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FOURIER INTEGRALS OF SERIES OF BESSEL FUNCTIONS
ARISING IN THE THEORY OF RESIDUAL CURRENTS IN
TIDAL AREAS

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Fourier integrals of series of Bessel functions arising in the theory of residual currents in tidal areas

by

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ABSTRACT

Fourier transforms of some series of Bessel functions are considered, which arise in the theory of residual currents in tidal areas. The integrals are evaluated by writing them in terms of Legendre functions. The series are expressible in these functions too.

KEY WORDS & PHRASES: *Bessel functions, Fourier integrals of generalized hypergeometric functions*

1. INTRODUCTION

In the statistical theory of residual currents in tidal areas, ZIMMERMAN (1978,1980) obtained for the nontransient solution of a differential equation, describing the vorticity of a two-dimensional velocity field, a representation in terms of Bessel functions. In TEMME (1978^a, 1978^b) some analytical and numerical aspects of functions arising in this theory were considered. In the present report we compute Fourier integrals of some of these functions.

Let us consider (1.3) and (1.4) of TEMME (1978^b) in first approximation (that is, with $\ell=0$) then the residual part of the velocity is given by

$$(1.1) \qquad \eta(\vec{k},t) = c_0(\vec{k})$$

where $\vec{k}$ is a two-vector, the wave number in a Fourier analysis. Here we take $\vec{k} = (k,0)$ and we are interested in

$$(1.2) \qquad \hat{\eta}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_0(k) e^{-ixk} dk$$

and in some functions related to it. The function $c_0(k)$ is given in terms of Bessel functions as the series (we take τ_1, τ_2 and b of our previous publication equal to 1)

$$(1.3) \qquad c_0(k) = ia(k)\ \{\psi_1(k) + \psi_2(k)\},$$

where $a(k)$ is related to a forcing field and

$$(1.4) \qquad \psi_1(k) = \frac{i}{k} \sum_{n=-\infty}^{\infty} \frac{nJ_n^2(k)}{in-1}, \quad \psi_2(k) = \sum_{n=-\infty}^{\infty} \frac{J_n'(k)J_n(k)}{in-1}.$$

$\psi_1(k)$ and $\psi_2(k)$ are real functions. By using some symmetry relations for the Bessel functions we obtain

$$(1.5) \qquad \psi_1(k) = \frac{2}{k} \sum_{n=1}^{\infty} \frac{n^2 J_n^2(k)}{n^2+1}, \quad \psi_2(k) = -\sum_{n=-\infty}^{\infty} \frac{J_n(k)J_n'(k)}{n^2+1}$$

In this paper we are interested in the Fourier transform of $k^{-1}\psi_1(k)$ and $k^{-1}\psi_2(k)$. Since these functions are even functions of k we write the transform as

$$(1.6)\qquad \hat{f}(x) = \sqrt{2/\pi}\int_0^\infty \cos kx\, f(k)\, dk,\qquad \hat{g}(x) = \sqrt{2/\pi}\int_0^\infty \cos kx\, g(k)\, dk.$$

with

$$(1.7)\qquad f(k) = k^{-1}\psi_1(k),\qquad g(k) = k^{-1}\,\psi_2(k).$$

The functions in (1.6) are required in the following analysis (ZIMMERMAN (1981)). The function a(k) occurring in (1.3) is in fact the distribution h(k) of fluctuations in water depth (up to a factor), that is, $a(k) = -\,ik\,h(k)$, and the Fourier transform

$$(1.8)\qquad \hat{h}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, h(k)\, dk$$

is the depth. Suppose that we are given the following representation of $\hat{h}(x)$.

$$(1.9)\qquad \hat{h}(x) = \int_{-\infty}^{x} \hat{\phi}(\xi)\, d\xi - \tfrac{1}{2},$$

where for example, the slope $\hat{\phi}(x) = \pi^{-\frac{1}{2}}\alpha\,\exp(-\alpha^2x^2)$ (this special form of $\hat{\phi}(x)$ will not be used here). The function $\hat{u}(x)$ defined by

$$(1.10)\qquad \hat{u}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\, u(k)\, dk,$$

with $u(k) = k^{-1}\,\eta(k)$, is of special interest in the theory. It can be written in the form

$$(1.11)\qquad \hat{u}(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\xi)[\hat{f}(x-\xi) + \hat{g}(x-\xi)]d\xi$$

