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Universal morphisms II

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April 1963

In this note we continue the dicussion started in "Universal morphisms I", [1]. We obtain results concerning the existence of universal morphisms or bimorphisms in $K(LO,\mathfrak{m})$ and $K(LO,\bar{\mathfrak{m}})$; the study of dual-universal morphisms in these categories is deferred to a subsequent note.

The same notation is used as in [1]; propositions, definitions and sections of [1] are referred to by their number there.

§ 3. The structure of order-preserving maps in linearly ordered sets.

In the next section we will prove that e.g. $K(LO,\aleph_0)$ contains universal morphisms and bimorphisms. In order to do so, we need some general results about order-preserving maps of a linearly ordered set into itself.

A mapping φ of a linearly ordered set X into a linearly ordered set Y is called <u>order-preserving</u> if for all $x_1, x_2, \in X$

$$x_1 \leqslant x_2 \Rightarrow x_1\varphi \leqslant x_2\varphi .$$

A map $\varphi : X \to X$ is called <u>increasing</u> if for all $x \in X$

$$x \leqslant x\varphi ,$$

and <u>decreasing</u> if for all $x \in X$

$$x \geqslant x\varphi .$$

The map φ will be called a <u>translation</u> if it is either increasing or decreasing.

In the remainder of this section, X denotes a linearly ordered set and $\varphi : X \to X$ an order-preserving map. Furthermore, N will designate the set of all integers, and N^+ the set of all non-negative integers.

<u>Definition 1</u>. If $S \subset X$, then $\hat{S} = \{x \in X : a \leqslant x \leqslant b$ for some $a,b \in S\}$. If $S=\hat{S}$, S is called an <u>interval</u> in X. For certain kinds of intervals we adapt the well-known bracket notation; e.g.

$[a;b) = \{x \in X:\ a \leqslant x < b\}$.

Definition 2. $x \,\Delta\, y \Longleftrightarrow (\exists n \in N^+)\ (x\varphi^n \in \widehat{TO(y)})$. *)

Lemma 1. $x \,\Delta\, y \Longleftrightarrow (\exists n,m \in N^+)\ (y\varphi^n \leqslant x\varphi^m \leqslant y\varphi^{n+1})$..

Proof: evident.

Proposition 1. The relation Δ is an equivalence relation in X.

Proof.

Certainly always $x \Delta x$. Suppose $x \Delta y$; say $y\varphi^m \leqslant x\varphi^n \leqslant y\varphi^{m+1}$. Then $x\varphi^n \leqslant y\varphi^{m+1} \leqslant x\varphi^{n+1}$; hence $y \Delta x$. Suppose next $x \Delta y$ and $y \Delta z$; say $y\varphi^m \leqslant x\varphi^n \leqslant y\varphi^{m+1}$ and $z\varphi^{m_1} \leqslant y\varphi^{n_1} \leqslant z\varphi^{m_1+1}$. Then $z\varphi^{m+m_1} \leqslant x\varphi^{n+n_1} \leqslant z\varphi^{m+m_1+2}$, hence $x \Delta z$.

Proposition 2. If φ is onto, then $x \Delta y \Longleftrightarrow x \in \widehat{TO(y)}$.

Proof.

Let $y\varphi^m \leqslant x\varphi^n \leqslant y\varphi^{m+1}$. Then there are $a \in (y\varphi^m)\varphi^{-n}$ and $b \in (y\varphi^m)\varphi^{-n+1}$ such that $a \leqslant x \leqslant b$; i.e. $x \in \widehat{TO(y)}$.

Definition 3. If $x \in X$, then $\Delta(x)$ denotes the Δ-equivalence class of x:

$$\Delta(x) = \{y \in X\ :\ x \Delta y\}\ ;$$

moreover

$$\Delta_1(x) = \{y \in \Delta(x):\ y \leqslant y\varphi\};$$

$$\Delta_2(x) = \{y \in \Delta(x):\ y \geqslant y\varphi\}.$$

It follows that $\Delta(X) := \{\Delta(x)\ :\ x \in X\}$ is a disjoint covering of X. For each x we have $\Delta(x) = \Delta_1(x) \cup \Delta_2(x)$, $\varphi|\Delta_1(x)$ is increasing, $\varphi|\Delta_2(x)$ is decreasing. Moreover, $\Delta_1(x) \cap \Delta_2(x)$ consists of all points of $\Delta(x)$ that are fixed under φ.

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*) For the definition of TO(y), see §2.

Proposition 3. $\Delta_1(x)$ and $\Delta_2(x)$ are intervals, and $y_1 \leq y_2$ for all $y_1 \in \Delta_1(x), y_2 \in \Delta_2(x)$. Moreover, φ maps $\Delta_i(x)$ into itself $(i=1,2)$.

Proof.

Let $y_2 \in \Delta(x)$, $y_2 < y_1 \in \Delta_1(x)$. As $y_2 \Delta y_1$, there are $n_1, n_2 \in N^+$ such that $y_1\varphi^{n_1} \leq y_2\varphi^{n_2}$. As $y_2 < y_1 \leq y_1\varphi^{n_1}$, it follows that $y_2 < y_2\varphi$; so $y_2 \notin \Delta_2(x)$.

It is easily seen that $\Delta(x)$ is an interval. It then follows that both $\Delta_1(x)$ and $\Delta_2(x)$ are intervals. If $y \leq y\varphi$, then $y\varphi \leq y\varphi^2$; hence $(\Delta_1(x))\varphi \subset \Delta_1(x)$. Similarly $(\Delta_2(x)\varphi \subset \Delta_2(x)$.

