by

O. Diekmann & D. Hilhorst

ABSTRACT

Using convex analysis we show that the solution u_ε of a nonlinear boundary value problem (depending on a parameter ε) converges to a limit u_0 as $\varepsilon \downarrow 0$. We characterize u_0 as the solution of a free boundary problem and we discuss some of its properties.

KEY WORDS & PHRASES : *nonlinear boundary value problem, integral condition, singular perturbation, convex analysis, duality theory, maximal monotone operator, free boundary problem*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

In this paper we study the nonlinear boundary value problem

$$\text{BVP}\quad \begin{cases} -\Delta u + h(\frac{u}{\varepsilon}) = f & \text{in } \Omega \\ \int_\Omega h(\frac{u(x)}{\varepsilon})dx = C & \\ u|_{\partial\Omega} \text{ is constant (but unknown)} & \end{cases}$$

where

(i) Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$

(ii) ε is a small positive parameter

(iii) $h : \mathbb{R} \to \mathbb{R}$ is a given continuous, strictly monotone increasing function with $h(0) = 0$

(iv) f is a given distribution in $H^{-1}(\Omega)$

(v) C is a given constant which satisfies the compatibility condition

$$h(-\infty)|\Omega| < C < h(+\infty)|\Omega|.$$

Here $|\Omega|$ denotes the measure of Ω.

The motivation for studying BVP partly stems from the physics of ionized gases and in this respect we continue earlier work [15, 16, 21, 22]. We refer to [22] and Appendix 2 for a discussion of this connection.

Our basic tools are the calculus of variations, convex analysis and the maximum principle.

We prove that BVP admits for each $\varepsilon > 0$ a unique solution u_ε which converges as $\varepsilon \downarrow 0$ to a limit u_0. Moreover, we give a variational characterization of u_0 which narrows down to the conclusion that u_0 solves a free boundary problem.

Our findings fit in with those of BRAUNER & NICOLAENKO [7, 8] in their study of related Dirichlet problems (we certainly have been inspired by their paper). In this connection it is also worth mentioning the work of FRANK & VAN GROESEN [18] and FRANK & WENDT [19] which analyses in particular the coincidence set. In Appendix 1 we give the analysis of the homogeneous Dirichlet problem.

In the physical problem of Appendix 2 the parameter ε naturally appears in the same way as in BVP. In other situations one may arrive at the equation

$$-\varepsilon\Delta v + h(v) = f.$$

Then our results bear on $\varepsilon v_\varepsilon$ and $h(v_\varepsilon)$.

In a recent paper [9] BRAUNER & NICOLAENKO stress the following point. Suppose one wants to analyse some free boundary problem, then it may be possible to view this problem as the limit when $\varepsilon \downarrow 0$ of a problem like BVP (with ε occurring in the argument of a smooth function). This smooth regularization can be used to solve problems of existence, regularity and approximation and it forms an alternative version of the usual penalization method. (see also [6]).

After these general remarks, let us describe the contents of the paper in some more detail. We shall interpret BVP as the subdifferential equation $\partial V_\varepsilon(u) = 0$, where V_ε is a proper, strictly convex, lower semicontinuous and coercive functional defined on the direct sum of $H_0^1(\Omega)$ and the constant functions on Ω. This is rather easy if h satisfies certain growth restrictions. For the general case we heavily lean upon some results of BREZIS [11]. These and some other preliminaries are collected in section 2. The functional V_ε is defined in section 3 and from its properties we deduce the existence and uniqueness of a solution u_ε for each $\varepsilon > 0$.

The functional V_ε depends monotonously on ε and therefore has a well-defined limit V_0. Moreover, V_ε is coercive uniformly in ε and consequently we deduce in section 4 that as $\varepsilon \downarrow 0$ u_ε converges to u_0, the minimizer of V_0. The subdifferential ∂V_0 is multivalued. We find that u_0 satisfies an operator inclusion relation if h is bounded and a variational inequality if h is unbounded. We emphasize that the reduced problem is piecewise linear: u_0 depends only on f,C and $h(\pm\infty)$.

Problem BVP has the form

$$Lu + N\left(\frac{u}{\varepsilon}\right) = f$$

where both L and N are maximal monotone operators. The variational approach suggests the introduction of a dual formulation (in section 5) which turns out to be of the form

$$(\varepsilon A + I)p = g$$

where A is a maximal monotone operator on $(L_2(\Omega))^n$ with a special structure, and where g is related to f by div g = f. This gives some further insight into the convergence. The limit p_0 equals the projection of g onto the closed convex set $\overline{\mathcal{D}(A)}$. Duality theory yields a characterization of $\overline{\mathcal{D}(A)}$ by inequalities which seems difficult to obtain directly. Duality theory has been applied to related problems by ARTHURS & ROBINSON [4] and ARTHURS [3]. For the basic theory we refer to EKELAND & TEMAM [17]

In section 6 we assume $f \in L_\infty(\Omega)$. We employ maximum principle arguments and make some estimates. We prove that u_ε and u_0 belong to $W^{2,p}(\Omega)$ for each $p \geq 1$ and that u_ε converges weakly to u_0 in $W^{2,p}(\mathcal{O})$ for each $\mathcal{O}$ with $\bar{\mathcal{O}} \subset \Omega$. Either one has convergence in $W^{2,p}(\Omega)$ itself, or a boundary layer develops as $\varepsilon \downarrow 0$. We present criteria in terms of the data f, $h(\pm\infty)$ and C from which it can be decided in many cases which of these two possibilities actually occurs. In section 7 we briefly discuss the one-dimensional case.

Our analysis reveals that BVP and the homogeneous Dirichlet problem have exactly the same variational structure. In order to emphasize this point we analyse the latter problem in Appendix 1. Finally, we discuss the physical background of BVP in Appendix 2.

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2. PRELIMINARIES

In this section we collect some definitions and results from the literature which we will use later. We state these in the form we need, which is not always the most general.

Let B be a Banach space and B^* its dual. Let $F : B \to (-\infty,+\infty]$ be a proper (i.e. $F \not\equiv +\infty$), lower semicontinuous (ℓ.s.c.), convex functional. The polar (or conjugate) functional $F^* : B^* \to (-\infty,+\infty]$ is defined by

$$(2.1) \qquad F^*(u^*) = \sup\{\langle u^*,u\rangle - F(u) \mid u \in \mathcal{D}(F)\}$$

where

$$(2.2) \qquad \mathcal{D}(F) = \{u \mid F(u) < +\infty\}$$

and where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between B^* and B. The subdifferential ∂F is a, possibly multivalued, mapping of X into X^* defined by

$$(2.3) \qquad u^* \in \partial F(u) \text{ if and only if } F(v) - F(u) \geq \langle u^*,v-u\rangle,\ \forall v \in B.$$

<u>LEMMA 2.1.</u>

$$u^* \in \partial F(u) \textit{ if and only if } F(u) + F^*(u^*) = \langle u^*,u\rangle.$$

<u>LEMMA 2.2.</u>

$$u^* \in \partial F(u) \textit{ if and only if } u \in \partial F^*(u^*).$$

A convenient reference for these items is EKELAND & TEMAM [17].

If B is a Hilbert space one can identify B and B^* and then ∂F becomes a mapping of B into itself. It is well-known that ∂F is maximal monotone.