where f and g and their Fourier transforms are introduced in (1.6) and (1.7). The respresentation (1.11) is obtained by the Fourier convolution theorem. To see this, write (1.10) via $u(k) = k^{-1}\eta(k)$, (1.1), (1.2), $a(k) = -ik\,h(k)$, as

$$\hat{u}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} kh(k)\ [f(k)+g(k)]\ e^{-ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}(\xi)\ [\hat{f}(x-\xi) + \hat{g}(x-\xi)]\,d\xi$$

with $\hat{v}(x)$ the Fourier transform of kh(k). This is written by using (1.8) and (1.9) as

$$\hat{v}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(k)\ e^{-ikx}dk =$$

$$\frac{i}{\sqrt{2\pi}}\ \frac{d}{dx}\int_{-\infty}^{\infty} h(k)\ e^{-ikx}dk = i\hat{\phi}(x).$$

The Fourier transform of $\hat{g}(x)$ in (1.6) is obtained by using the basic formula

$$(1.12)\qquad \int_0^{\infty} \cos kx\ J_n^2(k)\ dk = \begin{cases} \frac{1}{2}(-1)^n\ P_{n-\frac{1}{2}}(\frac{1}{2}x^2-1), & |x| \le 2 \\ 0, & |x| > 2, \end{cases}$$

where $P_\nu(z)$ is a Legendre function (GRADSHTEIN & RYZHIK (1965, p.732)). A remarkable feature is that this integral vanishes outside the x- interval (-2,2). The functions $\hat{f}(x)$ and $\hat{g}(x)$ share this property.

We will show in the following section that also $\hat{g}(x)$ can be expressed in terms of a Legendre function. For $\hat{f}(x)$ this is less obvious. In section 3 we give a relation which expresses $\hat{f}(x)$ in terms of an integral of $\hat{g}(x)$. In section 4 we given series expansions of $\hat{g}(x)$, which enable numerical evaluation of this function. In section 5 we write the functions and their Fourier transforms as generalized hypergeometric functions.

2. THE FUNCTION $\hat{g}(x)$

By using

$$\frac{d}{dx}\int_0^\infty \frac{\cos kx}{k} J_n(k)\, J_n'(k)\, dk =$$

$$-\tfrac{1}{2}\int_0^\infty \sin kx\, dJ_n^2(k) = \tfrac{1}{2}x\int_0^\infty \cos kx\, J_n^2(k)\, dk$$

we obtain for $\hat{g}(x)$ via (1.5),(1.6), (1.7) and (1.12)

$$\hat{g}'(x) = \frac{-x}{2\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n P_{n-\frac{1}{2}}(\tfrac{1}{2}x^2-1)}{n^2+1}. \tag{2.1}$$

The integral of this expression can be handled by using the well-known relation

$$P_\nu^{-1}(x) = -(1-x^2)^{-\frac{1}{2}}\int_1^x P_\nu(\xi)\, d\xi, \tag{2.2}$$

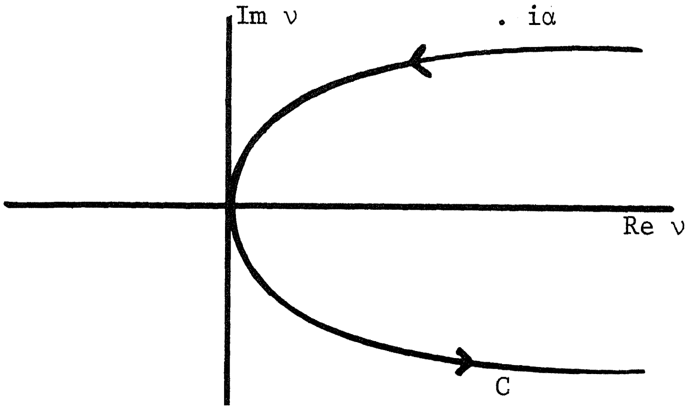
and so we obtain (using $\hat{g}(2) = 0$)

$$\begin{aligned} \hat{g}(x) &= \frac{-1}{2\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+1}\int_1^{\frac{1}{2}x^2-1} P_{n-\frac{1}{2}}(\xi)d\xi \\ &= \frac{x\sqrt{1-\frac{1}{4}x^2}}{2\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+1} P_{n-\frac{1}{2}}^{-1}(\tfrac{1}{2}x^2-1). \end{aligned} \tag{2.3}$$

According to $P_{-\nu-1}^{\mu}(x) = P_\nu^\mu(x)$, we infer that the Legendre functions in this sum are even in n. The series can be evaluated by considering the integral

$$\frac{1}{2\pi i}\int_C \frac{P_{\nu-\frac{1}{2}}^{\mu}(z)}{(\nu^2+\alpha^2)\sin\nu\pi}\, d\nu, \quad \mu \in \mathbb{R}, \ -1 < z < 1, \tag{2.4}$$

where the contour C runs as in the following figure. We suppose temporarily that α is a complex number such that $0 < \arg i\alpha < \frac{1}{2}\pi$. C encloses the poles 1,2,... of $1/\sin \alpha\pi$. It does not enclose the poles -1,-2,... and $\pm\, i\alpha$. Furthermore, it cuts the real ν-axis in $\nu=0$ perpendicularly.