Corollary. Every $\Delta(x)$ contains at most one fixed point. If $a \in \Delta(x)$ is a fixed point, then $y_1 \leq a \leq y_2$ for all $y_1 \in \Delta_1(x)$ and $y_2 \in \Delta_2(x)$.

Proposition 4. If $\Delta(x)$ contains a fixed point a, then $\Delta(x)=TO(a)$. If $\Delta(x)$ contains no fixed point, $\varphi|\Delta(x)$ is a translation.

Proof.

Assume $a \in \Delta(x)$ is fixed. As $x \Delta a$, $a\varphi^m \leq x\varphi^n \leq a\varphi^{m+1}$, for some $m,n \in N^+$. Then $x\varphi^n=a$, or $x \in TO(a)$.

Assume $\varphi|\Delta(x)$ is not a translation. Let $y_1 \in \Delta_1(x)$, $y_2 \in \Delta_2(x)$. Let $n_1, n_2 \in N^+$ such that $y_1\varphi^{n_1} \geq y_2\varphi^{n_2}$. Then, by prop.3, $y_1\varphi^{n_1} \in \Delta_1(x) \cap \Delta_2(x)$; hence $y_1\varphi^{n_1}$ is a fixed point under φ.

Definition 4. $\Delta(x) \leq \Delta(y) \iff (\exists a \in \Delta(x))\ (\exists b \in \Delta y)\ (a \leq b)$.

This is logically equivalent to:

(3.1) $\Delta(x) < \Delta(y) \iff (\forall a \in \Delta(x))(\forall b \in \Delta(y))\ (a < b)$.

As $\Delta(X)$ consists of disjoint intervals, the next proposition is evident.

Proposition 5. The set $\Delta(X)$ is linearly ordered by $\leq$.

This finishes the first stage of our analysis of φ . In order to know the behavior of φ , it is sufficient to know the ordering of $\Delta(X)$ and the behavior of the maps $\varphi|\Delta(x)$.

In the next stage we study the manner in which $\Delta(x)$ is built up from the total orbits $TO(y)$, $y \in \Delta(x)$.

<u>Proposition 6.</u> Let $x < x\varphi$. For every $y \in \Delta(x)$, the set $TO(y) \cap [x;x\varphi)$ is an interval; if $y \leq x$ and $y \notin TO(x)$, there is a unique $n \in N^+$ such that $y\varphi^n \in [x;x\varphi)$.

<u>Proof.</u>

Let $a,b \in TO(y) \cap [x;x\varphi)$ and $a \leq z \leq b$. There are $n,m \in N^+$ such that $a\varphi^n = b\varphi^m$; then $x\varphi^n \leq a\varphi^n \leq x\varphi^{n+1}$ and $x\varphi^m \leq b\varphi^m =$
$= a\varphi^n \leq x\varphi^{m+1}$.

It follows that $x\varphi^n \leq x\varphi^{m+1}$ and $x\varphi^m \leq x\varphi^{n+1}$.

If one of these two inequalities is an equality, we find that $z \in TO(x) = TO(y)$. If both $x\varphi^n < x\varphi^{m+1}$ and $x\varphi^m < x\varphi^{n+1}$, then $n < m+1$ and $m < n+1$, hence $n=m$, and $z\varphi^n = a\varphi^n \in TO(y)$.

Hence $TO(y) \cap [x;x\varphi)$ is an interval. Now let $y < x$. As $y \Delta x$, $x\varphi^{n_1} \leq y\varphi^{m_1}$, for some $n_1,m_1 \in N^+$. As $x \leq x\varphi^{n_1}$, we have $x \leq y\varphi^{m_1}$; let n be the smallest non-negative integer such that $x \leq y\varphi^n$. As $y < x$, $n \neq 0$; $y\varphi^{n-1} < x \Rightarrow y\varphi^n \leq x\varphi$. As $y \notin TO(x)$, $y\varphi^n < x\varphi < y\varphi^{n+1}$. This shows that for every $y \leq x, y \in \Delta(x) \setminus TO(x)$, there exists one and only one integer $n \in N^+$ such that $y\varphi^n \in [x;x\varphi)$.

If $x > x\varphi$, similar results are obtained, (with $[x;x\varphi)$ changed into $(x\varphi;x]$); in fact, we need only take into account that if we reverse the ordering of X, then X remains linearly ordered and φ remains order-preserving.

<u>Corollary.</u> Let $x < x\varphi$ and $y \in \Delta(x) \setminus TO(x)$. Then $TO(y) \cap [x;x\varphi]$ is an interval.

If $y \in TO(x)$, then $TO(y) \cap [x;x\varphi] = (TO(y) \cap [x;x\varphi)\ ) \cup \{x\varphi\}$ need not be an interval.

<u>Definition 4.</u> Let $\sum(x) = \{TO(y): y \in \Delta(x)\}$, and let $\leq_x$ be the binary relation in $\sum(x)$, defined as follows.

If $\Delta(x)$ contains a fixed point a, $\leq_x$ is the identity relation in $\sum(x) = \{TO(a)\}$.

