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S-maps

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1. Introduction and notation.

In this note we describe, generalize and apply a construction, used by J. de Groot to establish the existence of certain universal systems of mappings [7] , and to prove that continuous mappings of a metric space into itself can be considered as restrictions of linear maps ([8]; see also [5],[6]). The same construction has been used by G.C. Rota [15], [16] in order to derive universal bounded operators in Hilbert space.

If X,Y are sets, X^Y designates the set of all functions on Y with values in X. If $x \in X^Y$ and $\eta \in Y$, then x_η denotes the value of x in η ("the η-coordinate" of x): we often write

$$x = (x_\eta)_{\eta \in Y}.$$

The projection map $x \to x_\eta$ is denoted by π_η .

If $X_1 \subset X_2$, we consider $X_1{}^Y$ to be a subset of $X_2{}^Y$. If X is provided with some additional structure, we suppose X^Y to be provided with the product structure. For instance, if X is a topological space, X^Y is provided with the product topology (the weak topology, if one considers X^Y as a function space); if X is a group, X^Y is the full direct product. If X is a topological linear space, so is X^Y (by combination of the previous two conventions).

In the case of a linearly ordered set X and a well-ordered set Y, we take X^Y to be linearly ordered by means of the lexicographic order.

We make use of the language of category theory as exposed in [13] , and of the notation and conventions in [3] , [4]. Consequently, we write the argument of a function before the function symbol (there are a few exceptions, like x_η, card (G), H(A,B)); if $f,g \in A^A$, then fog designates the composite function $\alpha \to ((\alpha)f)g$.

The identity map of a set A into itself is denoted by i, or by i_A if it is necessary to call attention to its domain.

If $F \subset X^Y$ and $A \subset X$, then $F|A$ denotes the set $\{\varphi|A : \varphi \in F\}$. We say that A is invariant under F (or F-invariant) if $A\varphi \subset A$ for all $\varphi \in F$. If $F \subset X^X$ is a semigroup under composition and $A \subset X$ is F-invariant, then $F|A$ is a semigroup of mappings $A \to A$.

The category of all mappings of one set into another is denoted by K(S). If X is a set, K_X denotes the category, whose objects are all sets X^A, A a non-void set, and whose morphisms are all mappings of one such an object into another one.

If K is a category and A,B are objects of K, then H(A,B) denotes the set of all morphisms $\varphi \in K$ with A as first object and B as second object. If we use the notation H(A,B) without mentioning a specific category, then H(A,B) is formed in the category K(S).

Definition 1. Let K be a category, and let A,B be objects of K. Let $F \subset H(A,A)$ and let $G \subset H(B,B)$. We say that F and G are equivalent if there exists a bimorphism $\mu : A \to B$ such that the transformation

$$\varphi \to \mu^{-1}\varphi\mu$$

maps F onto G.

It is evident that in that case the transformation $\varphi \to \mu^{-1}\varphi\mu$ is 1-1. If F and G are semigroups of transformations, then $\varphi \to \mu^{-1}\varphi\mu$ is clearly a semigroup isomorphism.

If we want to stress the rôle of μ, we say that F and G are equivalent by means of μ. We will also write: the pairs (A,F) and (B,G) are equivalent. If F consists of one element φ and G of one element χ, we say that φ and χ are equivalent iff $\{\varphi\}$ and $\{\chi\}$ are.

If F and G are equivalent, and F is a semigroup, sc is G; if F is a group then G is a group.

By semigroup we mean in this note a semigroup with unit. In the case of a transformation semigroup, we assume that its unit is the identity transformation. The unit of an abstract semigroup is as a rule denoted by 1. By a homomorphism of a semigroup F into a semigroup G we always mean a semigroup homomorphism sending the unit of F onto the unit of G.

2. The star-functor.

Throughout this section and the subsequent two, X is a fixed non-void set.

Definition 1. If $\varphi \in K(S)$, say $\varphi : A \to B$, then φ^* designates the map $X^B \to X^A$ such that

$$(x\varphi^*)_\alpha = x_{\alpha\varphi}$$

for arbitrary $x = (x_\beta)_{\beta \in B} \in X^B$ and arbitrary $\alpha \in A$. In other words: $\varphi^* \circ \pi_\alpha = \pi_{\alpha\varphi}$.