Fif. 2.1 Contour for (2.4)

Deforming C into the axis Re ν=0, we pass the pole at ν=iα and the resulting integral vanishes since the integrand of (2.4) is an odd function of ν. By using the residues we arrive at

$$(2.5) \qquad \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{\mu}P_{n-\frac{1}{2}}(z)}{n^2+\alpha^2} = \frac{P^{\mu}_{i\alpha\ -\frac{1}{2}}(z)}{\alpha\ \sinh\ \alpha\pi}.$$

Using the principle of analytic continuation it follows that this formula is valid for all $\alpha \neq \pm i, \pm 2i, \ldots$.

As a consequence, the function $\hat{g}(x)$ of (2.3) can be written as

$$(2.6) \qquad \hat{g}(x) = \frac{\pi x\sqrt{1-\frac{1}{4}x^2}}{2\sqrt{2\pi}\ \sinh\pi}\ P^{-1}_{i-\frac{1}{2}}(\tfrac{1}{2}x^2-1).$$

A final step makes use of the relation (GRADSHTEIN & RYZHIK (1965, p.1008)

$$P^{-m}_{\nu}(x) = (-1)^m\frac{\Gamma(\nu+1-m)}{\Gamma(\nu+1+m)}\ P^{m}_{\nu}(x),$$

which results in

$$(2.7) \qquad \hat{g}(x) = \frac{x\sqrt{2\pi}\ \sqrt{1-\frac{1}{4}x^2}}{5\ \sinh\pi}\ P^{1}_{i-\frac{1}{2}}(\tfrac{1}{2}x^2-1),\ -\ 2 \le x \le 2.$$

The Legendre function is associated with the so called conical functions; it is real, although one of the parameters is complex.

The justification of considering (2.4) and deriving (2.5) follows from the asymptotic behaviour of $P^\mu_\nu(\cos\phi)$ for large complex values of ν. From GRADSHTEIN & RYZHIK (1965, p.1002) we obtain

$$P^\mu_\nu(\cos\phi) \sim \frac{2}{\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}\cos[(\nu+\tfrac{1}{2})\phi+\tfrac{1}{2}\pi(\mu-\tfrac{1}{2})]$$

and it follows that (2.4) converges absolutely for $0 < \phi < \pi$ and $\mu < 3/2$. Under these conditions the integral (2.4) yields (2.5).

A simple result as in (2.7) for $\hat{f}(x)$ is not obtained. In the next section we express $\hat{f}(x)$ in terms of $\hat{g}(x)$, from which we can obtain $\hat{f}(x)$ by integration.

3. THE FUNCTION $\hat{f}(x)$ AS AN INTEGRAL OF $\hat{g}(x)$.

By using the manipulations resulting into (2.1), we easily derive

$$x\hat{f}''(x) = -2\hat{g}'(x).$$

From this relation we obtain

$$(3.1)\qquad \begin{aligned} -2\hat{g}(x) &= \int_0^x \xi\hat{f}''(\xi)d\xi - 2\hat{g}(0) \\ &= x\hat{f}'(x) - \hat{f}(x) + \hat{f}(0) - 2\hat{g}(0). \end{aligned}$$