<u>LEMMA 2.3.</u> *Let* H *be a Hilbert space and* A *a maximal monotone operator on* H. *Then, for each* $\varepsilon > 0$, $(I+\varepsilon A)^{-1}$ *is a contraction defined on all of* H *and* $\lim_{\varepsilon\downarrow 0} (I+\varepsilon A)^{-1} h =$ *projection of* h *on* $\overline{\mathcal{D}(A)}$.

For this standard result we refer to BREZIS [10].

Let, as before, Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We shall write H_0^1, L_2 etc. to denote $H_0^1(\Omega), L_2(\Omega)$ etc. Also, we write $\int u$ to denote $\int_\Omega u(x)dx$.

Let $j : \mathbb{R} \to [0,+\infty]$ be a convex, l.s.c. function such that $j(0) = 0$. The convex, l.s.c. functional $J : H_0^1 \to [0,+\infty]$ is defined by

$$J(u) = \begin{cases} \int j(u) & \text{if } j(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

The following two lemmas are special cases of results due to BREZIS [11].

LEMMA 2.4. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then*

$$J^*(w) = \begin{cases} \int j^*(w) & \textit{if } w \in H^{-1} \cap L_1 \textit{ and } j^*(w) \in L_1 \\ +\infty & \textit{otherwise.} \end{cases}$$

LEMMA 2.5. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then* $w \in \partial J(u)$ *if and only if* $w \in H^{-1} \cap L_1$, $w.u \in L_1$ *and* $w(x) \in \partial j(u(x))$ *for almost all* $x \in \Omega$.

Finally, we quote a special case of a result of BREZIS & BROWDER [12, 13].

LEMMA 2.6. *Assume* $w \in H^{-1} \cap L_1$ *and* $u \in H_0^1$ *are such that* $w(x)u(x) \geq g(x)$ *for almost all* $x \in \Omega$ *and some* $g \in L_1$. *Then* $w.u \in L_1$ *and*

$$\langle w,u \rangle = \int w.u.$$

Here and in the following $\langle \cdot,\cdot \rangle$ denotes the duality pairing of H^{-1} and H_0^1. We observe that Lemma 2.6 implies that the condition $w.u \in L_1$ in Lemma 2.5 is automatically satisfied.

3. VARIATIONAL FORMULATION

Let X be the direct sum of H_0^1 and the constant functions: $X = H_0^1 \oplus \{c\}$. If u is some element of X, we write $u = \tilde{u} + u|_{\partial\Omega}$ for its decomposition. X is, provided with the topology inherited of H^1, a Hilbert space. Moreover, X is isomorphic to $H_0^1 \times \mathbb{R}$ and the H^1-norm is equivalent with the norm $\|\tilde{u}\|_{H_0^1} + |u|_{\partial\Omega}|$ on X. So we can realize the dual space X^* by

$$X^* = H^{-1} \times \mathbb{R}$$

the pairing being given by

$$<(w,k),u>_X = <w,\tilde{u}> + ku|_{\partial\Omega}.$$

Consider the functional W defined on X by

$$(3.1) \qquad W(u) = \begin{cases} \int H(u) - C\,u|_{\partial\Omega} & \text{if } H(u) \in L_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where by definition

$$(3.2) \qquad H(y) = \int_0^y h(\eta)d\eta.$$

<u>LEMMA 3.1.</u>

$$W^*(w,k) = \begin{cases} \int H^*(w) & \textit{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \textit{ and } \int w = k + C, \\ +\infty & \textit{otherwise.} \end{cases}$$

<u>PROOF</u>. The idea is to take first the supremum with respect to the H_0^1-component and to use Lemma 2.4.

$$\sup\{<w,\tilde{u}> + k\,u|_{\partial\Omega} - \int H(\tilde{u} + u|_{\partial\Omega}) + C\,u|_{\partial\Omega} \mid \tilde{u} \in H_0^1,\ u|_{\partial\Omega} \in \mathbb{R}\}$$

$$= \begin{cases} \sup\{ \int H^*(w) - u|_{\partial\Omega} \int w + (k+C)\, u|_{\partial\Omega} \;\Big|\; u|_{\partial\Omega} \in \mathbb{R} \} \\ \qquad\qquad \text{if } w \in L_1 \cap H^{-1} \text{ and } H^*(w) \in L_1 \\ +\infty \qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} \int H^*(w) & \text{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \text{ and } \int w = k + C \\ +\infty & \text{otherwise.} \end{cases}$$

LEMMA 3.2.

$$\partial W(u) = \begin{cases} (h(u), \int h(u) - C) & \textit{if } h(u) \in H^{-1} \cap L_1 \\ \emptyset & \textit{otherwise.} \end{cases}$$

PROOF. (i) Let $(w,k) \in \partial W(u)$ then

$$W(\tilde{v} + v|_{\partial\Omega}) - W(\tilde{u} + u|_{\partial\Omega}) \geq \langle w, \tilde{v} - \tilde{u} \rangle + k(v-u)|_{\partial\Omega}$$

for all $\tilde{v} \in H_0^1$ and all $v|_{\partial\Omega} \in \mathbb{R}$. By first taking $v|_{\partial\Omega} = u|_{\partial\Omega}$, we see that necessarily w belongs to the subdifferential of the functional $\tilde{u} \to W(\tilde{u}+u|_{\partial\Omega})$ defined on H_0^1. Hence, by Lemma 2.5, $w = h(u)$ and $w \in L_1$. Next, a combination of Lemma 2.1 and Lemma 3.1 shows that necessarily $k = \int w - C = \int h(u) - C$.
(ii) Conversely, let $h(u) \in H^{-1} \cap L_1$. Since h is the derivative of H we have

$$H(v) - H(u) \geq h(u)(v-u) = h(u)(\tilde{v} - \tilde{u} + (v-u)|_{\partial\Omega}).$$

So if $H(v)$ and $H(u) \in L_1$, we can invoke Lemma 2.6 and conclude that $h(u)(\tilde{v} - \tilde{u}) \in L_1$ and that the integral equals the duality pairing. Integration of the inequality then yields, after adding a term $-C(v-u)|_{\partial\Omega}$,

$$W(v) - W(u) \geq \langle h(u), \tilde{v} - \tilde{u} \rangle + \left(\int h(u) - C\right)(v-u)|_{\partial\Omega}. \qquad \square$$

We remark that, by Lemma 2.2, $\partial H^* = h^{-1}$. So, since h is strictly monotone,

(3.3) $$H^*(y) = \int_0^y h^{-1}(\eta)d\eta.$$

Let $g \in (L_2)^n$ be such that div g = f. The functional $G : (L_2)^n \to \mathbb{R}$ defined by

(3.4) $$G(p) = \int (\tfrac{1}{2}p^2 + g.p)$$

is Fréchet-differentiable with derivative p+g. The polar functional $G^* : (L_2)^n \to \mathbb{R}$ is given by

(3.5) $$G^*(p) = \tfrac{1}{2}\int (p-g)^2$$

and its derivative is p-g.

We define the bounded linear mapping $T : X \to (L_2)^n$ by

(3.6) $$T\,u = -\,\text{grad}\,u.$$

Its adjoint $T^*:(L_2)^n \to X^*$ is given by

(3.7) $$T^*p = (\text{div}\,p,0).$$

Clearly the functional $u \mapsto G(-Tu)$ defined on X is differentiable with derivative $-T^*G'(-Tu) = (-\Delta u - f,0)$.