We denote by K_X^* the subcategory of K_X consisting of all transformations φ^*, $\varphi \in K_X$; these transformations are called S-maps (with base space X).

Proposition 1. $K_X^* \neq K_X$ if X contains at least two distinct elements p,q.

Proof.
Let A,B be non-void sets. Let $x \in X^B$ such that $x_\beta = p$ for all $\beta \in B$, and let $y \in X^A$ such that $y_\alpha = q$ for all $\alpha \in A$. Any $\mu : X^B \to X^A$ such that $x\mu = y$ cannot be an S-map.

Proposition 2. The transform $\varphi \to \varphi^A$ is a contravariant functor $K(S) \to K_X$; if X contains at least two distinct elements p,q it is in fact an anti-isomorphism of K(S) into K_X.

Proof.

We first show that $(\varphi\circ\psi)^* = \psi^*\circ\varphi^*$. Let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$; then $X^C \xrightarrow{\psi^*} X^B \xrightarrow{\varphi^*} X^A$. Hence the product $\psi^*\circ\varphi^*$ is in any case defined. And for arbitrary $\alpha \in A$:

$$\psi^*\circ\varphi^*\circ\pi_\alpha = \psi^*\circ\pi_{\alpha\varphi} = \pi_{\alpha\varphi\psi} = (\varphi\circ\psi)^*\circ\pi_\alpha .$$

Thus $\varphi \to \varphi^*$ is a contravariant functor. As clearly $\varphi\circ\psi$ is defined as soon as $\psi^*\circ\varphi^*$ is, it only remains to show that the functor $\cdot\cdot^*$ is 1-1, if X contains at least two distinct elements.

Suppose $\varphi \neq \psi$. If $\varphi: A_1 \to B_1$ and $\psi: A_2 \to B_2$, where either $A_1 \neq A_2$ or $B_1 \neq B_2$, it is trivial that $\varphi^* \neq \psi^*$. Assume φ and ψ both belong to H(A,B). There is an $\alpha \in A$ such that $\alpha\varphi \neq \alpha\psi$ Let x be any point of X^B such that $x_{\alpha\varphi} = p$ and $x_{\alpha\psi} = q$. Then

$$(x\varphi^*)_\alpha = x_{\alpha\varphi} \neq x_{\alpha\varphi} = (x\psi^*)_\alpha .$$

Thus $\varphi^* \neq \psi^*$.

The functor $\cdot\cdot^*$ will be called the star-functor or STAR; if we want to emphasize the base space X, we will write $STAR_X$.

Corollary. For any set A, STAR maps the semigroup H(A,A) anti-isomorphically into $H(X^A, X^A)$.

It also follows that φ^* is a monomorphism (epimorphism) in K_X if φ is an epimorphism (monomorphism). As $K_X^* \neq K_X$, however, the next proposition still needs a proof.

Proposition 3. φ^* is 1-1 iff φ is onto; φ^* is onto iff φ is 1-1.

Proof.

If φ^* is 1-1, it is a monomorphism in K_X, and a fortiori it is a monomorphism in K_X^*; hence φ is onto. Similarly, if φ^* is onto, then φ is 1-1. Now suppose $\varphi: A \to B$ is an epimorphism in K(S). Let $x,y \in X^B$, $x \neq y$. Then $x_\beta \neq y_\beta$ for some $\beta \in B$; as φ is onto, $\beta = \alpha\varphi$ for some $\alpha \in A$. It follows that

$$(x\varphi^*)_\alpha = x_{\alpha\varphi} = x_\beta \neq y_\beta = (y\varphi^*)_\alpha ;$$

hence $x\varphi^* \neq y\varphi^*$. Thus φ^* is 1-1.

Finally, suppose $\varphi : A \to B$ is 1-1, and let $y \in X^A$. Define $x \in X^B$ as follows. If $\beta \notin A\varphi$, we put $x_\beta = p$ where p is a fixed point of X. If $\beta \in A\varphi$, there is exactly one $\alpha \in A$ such that $\beta = \alpha\varphi$; we put $x_\beta = y_\alpha$. Then $x\varphi^* = y$. Hence φ^* is onto.