If $\Delta(x)$ contains no fixed point, and $\varphi|\Delta(x)$ is increasing then, for $S_1, S_2 \in \sum(x)$,

$$S_1 \leq_x S_2 \Longleftrightarrow (\exists n \in N^+)\ (\exists a \in S_1)\ (\exists b \in S_2)(x\varphi^n \leq a \leq b < x\varphi^{n+1}).$$

If $\Delta(x)$ contains no fixed point, and $\varphi|\Delta(x)$ is decreasing then, for $S_1 S_2 \in \sum(x)$,

$$S_1 \leq_x S_2 \Longleftrightarrow (\exists n \in N^+)\ (\exists a \in S_1)\ (\exists b \in S_2)(x\varphi^{n+1} < a \leq b \leq x\varphi^n).$$

<u>Proposition 7.</u> The relation $\leq_x$ linearly orders $\sum(x)$.

<u>Proof.</u>

To simplify the notation, we will write $\sum$ and $\leq$ instead of $\sum(x)$ and $\leq_x$. It suffices to consider the case that $\Delta(x)$ contains no fixed point and $x < x\varphi$.

Evidently $S \leq S$, for all $S \in \sum$. Let $S_1, S_2 \in \sum$ such that $S_1 \leq S_2$ and $S_2 \leq S_1$. Take $n,m \in N^+$; $a,b \in S_1$; $c,d \in S_2$; such that

$$x\varphi^n \leq a \leq d < x\varphi^{n+1}$$

and

$$x\varphi^m \leq c \leq b < x\varphi^{m+1}.$$

Then $a\varphi^m$, $d\varphi^m$, $c\varphi^n$ and $b\varphi^n \in [x\varphi^{n+m}; x\varphi^{n+m+1}]$; hence it follows from prop.6 and its corollary that $S_1 = S_2$.

Suppose now that $S_1 \leq S_2$ and $S_2 \leq S_3$. Let $n,m \in N^+$, $a \in S_1$, $b,c \in S_2$ and $d \in S_3$ such that

$$x\varphi^n \leq a \leq b < x\varphi^{n+1};$$

$$x\varphi^m \leq c \leq d < x\varphi^{m+1}.$$

It follows that

$$x\varphi^{n+m} \leq a\varphi^{m} \leq b\varphi^{m} \leq x\varphi^{n+m+1};$$

$$x\varphi^{n+m} \leq c\varphi^{n} \leq d\varphi^{n} \leq x\varphi^{n+m+1}.$$

If $b\varphi^m = x\varphi^{n+m+1}$, then $b \in TO(x)$; both x and $b \in TO(x) \cap [x;x\varphi)$ hence, by prop.6, $a \in TO(x)$, and $S_1=S_2 \leq S_3$. Similarly, $d\varphi^n = x\varphi^{n+m+1}$ implies $S_1 \leq S_2 = S_3$. Assume

$$x\varphi^{n+m} \leq a\varphi^{m} \leq b\varphi^{m} < x\varphi^{n+m+1}$$

and

$$x\varphi^{n+m} \leq c\varphi^{n} \leq d\varphi^{n} < x\varphi^{n+m+1}.$$

Then, by prop.6, $a\varphi^m \leq d\varphi^n$; hence $S_1 \leq S_3$.

Finally we must show that the relation $\leq$ is total. Let S_1, $S_2 \in \Sigma$; take $y_i \in S_i$ (i=1,2). As $y_1 \Delta x$ and $y_2 \Delta x$, there are $n, n_1, n_2 \in N^+$ such that $y_i\varphi^{n_i} \leq x\varphi^n$ (i=1,2). It is easily seen that $S_i \cap [x\varphi^n; x\varphi^{n+1}) \neq \emptyset$ (i=1,2); hence either $S_1 \leq S_2$ or $S_2 \leq S_1$.

<u>Remark 1.</u> As $TO(x) \leq_x S$, for all $S \in \Sigma(x)$, the relation $\leq_x$ in general differs from $\leq_y$, even if $\Sigma(x) = \Sigma(y)$.

<u>Remark 2.</u> Suppose $\Delta(x)$ contains no fixed point; let e.g. $x < x\varphi$. Then by logical inversion, we have

(3.2) $S_1 <_x S_2 \Longleftrightarrow (\forall n \in N^+)(\forall a \in S_1 \cap [x\varphi^n;x\varphi^{n+1})$

$(\forall b \in S_2 \cap [x\varphi^n;x\varphi^{n+1}))\ (a < b)$.

From this we conclude:

(3.3) If $z \in TO(x)$, then $S_1 \leq_z S_2 \Longleftrightarrow S_1 \leq_x S_2$.

This remains true if $x > x\varphi$ or if $\Delta(x)$ has a fixed point.

Proposition 8. Let φ be 1-1 and onto. If $\Delta(x)$ contains a fixed point a, then $\Delta(x) = \{a\}$. If $x < x\varphi$, then for every $y \in \Delta(x)$ the set $TO(y) \cap [x;x\varphi)$ contains exactly one point. Moreover, there is a unique $n \in N$ such that $y\varphi^n \in [x;x\varphi)$. Similar results hold if $x > x\varphi$.

Proof.

If $a\varphi = a$, then $TO(a) = \{a\}$, as φ = 1-1, and hence $\Delta(a) = \{a\}$.

Suppose $x < x\varphi$; let $y \in \Delta(x)$. It is easily seen that $y\varphi^n \in TO(y) \cap [x;x\varphi)$, for some $n \in N$. As $y\varphi^{n-1} < x$ and $y\varphi^{n+1} \geqslant x\varphi$, the integer n is unique.

The third stage in these considerations about order-preserving maps consists of an analysis of one single total orbit TO(x).

