STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

WN 1

# Universal morphisms I

Preliminary note by P.C. Baayen and J. de Groot

March 1963

## § 1. General concepts and definitions

In this note the existence of several types of universal mappings is proved. In order to provide a general setting, we use the language of the theory of categories. The notation is the same as in Kurosh, Livshits and Shul'geifer [2]; in particular, the composition of two mappings $\varphi: A \to B$ and $\psi: B \to C$ is denoted by $\varphi\psi$. Accordingly, the image of $a \in A$ under $\varphi$ is denoted by $(a)\varphi$ (and also by $a\varphi$).

Let K be a category. An object a of K is called (K-) *universal* if for every other object b of K there exists a monomorphism $\mu: b \to a$. A morphism $\varphi: a \to a$ is said to be a (K-) *universal morphism* if for every morphism $\psi: b \to b$ in K there exists a monomorphism $\mu: b \to a$ such that $\mu\varphi = \psi\mu$.

$$\begin{tikzcd} a \arrow[r, "\varphi"] & a \\ b \arrow[u, "\mu"] \arrow[r, "\psi"] & b \arrow[u, "\mu"'] \end{tikzcd}$$

The morphism $\varphi$ is called a (K-) *universal bimorphism* if $\varphi$ is a bimorphism and if for every bimorphism $\psi: b \to b$ there exists a monomorphism $\mu: b \to a$ such that $\mu\varphi = \psi\mu$.

The dual concepts are called *dual-(K-) universal objects*, *morphisms* and *bimorphisms*. E.g. a *dual-universal morphism* is a morphism $\varphi: a \to a$ that is universal in the dual category $K^*$, i.e. for every $\psi: b \to b$ there exists a surjection $\nu: a \to b$ such that $\varphi\nu = \nu\psi$.

$$\begin{tikzcd} a \arrow[d, "\nu"] & a \arrow[l, "\varphi"'] \arrow[d, "\nu"] \\ b & b \arrow[l, "\psi"] \end{tikzcd}$$

It is trivial that the existence of (dual-) universal morphisms or bimorphisms implies the existence of (dual-) universal objects. The converse is not true, in general.

In this and in subsequent notes we will examine the existence of universal or dual universal morphisms and bimorphisms in a number of categories; the most important ones are listed below.

$K(S,\mathfrak{m})$: category of all mappings of a set of power $\mathfrak{m}$ into another set of power $\mathfrak{m}$.

$K(LO,\mathfrak{m})$: category of all order-preserving mappings of a linearly ordered set of power $\mathfrak{m}$ into another linearly ordered set of power $\mathfrak{m}$.

$K(PO,\mathfrak{m})$: category of all order-preserving mappings of a partially ordered set of power $\mathfrak{m}$ into another partially ordered set of power $\mathfrak{m}$.

$K(T,\mathfrak{m})$: category of all continuous mappings of a completely regular topological space of weight $\mathfrak{m}$ into another completely regular space of the same weight.

Also other categories of continuous mappings will be studied, as well as categories of homomorphisms between abelian groups, and of boolean homomorphisms between boolean algebras. Furthermore, we also define

$K(S,\overline{\mathfrak{m}})$: category of all mappings of a set of power less than or equal to $\mathfrak{m}$ into another set of power less than or equal to $\mathfrak{m}$.

Similarly $K(LO,\overline{\mathfrak{m}})$, $K(PO,\overline{\mathfrak{m}})$ and $K(T,\overline{\mathfrak{m}})$ are defined.

In this first note we treat the categories $K(S,\mathfrak{m})$ and $K(S,\overline{\mathfrak{m}})$, for arbitrary cardinal $\mathfrak{m}$.

## § 2. Universal morphisms in $K(S,\mathfrak{m})$ and $K(S,\overline{\mathfrak{m}})$.

The results proved in this note can be summarized as follows.

Theorem 1. For every transfinite cardinal $\mathfrak{m}$, the categories $K(S,\mathfrak{m})$ and $K(S,\overline{\mathfrak{m}})$ contain universal morphisms and bimorphisms, and dual-universal morphisms and bimorphisms.

The proof falls apart in a number of separate propositions. We start with the simplest ones. In all the following, $\mathfrak{m}$ is supposed to be transfinite.

Proposition 1. $K(S,\mathfrak{m})$ contains dual universal bimorphisms.

Proof.

Let S be a set of power $\mathfrak{m}$ , and let A be the set of all ordered pairs (x,n), with $x \in S$ and n an integer. Then also card(A) $= \mathfrak{m}$, as $\mathfrak{m}$ is transfinite. Define $\Phi : A \twoheadrightarrow A$ as follows:

$$(2.1) \qquad (x,n)\Phi = (x,n+1).$$

We will show that $\Phi$ is a dual-universal bimorphism in K(S, $\mathfrak{m}$).