Finally, let us put together the materials constructed above. Define $V_\varepsilon : X \to (-\infty,+\infty]$ by

(3.8) $$v_\varepsilon(u) = G(-Tu) + \varepsilon W(\tfrac{u}{\varepsilon}).$$

Then

(3.9) $$\partial V_\varepsilon(u) = \begin{cases} (-\Delta u- f + h(\frac{u}{\varepsilon}), \int h(\frac{u}{\varepsilon}) - C) & \text{if } h(\frac{u}{\varepsilon}) \in H^{-1} \cap L_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and, consequently, the problem BVP is equivalent with the variational problem

$$\text{VP} \qquad \inf_{u \in X} V_\varepsilon(u).$$

THEOREM 3.3. VP *has a unique solution* u_ε.

PROOF. G is convex, W is strictly convex and both functionals are l.s.c. (by Fatou's lemma). It remains to verify that V_ε is coercive on X. It is convenient to rewrite the functional V_ε as

$$V_\varepsilon(u) = \int \left(\tfrac{1}{2}(\text{grad}u)^2 + (g-a) \cdot \text{grad}u + \varepsilon\, H(\tfrac{u}{\varepsilon}) - \frac{C}{|\Omega|}\, u\right)$$

where $|\Omega|$ denotes the measure of Ω and a is such that $\text{div}a = C|\Omega|^{-1}$ (for instance take $a = C(n|\Omega|)^{-1}\,(x_1,\ldots,x_n)$). Since $C|\Omega|^{-1} \in (h(-\infty),h(+\infty))$, there exist positive constants δ and M_1 such that

$$\varepsilon\, H(\tfrac{y}{\varepsilon}) - \frac{C}{|\Omega|}\, y \geq \delta|y| - M_1.$$

By the inequalities of Hölder and Poincaré there exists a positive constant $M_2 = M_2(\Omega)$ such that

$$\int |\tilde{u}| \leq \sqrt{|\Omega|}\, \|\tilde{u}\|_{L_2} \leq M_2 \|\text{grad}\, \tilde{u}\|_{L_2} = M_2\|\text{grad}\, u\|_{L_2}.$$

Hence, using Hölder's inequality once more, we find

$$\begin{aligned} V_\varepsilon(u) &\geq \tfrac{1}{2}\|\text{grad}\,u\|^2_{L_2} - \|g-a\|_{L_2}\, \|\text{grad}\, u\|_{L_2} + \delta|\Omega|\, |\, u|_{\partial\Omega}| - \delta \int |\tilde{u}| - M_1 \\ &\geq \tfrac{1}{4}\|\text{grad}\, u\|^2_{L_2} + \delta|\Omega|\, |\, u|_{\partial\Omega}| - M_3 \end{aligned}$$

for some constant M_3. It should be noted that the right hand side is independent of ε. $\square$

4. LIMITING BEHAVIOUR OF u_ε AS $\varepsilon \downarrow 0$

In this section we show that u_ε converges as $\varepsilon \downarrow 0$. The limit u_0 is characterized as the unique solution of a variational problem. Equivalently one can characterize u_0 by an operator inclusion relation if h is bounded and by a variational inequality if h is unbounded. It turns out that u_0 depends only on $h(\pm\infty)$, f and C.

As $\varepsilon \downarrow 0$, the function $h(\frac{y}{\varepsilon})$ converges in the sense of graphs to the multivalued function

$$(4.1) \qquad h_0(y) = \begin{cases} h(+\infty), & y > 0 \\ [h(-\infty), h(+\infty)], & y = 0 \\ h(-\infty), & y < 0. \end{cases}$$

We define

$$(4.2) \qquad H_0(y) = \begin{cases} h(+\infty)y, & y > 0 \\ 0\ , & y = 0 \\ h(-\infty)y, & y < 0 \end{cases}$$

LEMMA 4.1. $\varepsilon\, H(\frac{y}{\varepsilon})$ *converges monotonously increasing to* $H_0(y)$.

PROOF. $h(\frac{\eta}{\varepsilon})$ increases towards $h_0(\eta)$ for $\eta > 0$ and decreases towards $h_0(\eta)$ for $\eta < 0$. Since $\varepsilon\, H(\frac{y}{\varepsilon}) = \int_0^y h(\frac{\eta}{\varepsilon})d\eta$ we can use Lebesgue's monotone convergence theorem. □

We note that, by Dini's theorem, the convergence is uniform on compact subsets if h is bounded and, for instance, uniform on compact subsets of $(-\infty,0]$ if $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. Motivated by Lemma 4.1 we define

$$(4.3) \qquad W_0(u) = \begin{cases} \int H_0(u) - C\, u|_{\partial\Omega} & \text{if } H_0(u) \in L_1 \\ +\infty & \text{otherwise} \end{cases}$$

and we introduce the reduced variational problem

$$\text{RVP} \quad \inf_{u \in X} \; G(-Tu) + W_0(u).$$

Exactly as in the proof of Theorem 3.3 it follows that RVP has a solution. The functional $G(-Tu) + W_0(u)$ is convex, but not strictly convex. Still we have

LEMMA 4.2. RVP *has a unique solution* u_0.

PROOF. Since $G(\operatorname{grad} u)$ is strictly convex on H_0^1, two minimizers can only differ by a constant. For arbitrary $u \in X$ define

$$\Omega_+(u) = \{x \mid u(x) > 0\}, \; \Omega_0(u) = \{x \mid u(x) = 0\}, \; \Omega_-(u) = \{x \mid u(x) < 0\}.$$

Then

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(+\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C$$

and

$$\lim_{\delta \uparrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(-\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C.$$

So if $W_0(u+\ell)$ is constant for $|\ell| \le \eta$ then necessarily for those values of ℓ

$$h(+\infty)|\Omega_+(u+\ell)| + h(+\infty)|\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| =$$

$$h(+\infty)|\Omega_+(u+\ell)| + h(-\infty)|\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| = C.$$

Since $h(+\infty) > h(-\infty)$ this implies that

$$\{x \mid -\eta \le u(x) \le \eta\}$$

has measure zero. Then, however, u has to be sign-definite (this follows, for instance, from the connection between Sobolev and Beppo Levi spaces; see DENY & LIONS [14]) and we arrive at the conclusion that either $h(+\infty)|\Omega| = C$ or $h(-\infty)|\Omega| = C$. Finally, the compatibility condition excludes both of these possibilities. □

THEOREM 4.3.

$$\lim_{\varepsilon \downarrow 0} \|u_\varepsilon - u_0\|_X = 0.$$

PROOF.

Step 1. We know that V_ε is coercive uniformly in ε (see the proof of Theorem 3.3). Hence $\|u_\varepsilon\|_X \le M$ for some constant M independent of ε and, consequently, the weak limit set of $\{u_\varepsilon\}$ is nonempty.