Definition 5. Let E be the following equivalence relation in TO(x):

$$yEz \Leftrightarrow (\exists n \in N^+)\ (y\varphi^n = z\varphi^n).$$

If $y \in TO(x)$, we denote by E(y) the equivalence class of y:

$$E(y) = \{ z \in TO(x) \ : \ yEz \}.$$

Proposition 9. Every E(y) is an interval in X. For each $n \in N^+$, $((E(y))\varphi^n \subset E(y\varphi^n))$. If TO(x) contains no fixed point, then $n,m \in N^+$, $n \neq m$ imply

$$(E(y))\varphi^n \cap (E(y))\varphi^m = \emptyset.$$

Proof.

Let $y_1 \leqslant z \leqslant y_2$; $y_1, y_2 \in E(y)$; $z \in X$. For some $n \in N^+$, $y_2\varphi^n = y_1\varphi^n \leqslant z\varphi^n \leqslant y_2\varphi^n$; hence $z \in E(y)$.

It is trivial that $(E(y))\varphi^n \subset E(y\varphi^n)$. Suppose $y \in TO(x)$; $n,m \in N^+$, $n \neq m$; and $y\varphi^n E y\varphi^m$.

There is a $k \in N^+$ such that $y\varphi^{n+k} = y\varphi^{m+k}$. It follows that $TO(y) = TO(x)$ contains a fixed point.

Corollary. If $TO(x)$ contains no fixed point, then $TO(x)$ can be written as the union of countable many intervals (possibly void), each of which an E-equivalence class:

$$TO(x) = \bigcup_{n \in N} E_n,$$

in such a way that $E_n\varphi \subset E_{n+1}$. Moreover, if $a \in E_n, b \in E_m, n < m$, then $a < b$ if $x < x\varphi$, and $a > b$ if $x > x\varphi$.

§4. Universal order-preserving mappings in linearly ordered sets.

In this section we consider the category $K(LO, \mathfrak{m})$ of all order-preserving maps of a linearly ordered set of power $\mathfrak{m}$ into another such a set. Except if the converse is explicitly stated, it is assumed that $\mathfrak{m}$ is transfinite.

The monomorphisms in $K(LO, \mathfrak{m})$, and also in $K(LO, \bar{\mathfrak{m}})$, are the one-to-one maps; the epimorphisms are the mappings onto.

In the introduction to [1] it was remarked already that the existence of universal morphisms or bimorphisms implies the existence of universal objects. For the categories that we want to consider in this section, we are in the sad position that the existence of universal objects is an open problem for all $\mathfrak{m} > \aleph_0$. (If $\mathfrak{m} = \aleph_0$, the set Q of all rational numbers, with the usual ordering, is a well-known universal object).

However, for those who are inclined to accept the generalized continuum hypothesis as valid, there is no problem after all. For it follows from results of W. Sierpinski [5] and L. Gillman [2], that $K(LO, \aleph_{\alpha+1})$ contains a universal object if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$; if α is a limit number, then $K(LO, \aleph_\alpha)$ contains universal objects as soon as $2^{\aleph_\beta} \leq \aleph_\alpha$ for all $\beta < \alpha$. (A very short proof of these facts is given in [4].)

We will prove that the existence of a universal object suffices to guarantee the existence of universal morphisms and bimorphisms:

<u>Theorem 1.</u> Let $\mathfrak{m}$ be a transfinite cardinal, and suppose $K(LO,\mathfrak{m})$ contains a universal object. Then the categories $K(LO,\mathfrak{m})$ and $K(LO,\bar{\mathfrak{m}})$ contain universal morphisms and bimorphisms.

In particular we find that the categories $K(LO,\aleph_0)$ and $K(LO,\bar{\aleph}_0)$ contain universal morphisms and bimorphisms. Let $\Phi_0 : S_0 \to S_0$ be a universal bimorphism . If $S_1 = S_0 \times Q$, lexicographically ordered and if $\Phi_1 : S_1 \to S_1$ is defined by

$$(x,r)\Phi_1 = (x\Phi_0, r)$$

then it is immediate that Φ_1 is again a universal bimorphism. But the order type of S_1 is $\eta\alpha$, where η is the order type of Q and α is the order type of S_0; and $\eta\alpha = \eta$ for every countable order type α (see e.g. [3] Ch.IV §7). Thus S_1 is order-isomorphic to Q.

Using similar arguments in the case of a universal morphism, we arrive at

<u>Theorem 2.</u> The categories $K(LO,\aleph_0)$ and $K(LO,\bar{\aleph}_0)$ admit a universal bimorphism $\Phi : Q \to Q$ and a universal morphism $\Psi : Q \to Q$ the object Q, mapped into itself by these morphisms, being the set of rational numbers.

The proof of theorem 1 will be given in several steps.

<u>Lemma 1.</u> Let $K = K(LO,\mathfrak{m})$ contain a universal object A. Then K contains a bimorphism $\tau : T \to T$ with the property: for every bimorphism $\varphi : B \to B$ in K, and for every $x \in B$, there exists a 1-1 order-preserving map $\mu : \Delta(x) \to T$ such that $\mu\tau = (\varphi|\Delta(x))\mu$.

Proof.

Let $E=\{-1,0,1\}$, ordered as usual; we put $T=E\times N\times A$, ordered lexicographically, and we define $\tau:T\to T$ by

$$(e,n,a)\tau = (e,n-e,a),$$

for arbitrary $(e,n,a)\in T$. Then τ is a bimorphism of K.

Let $\varphi:B\to B$ be an arbitrary bimorphism of K, and let $x\in B$. If $x=x\varphi$, then $\Delta(x)=\{x\}$; for $x\mu$ we may then take any point $(o,n,a)\in T$. Suppose $x\neq x\varphi$; i.e. $\Delta(x)$ is infinite.