<u>Definition.</u> Let $Y \supset X$. For $\varphi \in K(S)$, the map $STAR_Y \varphi$ is called the <u>canonical extension</u> of the map $STAR_X\varphi$ to the base Y.

<u>Proposition 4</u> For $Y \supset X$, the canonical extension $STAR_X\varphi \to STAR_Y\varphi$ is an isomorphism of K_X^* into K_Y^*.
<u>Proof:</u> immediate from prop. 2.

The next proposition is also evident:

<u>Proposition 5.</u> If $Y \supset X$, and if $\varphi : A \to B$, then $STAR_X \varphi = (STAR_Y \varphi) \mid X^B$.

3. <u>The fundamental embedding lemma.</u>

<u>Definition 1.</u> If G is a semigroup, and $\gamma \in G$, we will denote by $\bar{\gamma}$ the map $G \to G$ such that

$$(\xi)\bar{\gamma} = \gamma \cdot \xi ,$$

for arbitrary $\xi \in G$. The transformation semigroup of all $\bar{\gamma}$, $\gamma \in G$, is denoted by $\bar{G}$.

The next proposition is trivial.

<u>Proposition 1.</u> If G has a unit, the map $\gamma \to \bar{\gamma}$ is an anti-isomorphism of G onto $\bar{G}$.

<u>Corollary.</u> If G is a semigroup with unit, the transformation $\gamma \to \bar{\gamma}^*$ is an isomorphism of G into $H(X^G, X^G)$.

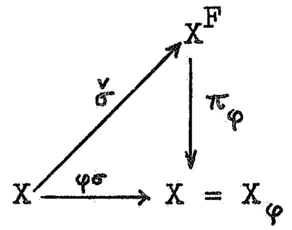
To simplify the notation, we will write $\hat{\gamma}$ instead of $\bar{\gamma}^*$. The subsemigroup $\{\hat{\gamma} : \gamma \in G\}$ of $H(X^G, X^G)$ will be denoted by $\hat{G}$. It should be kept in mind that this notation is ambiguous, as the notation $\bar{\gamma}$ is ambiguous. For the transformation $\bar{\gamma}$ is determined, not only by the semigroup element γ, but also by the semigroup G of which γ is considered to be an element: if $\gamma \in G_1 \subset G_2$, we would have to disting ish between a $\bar{\gamma} : G_1 \to G_1$ and a $\bar{\bar{\gamma}} : G_2 \to G_2$.

However, complications of this kind will not arise in the course of the considerations of this note.

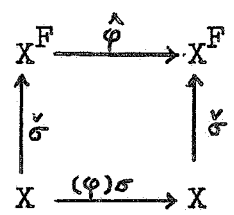
<u>Definition 2.</u> If σ is a homomorphism of a semigroup F into $H(X,X)$, then $\check{\sigma}$ designates the map $X \to X^G$ such that

$$(\xi\check{\sigma})_\varphi = (\xi)\ (\varphi\sigma),$$

for arbitrary $\xi \in X$ and $\varphi \in F$. In other words: $\check{\sigma} \circ \pi_\varphi = (\varphi)\sigma$.



<u>Proposition 2</u> (FUNDAMENTAL EMBEDDING LEMMA). If F is a semigroup with unit, the map $\check{\sigma}$ is one-to-one. Moreover, if $\varphi \in F$, then $\check{\sigma} \circ \hat{\varphi} = (\varphi\sigma) \circ \check{\sigma}$.



<u>Proof.</u>

As $\check{\sigma} \circ \pi_1 = (1)\sigma = i_X$, $\check{\sigma}$ is 1-1. And if $\varphi, \psi \in F$, then

$$(\check{\sigma} \circ \hat{\varphi}) \circ \pi_{\psi} = \check{\sigma} \circ (\hat{\varphi} \circ \pi_{\psi}) = \check{\sigma} \circ \pi_{\varphi\psi} =$$
$$= (\varphi \circ \psi)\sigma = (\varphi\sigma) \circ (\psi\sigma) = (\varphi\sigma) \circ \check{\sigma} \circ \pi_{\psi}.$$

<u>Remark.</u> An important special case is the one where F is a subsemigroup of H(X,X) and where σ is the natural injection $F \to H(X,X)$. In this case the diagram of proposition 2 simplifies to

$$\begin{array}{ccc} X^F & \xrightarrow{\hat{\varphi}} & X^F \\ \uparrow{\scriptstyle \check{\sigma}} & & \uparrow{\scriptstyle \check{\sigma}} \\ X & \xrightarrow{\varphi} & X \end{array}$$

and the definition of $\check{\sigma}$ reduces to : $\check{\sigma} \circ \pi_{\varphi} = \varphi$.