Step 2. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$ and suppose that $h(+\infty) = +\infty$. We claim that $\bar{u} \le 0$. Define $Q_0^\delta = \{x \mid \bar{u}(x) \ge \delta > 0\}$ and $Q_n^\delta = \{x \in Q_0^\delta \mid u_{\varepsilon_n}(x) \ge \frac{1}{2}\delta\}$. Then

$$\int |u_{\varepsilon_n} - \bar{u}|^2 \ge \int_{Q_0^\delta \setminus Q_n^\delta} |u_{\varepsilon_n} - \bar{u}|^2 \ge \frac{\delta^2}{4} |Q_0^\delta \setminus Q_n^\delta|.$$

Hence, since $u_{\varepsilon_n} \to \bar{u}$ strongly in L_2, necessarily $|Q_n^\delta| \to |Q_0^\delta|$. Furthermore,

$$\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right) \ge \varepsilon_n \int_{Q_n^\delta} H\left(\frac{\delta}{2\varepsilon_n}\right) = \varepsilon_n H\left(\frac{\delta}{2\varepsilon_n}\right) |Q_n^\delta|$$

Since $\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right)$ is bounded uniformly in n and since $\varepsilon_n H\left(\frac{\delta}{2\varepsilon_n}\right) \to +\infty$ as $n \to +\infty$, necessarily $|Q_n^\delta| \to 0$ as $n \to +\infty$. So we must have $|Q_0^\delta| = 0$. Since $\delta > 0$ was arbitrary we conclude that $\bar{u} \le 0$. Similarly, $h(-\infty) = -\infty$ implies $\bar{u} \ge 0$.

Step 3. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. We claim that $V_{\varepsilon_n}(u_{\varepsilon_n}) \to V_0(\bar{u})$. From $V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \langle \partial V_{\varepsilon_n}(\bar{u}), u_{\varepsilon_n} - \bar{u}\rangle_X$ we obtain, using step 2,

$$V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \int (\operatorname{grad} \bar{u} + g)(\operatorname{grad} u_{\varepsilon_n} - \operatorname{grad} \bar{u})$$

$$+ \int (h(\frac{\bar{u}}{\varepsilon_n})(u_{\varepsilon_n} - \bar{u})) - C(u_{\varepsilon_n} - \bar{u})|_{\partial\Omega}$$

Since the right-hand side converges to zero as $n \to +\infty$ we find

$$\liminf_{n\to+\infty} V_{\varepsilon_n}(u_{\varepsilon_n}) \geq \lim_{n\to\infty} V_{\varepsilon_n}(\bar{u}) = V_0(\bar{u}).$$

On the other hand, since u_{ε_n} minimizes V_{ε_n} and since $V_\varepsilon(v)$ is, for fixed v, monotone with respect to ε (Lemma 4.1), we have

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \leq V_{\varepsilon_n}(\bar{u}) \leq V_0(\bar{u}).$$

Step 4. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. Then

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \leq V_{\varepsilon_n}(u_0) \leq V_0(u_0)$$

and therefore $V_0(\bar{u}) \leq V_0(u_0)$. Hence $\bar{u} = u_0$.

Step 5. We now know that u_0 is the only point in the weak limit set of $\{u_\varepsilon\}$ and thus $u_\varepsilon \rightharpoonup u_0$ as $\varepsilon \downarrow 0$. From

$$\varepsilon \int \left(H\left(\frac{u_\varepsilon}{\varepsilon}\right) - H\left(\frac{u_0}{\varepsilon}\right)\right) \geq \int h\left(\frac{u_0}{\varepsilon}\right)(u_\varepsilon - u_0)$$

and Step 2 we conclude that

$$\liminf_{\varepsilon\downarrow 0} \int \varepsilon H\left(\frac{u_\varepsilon}{\varepsilon}\right) \geq \int H_0(u_0).$$

It then follows from the weak l.s.c. of G and Step 3 that necessarily $\|\mathrm{grad}\ u_\varepsilon\|_{L_2} \to \|\mathrm{grad}\ u_0\|_{L_2}$ as $\varepsilon \downarrow 0$. Consequently u_ε converges in fact strongly in X to u_0. □

In order to get more information about u_0 we first determine W_0^* and ∂W_0. We write $u \geq 0$ for some $u \in X$ if and only if $u(x) \geq 0$ for almost all $x \in \Omega$. Let C denote the closed, convex, positive cone corresponding to this ordering. By duality C induces a cone C^* in X^*: we write $(w,k) \geq 0$ if and only if $\langle (w,k),u\rangle_X \geq 0$ for all $u \in C$. For any $u \in X$ we define $u_+ = \max(u,0)$ and $u_- = \max(-u,0)$. Then $u_+ \in X$, $u_- \in X$ and at least one of these belongs to H_0^1 (see, for instance, KINDERLEHRER & STAMPACCHIA [23, Ch. II, Proposition 5.3]).

In the following we slightly abuse notation. But let us agree upon the convention that any inequality in which a quantity $+\infty$ appears is trivially fulfilled.

LEMMA 4.4.

$$W_0^*(w,k) = \begin{cases} 0 \quad \textit{if both} \quad \begin{array}{l} (h(+\infty)-w, h(+\infty)|\Omega| - C - k) \in C^* \\ (w - h(-\infty), k - h(-\infty)|\Omega| + C) \in C^* \end{array} \\ \\ +\infty \quad \textit{otherwise.} \end{cases}$$

PROOF.

$$W_0^*(w,k) = \sup\{<(w,k),u>_X - \int h(+\infty)u_+ + \int h(-\infty)u_- + C\, u|_{\partial\Omega} \mid u \in X\}$$

$$= \sup\{<(w-h(+\infty), k - h(+\infty)|\Omega| + C), u_+>_X$$

$$- <(w-h(-\infty), k - h(-\infty)|\Omega| + C), u_->_X \mid u \in X\}. \qquad \square$$

LEMMA 4.5. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then*

$$\partial W_0(u) = \{(w,k) \mid w \in L_1,\ w(x) \in h_0(u(x)) \textit{ for a.e. } x \in \Omega,\ k = \int w - C\}.$$

PROOF. (i) Suppose $(w,k) \in \partial W_0(u)$. As in the proof of Lemma 3.2 it follows that $w \in L_1$ and $w(x) \in h_0(u(x))$ a.e.. Let v_n be the solution of

$$\begin{cases} -\frac{1}{n}\Delta v_n + v_n = 0 \\ \quad v_n|_{\partial\Omega} = 1. \end{cases}$$

Then $v_n \geq 0$ and, as $n \to \infty$, v_n converges strongly in L_2 to zero. By Lemmas 2.1 and 4.4 we know that

$$<(h(+\infty) - w,\ h(+\infty)|\Omega| - C - k),\ v_n>_X \geq 0$$

and

$$<(h(-\infty) - w,\ h(-\infty)|\Omega| - C - k),\ v_n>_X \leq 0.$$

Taking into account that $w \in L_\infty$ (since $w \in h_0(u)$), we rewrite these inequalities as

$$\int (h(+\infty) - w)(v_n - 1) + h(+\infty)|\Omega| - C - k \geq 0$$

and

$$\int (h(-\infty) - w)(v_n - 1) + h(-\infty)|\Omega| - C - k \leq 0.$$

Upon passing to the limit $n \to +\infty$ we find that $\int w - C - k \geq 0$ and $\int w - C - k \leq 0$.