As A is universal, there exists a 1-1 order-preserving map $\sigma: S\to A$, where $S=[x;x\varphi)$ if $x<x\varphi$, and $S=(x\varphi;x]$ if $x>x\varphi$. We now define $\mu: \Delta(x)\to T$ as follows.

If $y\in\Delta(x)$, there is a unique $n\in N$ such that $y\varphi^n\in S$ (by §3 prop.8). Put

$$y\mu = (e,e.n,\ y\varphi^n\sigma),$$

where $e=-1$ if $x<x\varphi$ and $e=+1$ if $x>x\varphi$.

We will show that μ is a 1-1 order-preserving map. Let $y_1,y_2\in\Delta(x)$, $y_1<y_2$. Say $x<x\varphi$. There are $n_1,n_2\in N$ such that $y_i\varphi^{n_i}\in S(i=1,2)$. If $n_1=n_2$, then $y_1\varphi^{n_1}<y_2\varphi^{n_2}$, hence $y_1\mu<y_2\mu$. If $n_1\neq n_2$, we must have $n_1>n_2$ (as $n_1<n_2 \Rightarrow y_1\varphi^{n_1}<y_2\varphi^{n_1}<y_2\varphi^{n_2}\in S \Rightarrow y_1\varphi^{n_1}\notin S$); then $e.n_1<e.n_2$, and again $y_1\mu<y_2\mu$. Similar if $x>x\varphi$.

Finally, $\mu\tau=(\varphi|\Delta(x))\mu$. For let $y\in\Delta(x)$, and let $n\in N$ such that $y\varphi^n\in S$; then

$$y\mu\tau = (e,e.n,\ y\varphi^n\sigma)\tau =$$
$$= (e,e(n-1),(y\varphi)\varphi^{n-1}\sigma) = y\varphi\mu.$$

Proposition 1. If $K=K(LO,\mathfrak{m})$ contains a universal object, then K also contains universal bimorphisms.

Proof.

Let $\tau : T \to T$ be as described in lemma 1. Let A be a universal object of K, and let $S = A \times T$, ordered lexicographically . Define $\Phi : S \to S$ by

$$(a,t)\Phi = (a, t\tau)$$

for arbitrary $(a,t) \in S$. It is easily seen that Φ is a bimorphism of K.

Let $\varphi : B \to B$ be any bimorphism of K. The set $\Delta(B)$ is a linearly ordered set of power $\leq \mathfrak{m}$; hence there is a 1-1 order-preserving map $\delta : \Delta(B) \to A$. For every $D \in \Delta(x)$, let μ_D be a 1-1 order-preserving map $D \to T$ such that $\mu_D \tau = (\varphi | D)\mu_D$; the existence of μ_D is guaranteed by lemma 1.

We define map $\mu : B \to A$ in the following way: if $x \in B$, we put

$$x\mu = ((\Delta(x))\delta \, , \, x\mu_{\Delta(x)}).$$

Then μ is a monomorphism. For let $x,y \in B$, $x<y$. If $\Delta(x) < \Delta(y)$ in $\Delta(B)$, then $(\Delta(x))\delta < (\Delta(y))\delta$, hence $x\mu < y\mu$. If $x \Delta y$ then $x\mu_{\Delta(x)} < y\mu_{\Delta(x)}$, and again $x\mu < y\mu$.

Finally, $\mu\Phi = \varphi\mu$. For let $x \in B$; then

$$\begin{aligned} x\mu\Phi &= ((\Delta(x))\delta \, , \, x\mu_{\Delta(x)})\Phi = \\ &= ((\Delta(x))\delta \, , \, x\mu_{\Delta(x)}\tau) = \\ &= ((\Delta(x))\delta \, , \, (x\varphi)\mu_{\Delta(x)}) = x\varphi\mu , \end{aligned}$$

as $\Delta(x) = \Delta(x\varphi)$.

Corollary. If $K(LO,\bar{m})$ contains a universal object, it contains universal bimorphisms.

We have proved now half of theorem 1. The second half - the existence of universal morphisms - is considerably more complicated.

Lemma 2. Let $K=K(LO,m)$ contain a universal object A. then K contains a morphism $\pi_0: P_0 \to P_0$, with the following property. If $\varphi : B \to B$ is any morphism of K, and if a is a fixed point under φ, then there exists a 1-1 order-preserving map μ: $\Delta(a) \to P_0$ such that $\mu\pi_0=(\varphi|\Delta(a))\mu$.

Proof.

Let $A_0=\{0\}$, and, for $n \geq 0$, $A_{n+1} = A_n \times A$, ordered lexicograpically. Distinct sets A_n, A_m are disjoint. Let $S = \bigcup_{n=0}^{\infty} A_n$; if $x \in S$, then $\omega(x)$ designates the $n \in N$ such that $x \in A_n$.

It is immediate that S is linearly ordered by the relation $\leq$ defined by

$$x \leq y \iff (\omega(x) > \omega(y)) \text{ or } (\omega(x)=\omega(y)= n, \text{ and } x \leq y \text{ in the ordering of } A_n).$$

If $n>0$, we define $\sigma_n: A_n \to A_{n-1}$ as follows: if $a' \in A_{n-1}$ and $a \in A$, then

$$(a',a)\,\sigma_n = a' \quad ;$$

let σ_0 be the identity map $A_0 \to A_0$ Let $\sigma : S \to S$ be the "union" of the maps σ_n:

$$\sigma | A_n = \sigma_n \ , \ n=0,1,2,\ldots.$$

Then σ is increasing and order-preserving.