It is clear that $\Phi$ is a bimorphism. Now let B be any set of power $\mathfrak{m}$ , and let $\Phi$ be any bimorphism $B \twoheadrightarrow B$. Then we first choose any epimorphism $\tau : S \twoheadrightarrow B$; next we define $\nu : A \twoheadrightarrow B$ by

$$(2.2) \qquad (x,n)\nu = x\tau\varphi^n.$$

(For any map $\varphi$ , the map $\varphi^0$ is defined to be the identity map.) Then it is clear that $\nu$ maps A onto B (in fact, $\nu$ already maps the subset of A consisting of all pairs (x,0), $x \in S$, onto B); and $\Phi\nu = \nu\varphi$ :

$$(x,n)\Phi\nu = (x,n+1) \quad = x\tau\varphi^{n+1} = (x\tau\varphi^n)\varphi = (x,n)\nu\varphi .$$

Proposition 2. K(S, $\mathfrak{m}$ ) contains dual-universal morphisms.

Proof.

The proof is almost the same as that of proposition 1.

Let S again be a set of power $\mathfrak{m}$ . This time, let A exist of all ordered pairs (x,n), where $x \in S$ and n is a non-negative integer. A morphism $\Psi : A \twoheadrightarrow A$ is again defined by (2.1). If $\psi : B \twoheadrightarrow B$ is any morphism in K, we take again an epimorphism $\tau : S \twoheadrightarrow B$ and define an epimorphism $\nu : S \twoheadrightarrow B$ by (2.1). Then $\Psi\nu = \nu\psi$ ; hence $\Psi$ turns out to be a dual-universal morphism.

The same proofs can be used to show:

Proposition 3. K(S, $\bar{\mathfrak{m}}$) contains dual-universal morphisms and bimorphisms.

Proposition 4. K(S, $\mathfrak{m}$) contains universal bimorphisms.

Proof.

For any non-negative integer n, let $I_n$ be the set of all integers reduced modulo n, and let $\sigma_n : I_n \twoheadrightarrow I_n$ be the successor function:

$$(2.3) \qquad (k)\sigma_n = k+1, \quad \text{reduced modulo n.}$$

Let S be a set of power $\mathfrak{m}$, and let A be the set of all ordered triples $(x,n,k)$, where $x \in S$, n is a non-negative integer, and $k \in I_n$. We define a bimorphism $\Phi : A \to A$ as follows:

$$(2.4) \qquad (x,n,k)\Phi = (x,n,(k)\sigma_n).$$

We will prove that $\Phi$ is a universal bimorphism for K.

Let B be any set of power $\mathfrak{m}$, and let $\varphi : B \to B$ be a bimorphism. The orbits $O(x) = \{(x)\varphi^n : n=0, \pm 1, \pm 2, \ldots\}$ partition B into disjoint sets, each of them at most countable.

Let C be a choice set, containing exactly one point from every orbit $O(x)$; let $C_n = \{x \in C : \text{card}(O(x)) = n\}$ $(n=1,2,\ldots)$, and $C_o = \{x \in C: \text{card}(O(x)) = \aleph_o\}$. For each n, $\text{card}(C_n) \leq \mathfrak{m}$; hence for every n there is a 1-1 map $\tau_n$ of $C_n$ into S.

We define a mapping $\mu : B \to A$ in the following way. If $x \in B$, there is exactly one $y \in C$ such that $x \in O(y)$; there is exactly one n such that $y \in C_n$; and there is exactly one $k \in I_n$ such that $(y)\varphi^k = x$. We put

$$(2.5) \qquad (x)\mu = ((y)\tau_n, n, k).$$

Then $\mu$ is a monomorphism, and $\mu\Phi = \varphi\mu$.

<u>Corollary.</u> $K(S, \bar{\mathfrak{m}})$ contains universal bimorphisms.

<u>Proof.</u>

If $\varphi : B \to B$, and $\text{card}(B) < \mathfrak{m}$, then take any $B' \supset B$ such that $\text{card}(B') = \mathfrak{m}$, and define $\varphi' : B' \to B'$ by: $\varphi' | B = \varphi$, $\varphi' | B' \setminus B$ = identity map.

There remains to be shown that $K(S, \mathfrak{m})$ and $K(S, \bar{\mathfrak{m}})$ contain universal morphisms. In order to do this, we need some lemmas.

<u>Definitions</u>. The <u>two-sided orbit</u> $TO(x)$ of a point $x \in X$ under a mapping $\varphi : X \to X$ is the set

$$(2.6) \qquad TO(x) = \{ y \in X : (y)\varphi^n = (x)\varphi^m, \text{ for some non-negative integers } n,m \}.$$

A mapping $\varphi : X \to X$ is called <u>coherent</u> if $TO(x) = X$, for some $x \in X$.