(ii) is exactly the same as the second part of the proof of Lemma 3.2. □

COROLLARY 4.6. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then* RVP *is equivalent with the reduced boundary value problem*

$$\text{RBVP} \quad \begin{cases} \Delta u + f \in h_0(u) \\ \int (\Delta u + f) = C \\ u|_{\partial\Omega} \textit{ is constant (but unknown).} \end{cases}$$

Finally, let us consider a function h which is unbounded. We concentrate on the case $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. From the proof of Theorem 4.3 we know that $u_0 \leq 0$. Consequently RVP is equivalent to minimizing a differentiable functional on the cone $-\mathcal{C}$ and, therefore, with the variational inequality:

$$\text{VI} \quad \begin{cases} \text{Find } u \in -\mathcal{C} \text{ such that for all } v \in -\mathcal{C} \\ \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v - u\rangle_X \geq 0. \end{cases}$$

Unfortunately we cannot use Lemma 2.5 in this situation (see, however, [20]) but still we have

LEMMA 4.7. *Suppose* $h(-\infty) > -\infty$ *and* $h(+\infty) = +\infty$. *Then*

$$\partial W_0(u) = \begin{cases} \{(w,k) \mid (w-h(-\infty),\ k - h(-\infty)|\Omega| + C) \in C^* \textit{ and} \\ \quad \langle (w-h(-\infty),\ k - h(-\infty)|\Omega| + C), u \rangle_X = 0\} & \textit{if } -u \in C \\ \emptyset \ \textit{otherwise}. \end{cases}$$

PROOF. This follows directly from Lemma 2.1, Lemma 4.4 and the fact that W_0 is linear on the negative cone. □

5. THE DUAL FORMULATION

So far we have used polar functionals repeatedly, but we have not yet given a systematic presentation of duality theory as applied to our problem. This will be done now. We follow closely EKELAND & TEMAM [17, Ch. III, section 4, in particular Remarque 4.2].

The dual formulation of VP, corresponding to the splitting $V_\varepsilon(u) = G(-Tu) + \varepsilon W(\frac{u}{\varepsilon})$, is given by

$$\mathrm{VP}^* \quad \operatorname*{Inf}_{p \in (L_2)^n} \ \varepsilon\, W^*(T^* p) + G^*(p).$$

Since VP is stable (use [17, Proposition III.2.3]), VP^* has a (unique) solution p_ε. Furthermore, the infima are equal to each other and u_ε and p_ε are related by the so-called extremality relations

$$(5.1) \qquad T^* p_\varepsilon = \partial W\left(\frac{u_\varepsilon}{\varepsilon}\right)$$

$$(5.2) \qquad p_\varepsilon = \partial G(-Tu_\varepsilon).$$

By Lemma 3.2 and (3.4) these can be rewritten as

$$(5.3) \qquad \operatorname{div} p_\varepsilon = h\left(\frac{u_\varepsilon}{\varepsilon}\right) \text{ and } \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) = C$$

$$(5.4) \qquad p_\varepsilon = g + \operatorname{grad} u_\varepsilon.$$

Note that g is not uniquely determined by div g = f but that (5.3) and (5.4) define $p_\varepsilon - g$ and $\operatorname{div} p_\varepsilon$ unambiguously. One can view (5.3) and (5.4) as a canonical splitting of BVP into first order equations. Indeed, elimination of p_ε leads to BVP. On the other hand, we can also eliminate u_ε to find the subdifferential equation satisfied by p_ε:

$$(5.5) \qquad \varepsilon\, T(\partial W)^{-1}(T^* p_\varepsilon) + p_\varepsilon = g$$

or, more explicitly,

$$\mathrm{BVP}^* \quad \begin{cases} -\varepsilon\, \mathrm{grad}(h^{-1}(\mathrm{div}\, p_\varepsilon)) + p_\varepsilon = g \\ \int \mathrm{div}\, p_\varepsilon = C \\ h^{-1}(\mathrm{div}\, p_\varepsilon) \in X \end{cases}$$

By Lemmas 2.2, 3.2 and [17, Proposition I.5.7] the operator A from $(L_2)^n$ into itself defined by

$$(5.6) \qquad \begin{cases} Ap = -\mathrm{grad}(h^{-1}(\mathrm{div}\, p)) \\ \mathcal{D}(A) = \{p \in (L_2)^n \mid \mathrm{div}\, p \in L_1,\ \int \mathrm{div}\, p = C,\ \mathrm{div}\, p = h(u) \text{ for} \\ \qquad\qquad \text{some } u \in X\} \end{cases}$$

is the subdifferential of the convex l.s.c. functional $p \mapsto W^*(T^*p)$. Consequently, A is maximal monotone. (See Weyer [26] for related results). Rewriting (5.5) as

$$(5.7) \qquad (\varepsilon A + I)p_\varepsilon = g$$

and invoking Lemma 2.3, we find that p_ε converges, as $\varepsilon \downarrow 0$, strongly in $(L_2)^n$ to the projection of g onto $\overline{\mathcal{D}(A)}$. It does not seem easy to characterize $\overline{\mathcal{D}(A)}$ directly from (5.6). Therefore we use duality theory once more, but now for the reduced problem.

The dual formulation of RVP is given by

$$\mathrm{RVP}^* \qquad \inf_{p\in(L_2)^n} \quad W_0^*(T^* p) + G^*(p).$$

By (3.5) and Lemma 4.4 the solution of RVP* is the projection of g onto the closed convex set

$$(5.8) \qquad Q = \{p \in (L_2)^n \mid (h(+\infty) - \operatorname{div} p,\ h(+\infty)|\Omega| - C) \in \mathcal{C}^*$$
$$\text{and } (\operatorname{div} p - h(-\infty),\ C - h(-\infty)|\Omega|) \in \mathcal{C}^*\}$$

Denoting the (unique) solution of RVP* by p_0, we have the extremality relations

$$(5.9) \qquad T^* p_0 \in \partial W_0(u_0)$$

$$(5.10) \qquad p_0 = \partial G(-T u_0).$$

The second one, $p_0 = g + \operatorname{grad} u_0$, is identical to the extremality relation $p_\varepsilon = g + \operatorname{grad} u_\varepsilon$. Hence the fact that u_ε converges strongly in X to u_0, implies that p_ε converges strongly in $(L_2)^n$ to p_0. So we find that p_ε converges to a limit which is at the same time the projection of g onto $\overline{\mathcal{D}(A)}$ and onto Q. Since g is an arbitrary element of $(L_2)^n$, necessarily $\overline{\mathcal{D}(A)} = Q$. Thus we have shown that (5.8) gives an explicit characterization of $\overline{\mathcal{D}(A)}$.

The extremality relation (5.9) is easy to work with only in the case that h is bounded (see Lemmas 4.5 and 4.7). It then follows that RBVP is equivalent to (5.9) - (5.10). Likewise one can, by elimination of u_0, derive a subdifferential equation for p_0 similar to BVP*.

If $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ we deduce from Lemma 4.7 that u_0 is the solution of the following variant of VI:

$$\begin{cases} \text{Find } u \in -\mathcal{C} \text{ such that} \\ \text{(i)} \quad \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v\rangle_X \le 0, \ \forall v \in \mathcal{C}, \\ \text{(ii)} \quad \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ u\rangle_X = 0. \end{cases}$$

6. THE REDUCED PROBLEM AS A FREE BOUNDARY PROBLEM

In this section we assume that $f \in L_\infty$. We shall deal with the regularity of u_0 (and u_ε), with the free boundary value problem satisfied by u_0 and with sharp convergence results versus the occurrence of boundary layers. We shall write $C^{1,\alpha}$ to denote the Hölder space $C^{1,\alpha}(\bar{\Omega})$ and $W^{2,p}$ to denote the usual Sobolev space. We recall that $W^{2,p}$ is imbedded into $C^{1,\alpha}$ if $p(1-\alpha) \geq n$.