Let $\varphi : B \to B$ be any morphism of K, and let $a\varphi = a \in B$. Then $\Delta(a) = TO(a)$. We will define a 1-1 order-preserving map $\nu : \Delta_1(a) \to S$ such that $\nu\sigma = (\varphi|\Delta_1(a))\nu$.

Let $D_0 = \{a\}$, $D_1 = a\varphi^{-1} \setminus \Delta_2(a)$, $D_{n+1} = D_n\varphi^{-1}$ for $n \geq 1$. The sets D_n, $n \in N^+$, are disjoint and cover $\Delta_1(a)$.

We define $\nu_0 : D_0 \to A_0$ in the only possible way: $a\nu_0 = 0$. Suppose $\nu_n : D_n \to A_n$ $(n \geq 0)$ already defined in such a way that

(i) ν_n is 1-1 and order-preserving;

(ii) $\nu_n \sigma_n = (\varphi | D_n)\nu_{n-1}$ (put $\nu_{-1} = \nu_0$).

Then it is possible to define $\nu_{n+1} : D_{n+1} \to A_{n+1}$ such that ν_{n+1} also satisfies (i) and (ii) (with all n's changed in n+1's). For let $b \in D_n$; the set $b\varphi^{-1}$ has power $\leq \mathfrak{m}$. Hence there is a 1-1 order-preserving map $\tau_b : b\varphi^{-1} \to A$. If $x \in D_{n+1}$, we define

$$x\nu_{n+1} = (x\varphi\nu_n, x\tau_{x\varphi}).$$

Finally we define $\nu : \Delta_1(a) \to S$ by: $\nu | D_n = \nu_n$, $n = 0,1,2,\ldots$. Then ν is indeed a 1-1 order-preserving map, and $\nu\sigma = (\varphi|\Delta_1(a))\nu$.

It follows that there is also a 1-1 order-preserving map $\tau : T \to T$ in K such that if $\varphi : B \to B$ in K and if $a \in B$, $a = a\varphi$, there exists a 1-1 order-preserving map $\nu' . \Delta_2(a) \to T$ such that $\nu'\tau = (\varphi|\Delta_2(a))\nu'$. We can take care that $S \cap T$ consists of exactly one point 0, and that this point 0 is fixed under both σ and τ : $0\sigma = 0 = 0\tau$. Then we put $P_0 = S \cup T$, ordered such that every $a \in S$ precedes every $b \in T$, and we define $\pi_0 : P_0 \to P_0$ by

$$\pi_0 | S = \sigma ; \quad \pi_0 | T = \tau .$$

<u>Lemma 3:</u> Let $K = K(LO, \mathfrak{m})$ contain a universal object A. Then K contains a morphism $\sigma_0 : N_0 \to N_0$ with following property. If $\varphi : B \to B$ is any morphism of K, and if $x \in B$ such that $TO(x)$ contains no fixed point, while moreover $\varphi|TO(x)$ is increasing, then there exists a 1-1 order -preserving map $\mu : TO(x) \to N_0$ such that $\mu\sigma_0 = (\varphi|TO(x))\mu$.

Proof.

Let $A^* = A \times \{1,2\} \cup \{0\}$, ordered as follows:

$(a,1) < 0 < (b,2)$, for arbitrary $a,b \in A$;

$(a,i) \leq (b,i) \iff a \leq b \quad (i=1,2)$

Then A^* is again a K-universal object. Even more is true: if Y is any object of K, and y any point of Y, there exists a monomorphism $\eta : Y \to A^*$ such that $y\eta = 0$.

For $n \in N$, let $M_n = \{k \in N : k \geq n\}$, and let $C_n = A^{M_n}$, $C = \bigcup_{n \in N} C_n$.

If $n \neq m$, then $C_n \cap C_m = \emptyset$. For $x \in C$, we write $\omega(x) = n$ iff $x \in C_n$.

The set C is constructed in the same way as in the proof of §2 lemma 3; let $\sigma : C \to C$ be defined as over there: if $x \in C_n$, then $x\sigma$ is the point of C_{n+1} such that $(x\sigma)_i = x_i$ for all $i \geq n+1$.

Let x_0 be the element of C_0 such that $x_i = 0$ for all $i \geq 0$; let $N_0 = TO(x_0)$ and $\sigma_0 = \sigma | N_0$.

If $x \in C$, then $x \in N_0 \iff (\exists k \geq \omega(x))(\forall i \geq k)(x_i = 0)$. Hence if $x,y \in N_0$, the following integer is well defined:

(4.1) $k(x,y)$ = the smallest $k \in N$ such that $k \geq \omega(x), k \geq \omega(y)$, and $(\forall i > k)(x_i = y_i)$.

The set N_0 is linearly ordered by the binary relation $\leq$ such that

(4.2) $x \leq y \iff (\omega(x) < \omega(y))$ or $(\omega(x) = \omega(y)$ and $x_{k(x,y)} \leq y_{k(x,y)})$.

It is immediately verified that in this ordering the map σ_0 is increasing and order-preserving.

Now let $\varphi : B \to B$ be a morphism of K; let $x \in B$ such that $TO(x)$ contains no fixed point, and let $x < x\varphi$. We define a 1-1 map $\mu : TO(x) \to N_0$ as in the proof of §2 lemma 4, with slight modifications, in order to obtain an order-preserving map.

In detail: let $A_0 = \{x\varphi^n : n \in N^+\}$, $A_1 = A_0\varphi^{-1} \setminus A_0$, $A_{n+1} = A_n\varphi^{-1}$ $(n \geq 1)$.