<u>Corollary.</u> The subset $X\check{\sigma}$ of X^F is invariant under F.

<u>Proof.</u>

Immediate from the fact that $\check{\sigma} \circ \hat{\varphi} = (\varphi\sigma) \circ \check{\sigma}$.

<u>Proposition 3.</u> $(X, F\sigma)$ and $(X\check{\sigma}, \hat{F}|X\check{\sigma})$ are equivalent by means of $\check{\sigma}$.

<u>Proof.</u>

By prop. 2, $\check{\sigma}$ is invertible if considered as a map $X \to X\check{\sigma}$. Let $G = F\sigma$. If $\gamma \in G$, we define $\gamma' : X\check{\sigma} \to X\check{\sigma}$ by:

$$\gamma' = \check{\sigma}^{-1} \circ \gamma \circ \check{\sigma}.$$

Then $\gamma \to \gamma'$ maps G onto $\hat{F}|X\check{\sigma}$: if $\varphi \in F$, then

$$\hat{\varphi}|X\check{\sigma} = \check{\sigma}^{-1} \circ (\varphi\sigma) \circ \check{\sigma},$$

and hence $\hat{\varphi}|X\check{\sigma} = (\varphi\sigma)'$.

4. Universal systems of mappings.

As immediate consequences of § 3 prop.3 we obtain the following theorems.

Theorem 1. Let X be a non-empty set, and let F be a free semigroup with $\mathfrak{n}$ generators ($\mathfrak{n}$ an arbitrary cardinal number). For every semigroup G of transformations $X \to X$, acting effectively on X, such that card $(G) \leq \mathfrak{n}$, there exists a subset X_0 of X^F, invariant under $\hat{F}$, such that (X,G) and $(X_0, \hat{F}|X_0)$ are equivalent.

Theorem 2. Let X be a non-void set, and let F be a free group with $\mathfrak{n}$ generators. For every transformation group G, acting effectively on X, such that card $(G) \leq \mathfrak{n}$, there exists a subset X_0 of X^F, invariant under $\hat{F}$, such that (X,G) and $(X_0, \hat{F}|X_0)$ are equivalent.

Proofs.

If G acts effectively on X, it can be identified with a sub-(semi-)group of $H(X,X)$. Let σ be any homomorphism of F onto G, and take $X_0 = X\check{\sigma}$.

Remark. The condition card $(G) \leq \mathfrak{n}$ can of course be replaced by the weaker condition: G has a system of generators of power $\leq \mathfrak{n}$.

As F and $\hat{F}$ are isomorphic, we can also express part of the content of theorems 1,2 in the following way.

Theorem 1'. Let X be a set with card $(X) = \mathfrak{m}$; let $\mathfrak{n}$ be any cardinal number. If Y is any set with card $(Y) \geq \mathfrak{m}^{\mathfrak{n}.\aleph_0}$, then $H(Y,Y)$ contains a subsemigroup F with the following properties:

(i) the abstract semigroup F is a free semigroup with-unit with $\mathfrak{n}$ generators;

(ii) if G is any transformation semigroup acting effectively on X, with card $(G) \leq \mathfrak{n}$, then (X,G) is equivalent to $(X_o, F|X_o)$, for a suitable F-invariant subset X_o of Y.

Theorem 2': obtained from theorem 1' by substituting "group" for every occurrence of "semigroup".

Corollary ($\mathfrak{n} = 1$) If $\mathfrak{m} = \mathfrak{m}^{\aleph_0}$, the categories K(S,$\mathfrak{m}$) and K(S,$\bar{\mathfrak{m}}$) contain universal morphisms and bimorphisms.