THEOREM 6.1. *If* h *is bounded,* u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \geq 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

PROOF.

$$\|\Delta u_\varepsilon\|_{L_\infty} \leq \max\{-h(-\infty), h(+\infty)\} + \|f\|_{L_\infty}. \qquad \square$$

We can now interpret RBVP as a free boundary problem. The domain Ω consists of three subdomains:

$$\Omega_+ = \{x \in \Omega \mid u_0(x) > 0\} \text{ where } -\Delta u_0 + h(+\infty) = f \quad \text{a.e.}$$
$$\Omega_- = \{x \in \Omega \mid u_0(x) < 0\} \text{ where } -\Delta u_0 + h(-\infty) = f \quad \text{a.e.}$$
$$\Omega_0 = \{x \in \Omega \mid u_0(x) = 0\} \text{ which has to be a subset of}$$
$$\{x \in \Omega \mid h(-\infty) \leq f(x) \leq h\,(+\infty)\}.$$

These subdomains are unknown, possibly empty and such that

$$h(+\infty)\,|\Omega_+| + h(-\infty)\,|\Omega_-| + \int_{\Omega_0} f = C.$$

From the proof of Theorem 4.3 we know that $u_0 = 0$ if $h(\pm\infty) = \pm\infty$. So in that case we cannot have convergence in $W^{2,p}$ unless $\int f = C$.

Next, we concentrate on the most interesting case in which h is bounded from one and only one side. In the remaining part of this section we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. We emphasize that all theorems below have a counterpart in the case $h(-\infty) = -\infty$ and $h(+\infty) < +\infty$.

THEOREM 6.2. $u_\varepsilon \in W^{2,p}$ *for each* $p \geq 1$.

PROOF. We shall show that Δu_ε is bounded by finding an upper bound for u_ε. Let $\zeta \in H_0^1$ be the solution of $-\Delta\zeta + h(-\infty) = f$. Then, in fact, since $\Delta\zeta$ is bounded, we have $\zeta \in C^{1,\alpha}$. Define $\psi \in H_0^1$ by $\psi = u_\varepsilon - u_\varepsilon|_{\partial\Omega} - \zeta$. Then

$$\Delta\psi = \Delta u_\varepsilon - \Delta\zeta = h\left(\frac{u_\varepsilon}{\varepsilon}\right) - h(-\infty) \geq 0$$

and hence, by the weak maximum principle, $\psi \leq 0$. So u_ε is bounded from above by the bounded function $u_\varepsilon|_{\partial\Omega} + \zeta$. □

THEOREM 6.3. *If* $C \leq \int f$, u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \geq 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

PROOF. We show that $h(\frac{u_\varepsilon}{\varepsilon})$ and hence Δu_ε is bounded. Choose $\delta > 0$ and define

$$\Omega_\varepsilon = \{x \in \Omega \,|\, h\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) > \|f\|_{L_\infty} + \delta\}.$$

The points of $\partial\Omega_\varepsilon$ either belong to $\partial\Omega$ or are such that $h(\frac{u_\varepsilon(x)}{\varepsilon}) = \|f\|_{L_\infty} + \delta$. If $|\Omega_\varepsilon| \neq 0$ and $\partial\Omega_\varepsilon \cap \partial\Omega = \emptyset$, we find that simultaneously $\Delta u_\varepsilon > 0$ in Ω_ε and u_ε assumes, with respect to Ω_ε, its maximum in an interior point. Since this is impossible we conclude that either $|\Omega_\varepsilon| = 0$ or $\partial\Omega_\varepsilon \cap \partial\Omega \neq \emptyset$ and u_ε assumes its maximum at $\partial\Omega$ with $h(\frac{u_\varepsilon|_{\partial\Omega}}{\varepsilon}) > \|f\|_{L_\infty} + \delta$.

Suppose $|\Omega_\varepsilon| \neq 0$. Let $\tilde{\Omega}_\varepsilon$ be a domain with boundary $\partial\Omega \cup \Gamma$ and strictly contained in Ω_ε. We define $\tilde{u}_\varepsilon$ to be the solution of $\Delta\tilde{u} = \delta$, $\tilde{u}(x) = u_\varepsilon(x)$, $x \in \partial\tilde{\Omega}_\varepsilon$. Then $\tilde{u}_\varepsilon$ attains its maximum on $\partial\Omega$ and it follows from the Hopf maximum principle [24, Thm 7, p. 65] that $\frac{\partial\tilde{u}_\varepsilon}{\partial n}|_{\partial\Omega} > 0$. Also we have that $\Delta(\tilde{u}_\varepsilon - u_\varepsilon) = \delta - h(\frac{u_\varepsilon}{\varepsilon}) + f \leq 0$ and therefore $\tilde{u}_\varepsilon - u_\varepsilon \geq 0$ and, finally,

$$\frac{\partial u_\varepsilon}{\partial n}\Big|_{\partial\Omega} \geq \frac{\partial\tilde{u}_\varepsilon}{\partial n}\Big|_{\partial\Omega} > 0.$$

This leads to the contradiction

$$C - \int f = \int \Delta u_\varepsilon = \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} > 0. \qquad \square$$

The proof above shows that, if $h(\frac{u_\varepsilon}{\varepsilon})$ blows up somewhere, it does so at the boundary. If $u_0|_{\partial\Omega} < 0$ this can not happen, so we also have

THEOREM 6.4. *If* $u_0|_{\partial\Omega} < 0$ *then* u_ε *converges to* u_0 *weakly in* $W^{2,p}$, $p \geq 1$, *and strongly in* $C^{1,\alpha}$, $\alpha \in [0,1)$.

THEOREM 6.5. $u_0 \in W^{2,p}$ *for each* $p \geq 1$.

PROOF. If $u_0|_{\partial\Omega} < 0$ we can apply Theorem 6.4. If $u_0|_{\partial\Omega} = 0$, then u_0 is completely characterized by the restriction of RVP to H_0^1. The result then follows, for instance, from Appendix 1. □

THEOREM 6.6. u_0 *is completely characterized by*

$$\begin{cases} -\Delta u_0 + h(-\infty) - f \leq 0 & \text{a.e.} \\ u_0 \leq 0 & \text{a.e.} \\ u_0(-\Delta u_0 + h(-\infty) - f) = 0 & \text{a.e.} \\ \int(\Delta u_0 + f) - C \leq 0 & \\ u_0|_{\partial\Omega}(\int(\Delta u_0 + f) - C) = 0. & \end{cases}$$

PROOF. Because of Theorem 6.5 we can rewrite the variant of VI given at the end of section 5 in the form

$$\int(\Delta u_0 - h(-\infty) + f)v + (C - \int(\Delta u_0 + f))v|_{\partial\Omega} \geq 0, \quad \forall v \in C,$$

$$\int(\Delta u_0 - h(-\infty) + f)u_0 + (C - \int(\Delta u_0 + f))u_0|_{\partial\Omega} = 0,$$

and from this formulation the result easily follows. □

If $\int f \geq C$ then Theorem 6.3 implies that actually $\int(\Delta u_0 + f) = C$. We emphasize that $\int f < C$ does not preclude the possibility that $u_0|_{\partial\Omega} < 0$ and $\int(\Delta u_0 + f) = C$. However, if $\int(\Delta u_0 + f) < C$ we cannot have weak convergence in $W^{2,p}$. Next, we present some conditions on the data $h(-\infty)$, f and C under which this happens.