If $n \neq m$, $A_n \cap A_m = \emptyset$. We first define $\mu | (A_0 \cup A_1)$.

If $n\in N^+$, let

$$T_n = \{u\in N_o : u\sigma_o = x_o\sigma_o{}^n\}.$$

Then $u\in T_n \Leftrightarrow \omega(u) = n-1$ and $(\forall i\geq n)(u_i=0)$. Hence $u\rightarrow u_{n-1}$ is an order-isomorphism of T_n onto A^*; it follows that for every $n\in N^+$ there is a 1-1 order-preserving map $\tau_n : (x\varphi^n)\varphi^{-1}\rightarrow T_n$, while in case $n\geq 1$ we can take care that

$$x\varphi^{n-1}\tau_n = x_o\sigma_o{}^{n-1}.$$

We put

$$\mu|(x\varphi^n)\varphi^{-1} = \tau_n$$

for each $n\in N^{\tau}$. Then the map μ is defined on all of $A_o\cup A_1$. And $\mu|(A_o\cup A_1)$ is 1-1 and order-preserving. For let $y_1, y_2\in A_o\cup A_1$; $y_1<y_2$. Then $y_1\varphi = x\varphi^{n_1}\leq x\varphi^{n_2} = y_2\varphi$ (for certain $n_1, n_2\in N^+$). If $n_1<n_2$, then $\omega(y_1\mu)<\omega(y_2\mu)$, and hence $y_1\mu<y_2\mu$. And if $n_1 = n_2 = n$, then

$$y_1\mu = y_1\tau_n < y_2\tau_n = y_2\mu.$$

Moreover, one verifies at once that, for $y\in A_o\cup A_1$,

(4.3) $$y\mu\sigma_o = y\varphi\mu.$$

Assume now that $\mu|(A_o\cup A_1\cup\ldots\cup A_n)$, $n\geq 1$, is already defined, in such a way that it is a 1-1 and order-preserving map, and that (4.3) holds for all $y\in A_o\cup A_1\cup\ldots\cup A_n$. Let $y\in A_n$; as card $(y\varphi^{-1})\leq m$, and as $(y\mu)\sigma_o{}^{-1}$ is order-isomorphic to A^*, there exists a 1-1 order-preserving map $\tau_y : y\varphi^{-1}\rightarrow(y\mu)\sigma_o{}^{-1}$. We put

$$\mu|y\varphi^{-1} = \tau_y.$$

Then μ is defined, 1-1 and order-preserving on $A_0 \cup A_1 \cup \ldots \cup A_{n+1}$, and (4.3) holds for every $y \in A_0 \cup \ldots \cup A_{n+1}$.

In this way we construct inductively a 1-1 order-preserving map $\mu : TO(x) \to N_0$ such that $\mu \sigma_0 = \varphi \mu$.

Lemma 4. Let $K=K(LO,\mathfrak{m})$ contain a universal object A. Then K contains a morphism $\pi_1 : P_1 \to P_1$ with the following property. If $\varphi : B \to B$ is any morphism of K, and if $x \in B$ such that $\Delta(x)$ contains no fixed points, while $\varphi|\Delta(x)$ is increasing, then there exists a 1-1 order-preserving map $\nu : \Delta(x) \to P_1$ such that $\nu \pi_1 = (\varphi|\Delta(x))\nu$.

Proof:

Let $\sigma_0 : N_0 \to N_0$ be the map defined in lemma 3, and let P_1 be the set $N_0 \times A^*$, linearly ordered in the following way: if (x,a) and $(y,b) \in P_1$, then

(4.4) $(x,a) \leq (y,b) \Leftrightarrow (\omega(x) < \omega(y))$ or $(\omega(x)=\omega(y)$ and $a<b)$ or $(\omega(x)=\omega(y)$ and $a=b$ and $x \leq y)$.

Define $\pi_1 : P_1 \to P_1$ by

$$(x,a)\pi_1 = (x\sigma_0, a).$$

Then π_1 is an increasing morphism of K.

Let $\varphi : B \to B$ be an arbitrary morphism of K, and let $x \in B$ such that $\Delta(x)$ contains no fixed point and $x < x\varphi$. The set $\sum(x)$, ordered by $\leq_x$ (cf. §3 def. 4) is an object of $K(LO,\overline{\mathfrak{m}})$; hence there exists a 1-1 order-preserving map $\lambda : \sum_x \to A$. In the remainder of this proof we will just write $\sum$ and $\leq$ for $\sum(x)$ and $\leq_x$.

For every $S \in \sum$ we choose an $n(S) \in N^+$ and a $y_S \in S$ such that

(4.5) $$x\varphi^{n(S)} \leq y_S < x\varphi^{n(S)+1}.$$

In case $S=TO(x)$, we take $n(S) = 0$ and $y_S = x$.
By lemma 3 there exists a 1-1 order-preserving map μ_S: $S \to N_o$ such that

$$y_S \mu_S = x_o \sigma_o^{\ n(S)}$$

while

$$\mu_S \sigma_o = (\varphi | S)\mu_S .$$

We define $\nu : \Delta(x) \to P_1$ as follows: if $y \in \Delta(x)$, and $S=TO(y)$, then

$$y\nu = (y\mu_S, S\lambda).$$

We will show that ν satisfies the requirements set forth in the lemma.