From (2.6) we cannot conclude that $\hat{g}(0) = 0$, since the Legendre function is not finite at $x = 0$. We can evaluate $\hat{f}(0)$ and $\hat{g}(0)$ from (1.5), (1.6) and (1.7), giving

$$(3.1)\qquad \begin{aligned} \hat{f}(0) &= 2\sqrt{2/\pi}\sum_{n=1}^{\infty}\frac{n^2}{n^2+1}\int_0^\infty k^{-2}J_n^2(k)dk \\ \hat{g}(0) &= \sqrt{2/\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}\int_0^\infty k^{-1}J_n(k)J_{n+1}(k)dk \end{aligned}$$

where we used for the second case $J_n'(k) = \frac{n}{k}J_n(k)-J_{n+1}(k)$. The integrals in (3.1) are easily evaluated by using the well-known Weber-Shafheitlin integrals, with as special case

$$\int_0^\infty J_\nu(t)J_\mu(t)t^{-\lambda}dt = 2^{-\lambda}\Gamma(\lambda)\Gamma(\frac{\nu+\mu-\lambda+1}{2}) /$$
$$\left[\Gamma(\frac{-\nu+\mu+\lambda+1}{2})\Gamma(\frac{\nu+\mu+\lambda+1}{2})\Gamma(\frac{\nu-\mu+\lambda+1}{2})\right].$$

Then it follows that

$$\hat{f}(0) = \frac{1}{5}(\frac{2}{\pi})^{3/2} \sum_{n=1}^{\infty} \{\frac{4}{n^2+1} + \frac{1}{n^2-\frac{1}{4}}\} = \frac{4}{5}\sqrt{\frac{2}{\pi}} \coth \pi.$$

The result for $\hat{g}(0)$ is

$$\hat{g}(0) = \frac{2}{5}\sqrt{\frac{2}{\pi}} \coth \pi \tag{3.2}$$

and we conlude that $\hat{f}(0) = 2\hat{g}(0)$, which reduces (3.1) to

$$\frac{d}{dx}\frac{1}{x}\hat{f}(x) = -2x^{-2}\hat{g}(x). \tag{3.3}$$

Upon integrating we obtain (where we use $\hat{f}(2) = 0$)

$$\hat{f}(x) = 2x\int_x^2 \xi^{-2}\hat{g}(\xi)d\xi, \tag{3.4}$$

which is the desired relation between $\hat{f}(x)$ and $\hat{g}(x)$. When we integrate (3.3) with initial value x=0, the result is

$$\hat{f}(x) = cx + \hat{f}(0) + 2x\int_0^x \xi^{-2}[\hat{g}(0)-\hat{g}(\xi)]d\xi, \tag{3.5}$$

where the constant can be computed by using more information about the behaviour of $\hat{f}(x)$ and $\hat{g}(x)$ near x=0. It can be shown by using the representation of section 5 that

$$c = -\sqrt{\pi/2}. \tag{3.6}$$

4. POWER SERIES FOR $\hat{g}(x)$

By writing $P_\nu^\mu(x)$ in terms of Gauss' hypergeometric ${}_2F_1$-function it is possible to derive series expansions, which enable numerical evalution.

The starting point is GRADSHTEIN & RYZHIK (1964, p.999, formula 8.704)

$$P_\nu^{-1}(x) = \left[\frac{1-x}{1+x}\right]^{\frac{1}{2}} {}_2F_1(-\nu,1+\nu;2;\frac{1-x}{2}), \quad -1 < x < 1,$$

and so (2.6) becomes

$$\hat{g}(x) = \sqrt{\frac{\pi}{2}}\frac{1-\frac{1}{4}x^2}{\sinh\pi}\, {}_2F_1(\tfrac{1}{2}-i,\tfrac{1}{2}+i;2;1-\tfrac{1}{4}x^2).$$

With the familiar expansion of the ${}_2F_1$-function we can write this as

$$\text{(4.1)} \qquad \hat{g}(x) = \sqrt{\frac{\pi}{2}}\,\frac{1-\frac{1}{4}x^2}{\sinh\pi} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-i)_n(\frac{1}{2}+i)_n(1-\frac{1}{4}x^2)^n}{(2)_n\; n!}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ $(n=0,1,2...)$. This series is useful for small values of $(1-x^2/4)$, i.e., for x in the neighbourhood of 2. Remark that for $x=0$ the value of $\hat{g}(0)$ given in (3.2) is obtained. In this evaluation we use

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

$c\neq 0,-1,\ldots,$ $\mathrm{Re}(c-a-b)>0$.

For $x \to 0^+$ we can use a transformation for the ${}_2F_1$-function, for instance, formula 15.3.11 of ABRAMOWITZ AND STEGUN (1964, p.559), where we must take m=1. The result is in our case

$$\text{(4.2)} \qquad \hat{g}(x) = \hat{g}(0) + \frac{\frac{1}{4}x^2\coth\pi}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(3/2-i)_n(3/2+i)_n}{n!\,(n+1)!}(\tfrac{1}{4}x^2)^n \times$$

$$[\ln\tfrac{1}{4}x^2 - \psi(n+1) - \psi(n+2) + \psi(3/2-i+n) + \psi(3/2+i+n)].$$

When $\psi(1/2-i) + \psi(1/2+i)$ is available, the remaining terms are easily computed by recursion.