THEOREM 6.7. *Any of the three assumptions*

(i) $f(x) \le h(-\infty)$ a.e.

(ii) $f(x) \ge h(-\infty)$ a.e. *and* $\int f < C$

(iii) $\int_{\tilde\Omega} f < C$ *for all* $\tilde\Omega \subset \Omega$

implies that $\int(\Delta u_0 + f) < C$.

PROOF. (i) Let $v \in H_0^1$ be the solution of $\Delta v = h(-\infty) - f$. Then $v \le 0$ and $\int(\Delta v + f) = h(-\infty)|\Omega| < C$. By Theorem 6.6 $u_0 = v$.

(ii) Again by Theorem 6.6, $u_0 = 0$.

(iii)

$$\int(\Delta u_0 + f) = \int_{\bar\Omega} h(-\infty) + \int_{\Omega\setminus\bar\Omega} f = h(-\infty)|\bar\Omega| + \int_{\Omega\setminus\bar\Omega} f < C$$

where $\bar\Omega = \{x \mid u_0(x) < 0\}$. □

In the proof of Theorem 6.3 it was already shown that if u_ε displays a layer of rapid change somewhere, it certainly does so near to the boundary. Next we prove that it can do so *only* near to the boundary. The estimates below have been indicated to us by H. BREZIS.

THEOREM 6.8. *Assume* h *is* C^1. *Then* u_ε *converges to* u_0 *weakly in* $W^{2,p}(\mathcal{O})$ *for any open set* $\mathcal{O}$ *with* $\bar{\mathcal{O}} \subset \Omega$ *and any* $p \ge 1$.

PROOF.

Step 1. Since $h(y) > h(-\infty)$ we have

$$\int \left|h\left(\frac{u_\varepsilon}{\varepsilon}\right)\right| \le \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) - 2h(-\infty)|\Omega| = C - 2h(-\infty)|\Omega|.$$

Step 2. Since u_ε is bounded uniformly in ε in H^1, it follows from the Sobolev imbedding theorem (see, for instance, ADAMS [1, p. 97]) that u_ε is bounded uniformly in ε in $L_r(\Omega)$, where $r = \frac{2n}{n-2}$ if $n > 2$ and $r \ge 1$ if $n \le 2$.

Step 3. (Proof by recursion). We suppose that $h\left(\frac{u_\varepsilon}{\varepsilon}\right)$ is bounded uniformly in ε in $L_q(\mathcal{U}_1)$ for some $q \ge 1$ and $\mathcal{U}_1$ such that $\overline{\mathcal{U}_1} \subset \Omega$. Let ζ be a C^∞-function with compact support in $\mathcal{U}_1$. We multiply the differential equation by $|h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} h(\frac{u_\varepsilon}{\varepsilon})\ |\zeta|^t$ and we integrate. Thus we obtain

$$\int -\Delta u_\varepsilon\ |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} h(\frac{u_\varepsilon}{\varepsilon})|\zeta|^t + \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \le \int |f\zeta|\ |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^{t-1}.$$

Integrating the first term by parts and using the inequality $ab \le \frac{1}{\alpha} a^\alpha + \frac{1}{\beta} b^\beta$ with $a,b > 0$, $\alpha,\beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, for the term at the right hand side we deduce

$$\frac{t-1}{\varepsilon} \int |\mathrm{grad}\ u_\varepsilon|^2\ |\zeta|^t\ |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2}\ h'(\frac{u_\varepsilon}{\varepsilon}) + \frac{1}{t} \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$$

$$\le \frac{1}{t} \int |f\zeta|^t - \int |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2}\ h(\frac{u_\varepsilon}{\varepsilon})\ \mathrm{grad}\ u_\varepsilon \cdot \mathrm{grad}\ |\zeta|^t.$$

We observe that the first term at the left hand side is nonnegative (so we delete this term). Now let $\gamma(x) = |h(x)|^{t-2} h(x)$ and $\Gamma(x) = \int_0^x \gamma(\tau)d\tau$. Then $\Gamma(x) \le x\gamma(x)$ for all x and hence

$$-\int \gamma(\frac{u_\varepsilon}{\varepsilon})\mathrm{grad}\ u_\varepsilon \cdot |\mathrm{grad}\ \zeta|^t = \varepsilon \int \Gamma(\frac{u_\varepsilon}{\varepsilon})\Delta|\zeta|^t \le \int u_\varepsilon \gamma(\frac{u_\varepsilon}{\varepsilon})\Delta|\zeta|^t$$

So finally

$$\text{(6.1)} \qquad \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \le K_1 + K_2 \int_{\mathcal{U}_1} |u_\varepsilon|\ |h(\frac{u_\varepsilon}{\varepsilon})|^{t-1}$$

We now distinguish different cases:

1st case $q = 1$. If $n > 2$, we choose $t = 1 + \frac{n+2}{2n}$ in (6.1) and apply Hölder's inequality with conjugate exponents $\frac{2n}{n-2}$ and $\frac{2n}{n+2}$; also using the results of Steps 1 and 2 we deduce that $\int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$ is bounded uniformly in ε. If $n \le 2$, we choose $t = 1 + \frac{r-1}{r}$ for some $r > 1$ and apply Hölder's inequality with conjugate exponents r and $\frac{r}{r-1}$ to obtain a similar result. So we know in both cases that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$ for some $t > 1$ and any open set $\mathcal{U}_2$ with $\overline{\mathcal{U}_2} \subset \mathcal{U}_1$. Consequently u_ε is bounded uniformly in $W^{2,t}(\mathcal{U}_2)$ (cf. AGMON [2]).

<u>2nd case $q > \frac{n}{2}$</u>. It follows from the Sobolev imbedding theorem that u_ε is bounded uniformly in ε in $L_\infty(\mathcal{U}_1)$. Choosing $t = q+1$ in (6.1), we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in ε in $L^{q+1}(\mathcal{U}_2)$. The result of the theorem follows then from a bootstrap argument.

<u>3rd case $q \le \frac{n}{2}$</u>. By the Sobolev imbedding theorem u_ε is bounded uniformly in $L_{q^*}(\mathcal{U}_1)$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$ (or $\frac{1}{q^*} = \frac{1}{q} - \alpha$ for any $\alpha \in (0,\frac{1}{q})$ if $q = \frac{2}{n}$). Let q^{**} be the conjugate exponent of q^* and choose $t = 1 + \frac{q}{q^{**}}$. Applying Hölder's inequality (with exponents q^* and q^{**}) to (6.1) we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$. Now a bootstrap argument either yields the result or leads to the 2nd case. $\square$

7. THE ONE DIMENSIONAL CASE

Again we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. The results of section 5 imply that p_0 is the projection of g onto the set

$$\overline{\mathcal{D}(A)} = \{p \in L_2 \mid (p' - h(-\infty),\ C - h(-\infty)|\Omega|) \in \mathcal{C}^*\}.$$

A simple calculation shows that, with $\Omega = (-1,+1)$,

$$\overline{\mathcal{D}(A)} \cap H^1 = \{p \mid p' \ge h(-\infty) \text{ and } p(1) - p(-1) \le C\}.$$

We found in section 6 that $p_0 \in \overline{\mathcal{D}(A)} \cap H^1$ if $f \in L_\infty$. So we can find p_0 by minimizing the L_2-distance to g subject to two constraints: an inequality for the derivative and a bound for the total variation. This is more or less a combinatorial problem which is rather easy to solve for some given smooth g, but whose general solution is cumbersome. We refer to [16, section 4] for a more detailed discussion of the symmetric case, noting that the result presented there covers the general case after some minor modifications. Finally, we remark that, once p_0 is found, u_0 can be calculated from the extremality relations.