A loop under a mapping $\varphi : X \to X$ is a finite set of points $x_1, x_2, \ldots, x_n$ $(n \geq 1)$, such that

$$(x_k)\varphi = x_{k+1} \qquad (k=1,2,\ldots,n-1);$$
$$(x_n)\varphi = x_1 \ .$$

The following two lemmas are evident.

Lemma 1. The two-sided orbits under a mapping $\varphi : X \to X$ constitute a partition of the set X.

Lemma 2. A two-sided orbit TO(x) under a mapping $\varphi$ contains at most one loop.

In the next lemma, the existence is established of certain mappings needed for the construction of a universal morphism.

Lemma 3. Let $\mathfrak{m}$ be any transfinite cardinal number. For every non-negative integer n, there exists a coherent mapping $\sigma_n : N_r \to N_n$ of a set $N_n$ of power $\mathfrak{m}$ into itself with the following properties:

(i) there is a loop of exactly n points (in case n=0, this means that there is no loop at all);

(ii) $\mathrm{card}((x)\sigma_n^{-1}) = \mathfrak{m}$, for each $x \in N_n$.

Proof.

First take n=0.

Let A be any set of power $\mathfrak{m}$. Consider the set C of all indexed sequences $a_k a_{k+1} a_{k+2} \ldots a_{k+n} \ldots$, where k is an arbitrary integer (possibly negative or zero) and each $a_i$ belongs to A. Define $\sigma : C \to C$ by

$$(a_k a_{k+1} a_{k+2} \ldots)\sigma = a_{k+1} a_{k+2} \ldots$$

Then there are no loops under $\sigma$, and $\mathrm{card}((x)\sigma^{-1}) = \mathfrak{m}$, for every $x \in C$.

The power of C is equal to $\mathfrak{m}^{\aleph_0}$; this may or may not be equal to $\mathfrak{m}$. But fortunately this does not matter. For if we choose any $x_0 \in C$, and put $N_0 = TO(x_0)$, then it is easy to show that $\mathrm{card}(N_0) = \mathfrak{m}$. Finally we may define $\sigma_0$ as $\sigma | N_0$.

Next we consider the case n=1.

Choose again a point $x_1 \in C$, and let

$$N_1 = \{ x \in C : (x)\sigma^n = x_1, \text{ for some non-negative integer } n \}.$$

Let $\sigma_1 | N_1 \setminus \{x_1\} = \sigma | N_1 \setminus \{x_1\}$ , and let $(x_1)\sigma_1 = x_1$. Then $\sigma_1 : N_1 \to N_1$ satisfies the requirements.

Finally let n be an integer $> 1$. Let $M_1, M_2, \ldots, M_n$ be disjoint sets, each of power $\mathfrak{m}$ , and for each i, $1 \leq i \leq n$, let $\tau_i : M_i \to M_i$ be a mapping with the properties required for $\sigma_1$. Furthermore, let the one-point loop under $\tau_i$ in $M_i$ consist of the single point $x_i$. Then we can put $N_n = M_1 \cup M_2 \cup \ldots \cup M_n$, and

$$\sigma_n | M_i \setminus \{x_i\} = \tau_i | M_i \setminus \{x_i\} \qquad (1 \leq i \leq n);$$

$$(x_i)\sigma_n = x_{i+1} \qquad (1 \leq i \leq n-1);$$

$$(x_n)\sigma_n = x_1.$$

Next we show that the mapping $\sigma_n$ is universal for all coherent mappings having a loop of n points.

Lemma 4. Let $\mathfrak{m}$ be a transfinite cardinal. Let $\mathrm{card}(X) = \mathfrak{m}$, and suppose $\varphi : X \to X$ is a coherent mapping with a loop of n points. If $\sigma_n : N_n \to N_n$ is a mapping meeting the requirements of lemma 3, then there exists a 1-1 mapping $\mu : X \to N_n$ such that $\mu \sigma_n = \varphi \mu$.

Proof.

First suppose n=0.

Choose an arbitrary $x_o \in X$ and an arbitrary $y_o \in N_o$; we put

$$(x_o \varphi^m)\mu = (y_o)\sigma_o^{\ m} \qquad (m=0,1,2,\ldots).$$

Let $A_1 = \{(x_o)\varphi^m : m = 0,1,2,\ldots\}$ , $A_2 = (A_1)\varphi^{-1} \setminus A_1$, and $A_{m+2} = (A_{m+1})\varphi^{-1}$ (m=1,2,...). The sets $A_1, A_2, \ldots$ are disjoint, have powers $\leq \mathfrak{m}$ , and $X = \bigcup_{m=1}^{\infty} A_m$. We have defined $\mu | A_1$; suppose $\mu | A_k$ already defined, for k=1,2,...,m, in such a way that

$$(2.7) \qquad (x)\mu\sigma_o = (x)\varphi\mu$$

for all $x \in A_1 \cup A_2 \cup \ldots \cup A_m$, while also $\mu$ is 1-1 on $A_1 \cup A_2 \cup \ldots \cup A_m$.