First we show that $\nu\pi_1 = (\varphi|\Delta(x))\nu$. Let $y \in \Delta(x)$, and let $S=TO(y)$. Then

$$y\nu\pi_1 = (y\mu_S, S\lambda)\pi_1 = (y\mu_S\sigma_o, S\lambda) =$$

$$= (y\varphi\mu_S, S\lambda) = y\varphi\nu,$$

as $TO(y\varphi) = S$.

Now we show that ν is 1-1 and order-preserving. Let $y_1 < y_2$, $y_1, y_2 \in \Delta(x)$. Put $TO(y_i) = S_i (i=1,2)$. If $S_1=S_2=S$, then $y_1\mu_S < y_2\mu_S$; it follows that either $\omega(y_1\mu_S) < \omega(y_2\mu_S)$ - implying $y_1\nu < y_2\nu$ - or $\omega(y_1\mu_S) = \omega(y_2\mu_S)$, in which case also $y_1\nu < y_2\nu$ ((4.4), third clause).

Therefore suppose $S_1 \neq S_2$. Let $\omega(y_i\mu_{S_i}) = m_i$ $(i=1,2)$.

In order to simplify the notation, we will write μ_i instead of μ_{S_i} and n_i instead of $n(S_i)$ $(i=1,2)$.

If $m_1 < m_2$, then $y_1\nu < y_2\nu$. Suppose $m_1=m_2=m$; we must show that $S_1 < S_2$.

Let $k \geq k(y_i\mu_i, y_{S_i})$, $i=1,2$ (cf. (4.1)). Then $k \geq m, n_1, n_2$.

Moreover,

$$y_1\varphi^{k-m}\mu_1 = y_1\mu_1\sigma_o^{k-m} = y_{S_1}\mu_1\sigma_o^{k-n_1} = y_{S_1}\varphi^{k-n_1}\mu_1 \;;$$

as μ_1 is 1-1, it follows that

$$y_1\varphi^{k-m} = y_{S_1}\varphi^{k-n_1}.$$

By (4.5) and the fact that $y_1 < y_2$ and $S_1 \neq S_2$, we conclude that

$$x\varphi^k \leq y_1\varphi^{k-m} < y_2\varphi^{k-m} \leq x\varphi^{k+1}.$$

If $y_2\varphi^{k-m} < x\varphi^{k+1}$, it follows from the definition of $\leq_x$ that $S_1 < S_2$. We will conclude the proof by showing that the assumption $y_2\varphi^{k-m} = x\varphi^{k+1}$ leads to a contradiction.

If $y_2\varphi^{k-m} = x\varphi^{k+1}$, then $S_2 = TO(x)$, hence $n_2 = 0$ and $y_{S_2} = x$.

It follows that

$$x\varphi^k = y_{S_2}\varphi^{k-n_2} = y_2\varphi^{k-n} = x\varphi^{k+1},$$

and hence that $\Delta(x)$ contains a fixed point. This contradicts our assumptions.

<u>Lemma 5.</u> If $K=K(LO,\bar{m})$ contains a universal object, then K contains amorphism $\tau: T \to T$ with the following property. If $\varphi : B \to B$ is any morphism of K, and if $x \in B$, then there exists a 1-1 order-preserving map $\mu : \Delta(x) \to T$ such that $\mu\tau = (\varphi|\Delta(x))\mu$.

<u>Proof.</u>

It follows from lemma 4 (reversing orderings) that there exists a morphism $\pi_2: P_2 \to P_2$ with the property: if $\varphi : B \to B$ in K, $x \in B$, $x > x\varphi$, and if $\Delta(x)$ contains no fixed point, then there exists a 1-1 order-preserving map $\nu : \Delta(x) \to P_2$ such that $\nu \pi_2 = (\varphi | \Delta(x))\nu$.

Let $\tau_i : T_i \to T_i$ $(i=0,1,2)$ be a copy of $\pi_i : P_i \to P_i$ (where $\pi_o : P_o \to P_o$ is the morphism defined in lemma 2, and $\pi_1 : P_1 \to P_1$ is the morphism described in lemma 4), such that the sets T_o, T_1 and T_2 are pairwise disjoint: we take as T the set $T_o \cup T_1 \cup T_2$, ordered such that

$$x_o < x_1 < x_2$$

for arbitrary $x_i \in T_i$ $(i=0,1,2)$, while on T_i the ordering of T coincides with the ordering of T_i. The map $\tau : T \to T$ is defined by

$$\tau | T_i = \tau_i \quad (i=0,1,2).$$

We have now the means by which to prove the second half of theorem 1.

Proposition 2. If $K = K(LO, m)$ contains a universal object, then K contains universal morphisms.

Proof.

The proof is exactly parallel to the proof of prop. 9, using lemma 5 instead of lemma 1.

Corollary. If $K(LO, \bar{m})$ contains a universal object, it contains universal morphisms.

Remark 1. If $m > 1$ is a finite cardinal, $K(LO, m)$ evidently contains no universal objects and hence no universal morphisms or bimorphisms.

Remark 2. At the end of section 2 we remarked that theorem 1 of that section could be proved in a much simpler way, using S-maps, if the cardinal number m has the property

$$m^{\aleph_o} = m .$$

The same is true for theorem 1 of the present section. As the proof of this theorem is so much more complicated, the remark is even more relevant.

However, the class of all cardinals $\mathfrak{m}$ such that $\mathfrak{m} < \mathfrak{m}^{\aleph_0}$ is cofinal in the class of all cardinals; and it contains $\aleph_0$, the only cardinal number $\mathfrak{m}$ for which we are really sure that K(LO,$\mathfrak{m}$) contains universal objects. Hence the proofs of this section are worthwhile.

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