APPENDIX 1. THE HOMOGENEOUS DIRICHLET PROBLEM

In this appendix we present some results about the problem

$$-\Delta u + h(\tfrac{u}{\varepsilon}) \ni f,$$

where by assumption h is the subdifferential of a convex, l.s.c. function $H : \mathbb{R} \to [0,\infty)$, with $H(0) = 0$ and $H(y) < +\infty$ for all $y \in \mathbb{R}$. Here $f \in H^{-1}$ is given and $u \in H_0^1$ is sought. We use some of the notation defined in the preceding pages and omit all proofs since these are similar to (and in fact easier than) those already given. In contravention of prior definitions we now have:

$$T : H_0^1 \to (L_2)^n, \qquad Tu = -\text{grad } u$$

$$T^*: (L_2)^n \to H^{-1}, \qquad T^*p = \text{div } p$$

$$W : H_0^1 \to [0,\infty], \qquad W(u) = \begin{cases} \int H(u) & \text{if } H(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

The problem can be rewritten as

$$\partial V_\varepsilon(u) \ni 0$$

where

$$V_\varepsilon(u) = G(-Tu) + \varepsilon\, W(\tfrac{u}{\varepsilon}).$$

It admits a unique solution u_ε which converges as $\varepsilon \downarrow 0$ strongly in H_0^1 to u_0, the unique solution of

$$\operatorname*{Inf}_{u \in H_0^1} \; G(-Tu) + W_0(u).$$

If h is bounded u_0 satisfies

$$-\Delta u + h_0(u) \ni f$$

and if, for instance, $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ then u_0 solves the variational inequality: find $u \leq 0$ such that

$$<-\Delta u + h(-\infty) - f,\ v-u> \geq 0, \quad \forall v \leq 0.$$

The dual formulation is obtained by the transformations

$$p = g - Tu$$
$$u \in \varepsilon h^{-1}(T^* p)$$
$$f = T^* g$$

and reads

$$\varepsilon\, T(h^{-1}(T^* p)) + p \ni g$$

or, equivalently,

$$(\varepsilon A + I)p \ni g$$

where $A : (L_2)^n \to (L_2)^n$ is defined by

$$Ap = T(h^{-1}(T^* p))$$

$\mathcal{D}(A) = \{p \in (L_2)^n \mid T^* p \in L_1$ and there exists $u \in H_0^1$ such that $T^* p \in h(u)\}$.

As $\varepsilon \downarrow 0$, p_ε converges to the projection of g onto

$$\overline{\mathcal{D}(A)} = \{p \in (L_2)^n \mid h(-\infty) \leq T^* p \leq h(+\infty)\},$$

where the inequalities are defined by the positive cone in H_0^1 and the duality of H_0^1 and H^{-1}.

If $f \in L_\infty$, u_ε converges to u_0 weakly in $W^{2,p}$ for each $p \geq 1$ and strongly in $C^{1,\alpha}$ for each $\alpha \in [0,1)$. This follows most easily from the observation that, by the maximum principle, u_ε equals the solution of the "truncated" problem

$$-\Delta u + \tilde{h}\left(\frac{u}{\varepsilon}\right) \ni f$$

where

$$\tilde{h}(y) = \begin{cases} \|f\|_{L_\infty} & \text{if } h(y) \geq \|f\|_{L_\infty} \\ h(y) & \text{if } -\|f\|_{L_\infty} \leq h(y) \leq \|f\|_{L_\infty} \\ -\|f\|_{L_\infty} & \text{if } h(y) \leq -\|f\|_{L_\infty}. \end{cases}$$

For sharper estimates under additional assumptions we refer to [7], [8], [5] and [25].

APPENDIX 2. THE PHYSICAL BACKGROUND OF THE PROBLEM

Consider a bounded domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a charge distribution inside Ω with two components:

(i) a fixed ionic charge density en_i

(ii) a mobile electronic charge density $-en_e$ such that

$$\text{(A.1)} \qquad \int n_e = N_e.$$

Here e is the unit charge, n_i and n_e are number densities and N_e is a number. N_e and n_i are given, but n_e is unknown.

Let the region outside Ω be a conductor. Then we have the condition

(A.2) the potential Φ is constant outside Ω.

Physically this condition is realized by the formation of a surface charge density which, however, will be of no further concern.

The equation for the potential Φ in Ω can be deduced from two physical laws:

$$\text{(A.3)} \qquad \Delta\Phi = -4\pi e(n_i - n_e), \qquad \text{Poisson's equation,}$$

and

(A.4) $$n_e = K\, e^{\frac{e\Phi}{k_B T}},$$ Boltzmann's formula.

Here K is a normalization constant, T is the temperature of the system and k_B is Boltzmann's constant.

Substituting (A.4) into (A.3) and (A.1) we obtain the problem

$$\begin{cases} -\Delta\Phi + 4\pi e K\, e^{\frac{e\Phi}{k_B T}} = 4\pi e\, n_i \\ K \int e^{\frac{e\Phi}{k_B T}} = N_e \\ \Phi|_{\partial\Omega} \text{ is constant (but unknown)} \end{cases}$$

which, up to a renaming of the constants and variables, is the special case of BVP in which $h(y) = e^y - 1$.

Alternatively, one can argue that n_e should be such that the free energy F of the system be minimized under the constraint (A.1). The free energy is defined by

$$F = U - TS$$

where U is the electrostatic energy given by

$$U = \frac{1}{8\pi} \int (\text{grad } \Phi)^2,$$

T is the temperature and S the entropy given by

$$S = - k_B \int n_e \, \ell n \, n_e .$$

So if E_i denotes the electric field created by the ions and E_e the electric field created by the electrons, it comes to solve the minimization problem

$$\operatorname*{Inf}_{E_e} k_B T \int \text{div } E_e \, \ell n \, (\text{div } E_e) + \frac{1}{8\pi} \int (E_i - E_e)^2$$

subject to the constraint

$$\int \operatorname{div} E_e = N_e.$$

Clearly this problem corresponds to VP*.

The main results of this paper concern the limiting behaviour of the potential Φ and the electrical field E_e due to the electrons, as the temperature T tends to zero. For instance, we find that at $\partial\Omega$ no boundary layer occurs if the total charge density $\int n_i$ of the ions exceeds N_e. In the limit $T \to 0$ there may be regions where electrons are absent. If such a region $\bar{\Omega}$ is strictly contained in Ω it necessarily must be such that $\int_{\bar{\Omega}} n_i = 0$. For such a region which extends up to $\partial\Omega$ there is a more complicated condition. If $n_i \geq 0$ and $\int n_i < N_e$, necessarily a boundary layer arises: the electrons are repelled against the conductor.

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