This is a result obtained in [4]. Our present proof is more simple and elegant then the one presented in [4]; on the other hand, the results in [4] are more general, as the assumption $\mathfrak{m} = \mathfrak{m}^{\aleph_0}$ is replaced there by the weaker condition $\mathfrak{m} \geq \aleph_0$. (The class of those cardinals $\mathfrak{m}$ for which $\mathfrak{m}^{\aleph_0} \neq \mathfrak{m}$ is cofinal in the class of all cardinals).

However, we have now considerably generalized the results of [4] in another direction: the theorems 1' and 2' show that if $\mathfrak{m}^{\mathfrak{n}\aleph_0} = \mathfrak{m}$, K(S,$\mathfrak{m}$) contains semigroups of morphisms and groups of bimorphisms (even free semigroups and groups) that are universal (cf.§7) for all semigroups of at most $\mathfrak{n}$ morphisms (all groups of at most $\mathfrak{n}$ bimorphisms).

5. S-maps in abstract categories.

As has been mentioned already in the introduction, the terminology of [13] and [4] is used.

Definition 1. Let K be a category. If X is an object of K and A is a set, then $\sum(X,A)$ is the family $(X_\alpha)_{\alpha \in A}$ with $X_\alpha = X$ for all $\alpha \in A$. The class of all direct joins in K of $\sum(X,A)$ is denoted by $\Delta(X,A)$; of course $\Delta(X,A)$ may be empty.

Let K be a fixed category, and let X be a fixed object of K. Let A,B be non-void sets, and suppose $\Delta(X,A)$ and $\Delta(Y,A)$ are non-void. Suppose

$$Y = \prod(\sum(X,A)) = \prod_{\alpha\in A} X_\alpha \ (\pi_\alpha),$$

$$Z = \prod(\sum(X,B)) = \prod_{\beta\in B} X_\beta \ (\pi'_\beta).$$

If $\varphi: A\to B$ is any map, then $\pi_{\alpha\varphi}: Z\to X_{\alpha\varphi} = X = X_\alpha$, for every $\alpha\in A$. Hence, by the definition of direct join ([12] § 12.1) there exists a <u>unique</u> $\tau : Z\to Y$ such that

$$\tau\pi_\alpha = \pi_{\alpha\varphi},$$

for all $\alpha\in A$. This morphism τ will be denoted by φ^*; all morphisms of K, obtained in such a way, will again be called S-maps. The fact that S-maps are again morphisms of the same category K is stressed, as it is quite essential.

<u>Examples:</u> if K is the category of all topological spaces, or of all groups, or of all abelian groups, or of all topological vector spaces, then direct joins always exist in K; hence we can always construct S-maps.
Thus:

<u>Proposition 1.</u> If X is a topological space, every S-map $\varphi^*: X^B\to X^A$ is continuous. If X is a group, φ^* is a homomorphism. If X is a topological vector space, φ^* is a continuous linear operator.

The following observations will be useful. Let $\mu: X\to X'$ be a monomorphism of K. Suppose $Y\in\Delta(X,A)$ and $Y'\in\Delta(X',A)$, where A is a non-void set; say

$$Y = \prod_{\alpha \in A} X_\alpha \qquad (\pi_\alpha);$$

$$Y' = \prod_{\alpha \in A} X'_\alpha \qquad (\pi'_\alpha).$$

Then there exists a unique morphism τ : $Y \to Y'$ such that $\tau \pi'_\alpha = \pi_\alpha \mu$, for all $\alpha \in A$. This morphism we denote by $\bar{\mu}$.

<u>Proposition 2.</u> $\bar{\mu}$ is a monomorphism.

<u>Proof.</u>

If $\rho_1 \bar{\mu} = \rho_2 \bar{\mu}$, then $\rho_1 \pi_\alpha \mu = \rho_1 \bar{\mu} \pi'_\alpha = \rho_2 \bar{\mu} \pi'_\alpha =$ $= \rho_2 \pi_\alpha \mu$, for all $\alpha \in A$. As μ is a monomorphism, it follows that $\rho_1 \pi_\alpha = \rho_2 \pi_\alpha$, for all $\alpha \in A$. This implies that $\rho_1 = \rho_2$ (cf. [13] §13.2).