Representations (4.1) and (4.2) are convenient starting points for numerical evalution of $\hat{g}(x)$. The function $\hat{f}(x)$ was computed by using (3.4) or (3.5). The graphs of the functions $\hat{f}(x)$ and $\hat{g}(x)$ are shown in the figures 4.1 and 4.2.

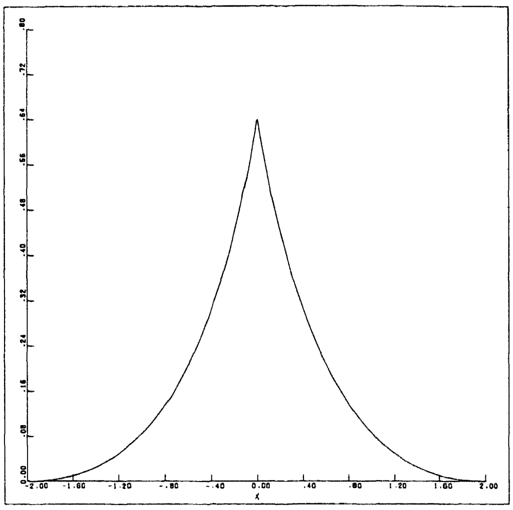


Fig. 4.1 The function $\hat{f}(x)$

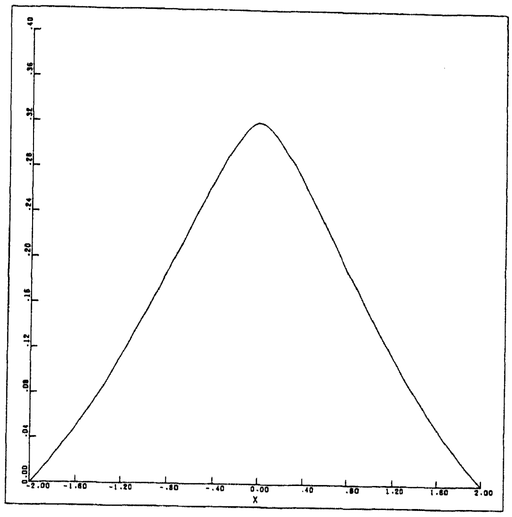


Fig. 4.2 The function $\hat{g}(x)$

5. $\hat{f}(x)$ AND $\hat{g}(x)$ EXPRESSED IN TERMS OF ${}_pF_q$-functions

We first write f and g introduced in (1.7) in terms of ${}_pF_q$-functions. In TEMME (1978[a]) we showed that

$$\sum_{n=-\infty}^{\infty} \frac{b^2}{n^2+b^2} J_n^2(k) = \frac{\pi b}{\sinh \pi b} J_{-ib}(k)\, J_{ib}(k)$$

where k and b are complex numbers, $ib \notin \mathbb{Z}$. The product of the Bessel functions can be expressed in terms of the ${}_1F_2$-functions and so we obtain

$$\text{(5.1)} \qquad \begin{aligned} f(k) &= \frac{1}{k^2}\,[1-{}_1F_2(\tfrac{1}{2};1+i,1-i;-k^2)] \\ &= \frac{1}{4}\,{}_2F_3(3/2,1;2+i,2-i,2;-k^2). \end{aligned}$$

Remarking that in fact the series for g in the second of (1.5) contains (up to a factor) the derivatives of the functions $J_n^2(k)$, we obtain

$$g(k) = -\frac{1}{2k}\frac{d}{dk}\ {}_1F_2(\tfrac{1}{2};1+i,1-i;-k^2) =$$

(5.2)

$$\frac{1}{4}\ {}_1F_2(3/2;2+i,2-i;-k^2).$$

Let us introduce

(5.3) $$h_\alpha(k) = \frac{1}{4}\,{}_2F_3(3/2,1;2+i,2-i,\alpha;-k^2).$$

Then we have

(5.4) $$f(k) = h_2(k),\ g(k) = h_1(k)$$

and we proceed with the function $h_\alpha(k)$.