The sets $(x)\varphi^{-1}$, $x \in A_m$, partition $A_{m+1}$ into at most $\mathfrak{m}$ disjoint sets. (In the case m=1, we must take the sets $(x)\varphi^{-1} \cap A_2$ instead). Let $B \subset A_m$ such that the sets $(x)\varphi^{-1}$, $x \in B$, are pairwise disjoint and cover

$A_{m+1}$. For each $x \in B$ there exists a 1-1 map $\tau_x$ of $(x)\varphi^{-1}$ into $(x\mu)\sigma_o^{-1}$, as the latter set has power $\mathfrak{m}$, while the first has a power at most $\mathfrak{m}$. We define

$$\mu \,|\, (x)\,\varphi^{-1} = \tau_x,$$

for each $x \in B$. Then $\mu$ is defined on all of $A_{m+1}$; $\mu$ is 1-1 on $A_1 \cup A_2 \cup \ldots \cup A_m$; and (2.7) holds, for all $x \in A_1 \cup A_2 \cup \ldots \cup A_m$. Using induction, the assertions of the lemma for the case n = 0 follow.

The cases $n \geqslant 1$.

The proof in the case $n \geqslant 1$ runs along similar lines; the only difference is that we do not start with an arbitrary $x_o \in X$, but with a point $x_o$ belonging to the loop of $\varphi$.

Now we are able to prove the existence of universal morphisms in $K(S, \mathfrak{m})$.

Proposition 5. $K(S, \mathfrak{m})$ contains universal morphisms.

Proof.

For each non-negative integer n, let $\sigma_n : N_n \to N_n$ be a mapping as described in lemma 3. Let S be any set of power $\mathfrak{m}$. Consider the set A of all ordered triples (s,n,x), where $s \in S$, n is a non-negative integer, and $x \in N_n$. We define a mapping $\Psi : A \to A$ in the following way:

$$(s,n,x)\Psi = (s,n,(x)\sigma_n).$$

Contention: $\Psi$ is a universal morphism for K.

It is clear that card(A) $= \mathfrak{m}$. Let B be any other set of power $\mathfrak{m}$, and let $\psi : B \to B$. Let C be a choice set in B, containing exactly one point from every two-sided orbit TO(x), $x \in B$; let $C_n$ be the subset of C consisting of all x such that TO(x) contains a loop of n points (contains no loop, if n=0). For each n, card$(C_n) \leqslant \mathfrak{m}$; hence for each n there is a 1-1 map $\tau_n$ of $C_n$ into S. Furthermore, by lemma 4, for each n and each $x \in C_n$, there exists a 1-1 mapping $\mu_{x,n}$ of TO(x) into $N_n$ with the property that

$$(y)\mu_{x,n}\sigma_n = (y)\psi\mu_{x,n}$$

for all $y \in$ TO(x).

We define a mapping $\mu : B \to A$ as follows: if $y \in B$, say $y \in TO(x)$, $x \in C_n$, we define

$$(y)\mu = (x\tau_n, n, y\mu_{x,n}).$$

Then is a 1-1 mapping defined on all of B, and $\mu\Psi = \psi\mu$ .

Corollary. $K(S,\overline{\mathfrak{m}})$ contains universal morphisms.

This finishes the proof of theorem 1.

Remark 1. If the cardinal number $\mathfrak{m}$ has the property

$$\mathfrak{m}^{\aleph_0} = \mathfrak{m},$$

it is possible to prove propositions 4 and 5 in an entirely different (and, in the case of proposition 5, much simpler) way, using a method described already in [1] . This method will be treated extensively in subsequent notes on universal continuous and topological mappings and on universal families of morphisms.

Remark 2. It is trivial that theorem 1 also holds for $\mathfrak{m} = 1$. If $\mathfrak{m}$ is a finite cardinal different from 1, it is easily seen that $K(S,\mathfrak{m})$ and $K(S,\overline{\mathfrak{m}})$ do not contain universal or dual-universal morphisms or bimorphisms.

## References

[1] P.C. Baayen, Toepassingen van een lineariseringsprincipe. Rapport ZW 1962-011, Mathematisch Centrum, Amsterdam.

[2] A.G. Kurosh, A.Kh. Livshits en E.G. Shul'geifer, Foundations of the theory of categories, Russian Math. Surveys 15 (1960), 1-46.