The ${}_2F_3$-function can be defined as a Mellin-Barnes-type integral

(5.5) $${}_2F_3\left({a,b \atop c,d,e}\middle|-k^2\right) = \frac{1}{2\pi i}\int_L k^{2s}\Gamma(-s)\,\frac{\Gamma(s+a)}{\Gamma(a)}\,\frac{\Gamma(s+b)}{\Gamma(b)}\,\frac{\Gamma(c)}{\Gamma(s+c)}\,\frac{\Gamma(d)}{\Gamma(s+d)}\,\frac{\Gamma(e)}{\Gamma(s+e)}\,ds,$$

where L starts at $-i\infty$ and ends at $+i\infty$ and L devides the complex s-plane into two parts: the poles of $\Gamma(-s)$ lie to the right of L, and all poles of $\Gamma(s+a)$, $\Gamma(s+b)$ lie to the left of it. For the hypergeometric functions the reader is referred to LUKE (1969, vol. I).

The Fourier transform

(5.6) $$\hat{h}_\alpha(x) = (2/\pi)^{\frac{1}{2}}\int_0^\infty \cos kx\ h_\alpha(k)dk$$

is computed substituting (5.5) for the special case (5.3) and by interchanging the order of integration. This is permitted when on L we take $-\frac{1}{2} < \mathrm{Re}\ s < 0$. We use the known transform

$$\int_0^\infty \cos kx\ k^{2s}ds = -\ x^{-1-2s}\Gamma(1+2s)\sin\pi s,$$

valid in the indicated domain for Re s on L, and we obtain

$$(5.7)\quad \hat{h}_\alpha(x) = -\frac{(2/\pi)^{\frac{1}{2}}}{4x}\,\frac{\Gamma(\alpha)\Gamma(2-i)\Gamma(2+i)}{\Gamma(3/2)\,2\pi i}\int_L x^{-2s}\,\frac{\Gamma(1+2s)\sin\pi s\,\Gamma(-s)\Gamma(s+3/2)\Gamma(s+1)}{\Gamma(2+i+s)\Gamma(2-i+s)\Gamma(\alpha+s)}\,ds.$$

Using $\sin\pi s\,\Gamma(-s) = -\pi/\Gamma(1+s)$, $\Gamma(1+2s)=2^{2s}\pi^{-\frac{1}{2}}\Gamma(s+1)\Gamma(s+\frac{1}{2})$ we obtain finally

$$(5.8)\quad \hat{h}_\alpha(x) = \frac{\Gamma(\alpha)\Gamma(2-i)\Gamma(2+i)}{x\sqrt{2\pi}\quad 2\pi i}\int_L \frac{(2/x)^{2s}\Gamma(s+\frac{3}{2})\Gamma(s+1/2)\Gamma(s+1)}{\Gamma(2+i+s)\Gamma(2-i+s)\Gamma(s+\alpha)}\,ds.$$

This representation can be evaluated by writing it as a series of residues due to the poles of the gamma functions. At the left of L we have the following poles

$$(5.9)\quad \begin{aligned} &s = -1/2\ , \text{ a simple pole}\\ &s = -3/2, -5/2\ldots, \text{ double poles}\\ &s = -1, -2, \ldots, \text{ simple poles.}\end{aligned}$$

The third group of poles are cancelled by the zeros of $\Gamma(s+\alpha)$ when $\alpha=1$; when $\alpha=2$ only the pole $s=-1$ of the third group has to be considered.

At the right of L the function under the integral sign in (5.8) has no poles. It follows that we can move the path L to the right as far as we please, without crossing a singularity. Since

$$\frac{\Gamma(s+3/2)\Gamma(s+1/2)\Gamma(s+1)}{\Gamma(s+2+i)\Gamma(s+2-i)\Gamma(s+\alpha)} \sim s^{-1-\alpha}$$

as $s \to \infty$ in $|\arg s| < \pi$, it is easily shown that for Re $\alpha>0$ and for real values of x satisfying $|2/x|\leq 1$, the function $\hat{h}_\alpha(x)$ vanishes.

From this we obtain immediately that

$$(5.10)\quad \hat{f}(x) = 0,\ \hat{g}(x) = 0, \qquad x \leq -2 \text{ or } x \geq 2.$$

For the remaining x-values the path L in (5.8) can be shifted to the left. Picking the residues due to the poles in (2.9) we obtain a representation in terms of an infinite series. For $\alpha = 1$ (i.e., for $\hat{g}(x)$) this series can be written in terms of ${}_2F_1$-series, which reduce to Legendre functions. For the case $\alpha = 2$ the situation is more complicated. To compute the function $\hat{f}(x)$ it is better to use the relation between $\hat{f}(x)$ and $\hat{g}(x)$, which is derived in section 3.

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