Now let φ : $A \to A$; let φ^* and φ'^* denote the corresponding S-maps $Y \to Y$ and $Y' \to Y'$, respectively.

<u>Proposition 3.</u> $\bar{\mu} \varphi'^* = \varphi^* \bar{\mu}$.

<u>Proof.</u>

For arbitrary $\alpha \in A$ we have

$$\bar{\mu} \varphi'^* \pi'_\alpha = \bar{\mu} \pi'_{\alpha\varphi} = \pi_{\alpha\varphi} \mu = \varphi^* \pi_\alpha \mu = \varphi^* \bar{\mu} \pi'_\alpha .$$

<u>Remark.</u> Prop. 3 can be considered as an abstract analogue of §2 prop. 4.

The case of an index set F which is itself a semigroup is again of special interest. If we write once more $\hat{\varphi}$ instead of $\bar{\varphi}^*$, we have:

<u>Proposition 4.</u> If $Y = \prod(\sum(X,F))$, then $\varphi \to \hat{\varphi}$ is an isomorphism of F into $H(Y,Y)$; in particular, $\hat{1} = \varepsilon_Y$.

<u>Proof.</u>

If $\varphi_1, \varphi_2, \psi \in F$, then

$$\hat{\varphi}_1 \cdot \hat{\varphi}_2 \cdot \pi_\psi = \hat{\varphi}_1 \cdot \pi_{\varphi_2\psi} = \pi_{\varphi_1\varphi_2\psi} .$$

As $\widehat{\varphi_1\varphi_2}$ is the unique τ such that $\tau\pi_\psi = \pi_{\varphi_1\varphi_2\psi}$, for all $\psi \in F$, we conclude that $\hat{\varphi}_1 \cdot \hat{\varphi}_2 = \widehat{\varphi_1\varphi_2}$.

Corollary. If F is a group, each $\hat{\varphi}$, $\varphi \in F$, is a bimorphism in K.

Proof.

$$\hat{\varphi} \cdot \widehat{\varphi^{-1}} = \widehat{\varphi^{-1}} \cdot \varphi = \varepsilon_Y .$$

6. Abstract analogue of the fundamental embedding lemma.

Let K be a category, X an object of K, and let F be a semigroup. If σ is a homomorphism of F into H(X,X) such that $(1)\sigma = \varepsilon_X$, and if $Y \in \Delta(X,F)$,

$$Y = \prod_{\varphi \in F} X_\varphi \ (\pi_\varphi),$$

then there exists a unique $\tau \in K$ such that

$$\tau \cdot \pi_\varphi = (\varphi)\sigma ,$$

for all $\varphi \in F$. This morphism τ will again be denoted by $\check{\sigma}$.

Proposition 1. $\check{\sigma}$ is a monomorphism; moreover, $\check{\sigma}\hat{\varphi} = (\varphi\sigma)\check{\sigma}$, for all $\varphi \in F$.

This proposition is a special case of the next one, which can be considered as a l.u.b. of prop. 1 and § 5 prop. 3.

Proposition 2. Let σ be a homomorphism of the semigroup F into H(X,X) such that $(1)\sigma = \varepsilon_X$, and let $\mu : X \to U$ be a monomorphism. Suppose $V \in \Delta(U,F)$:

$$V = \prod_{\varphi \in F} U_\varphi \quad (\pi_\varphi).$$

The unique map $\tau : X \to V$ such that $\tau\pi_\varphi = (\varphi)\sigma\mu$ for all $\varphi \in F$, is a monomorphism; moreover, $\tau\hat{\varphi} = (\varphi\sigma)\tau$, for all $\varphi \in F$ (here $\hat{\varphi}$ is constructed with U as base, $\hat{\varphi} : V \to V$).

Proof.

If $\rho_1\tau = \rho_2\tau$, then $\rho_1\mu = \rho_1 \varepsilon_X\mu = \rho_1(1\sigma)\mu = \rho_1\tau\ \pi_1 =$ $= \rho_2\tau\ \pi_1 = \dots = \rho_2\mu$; hence $\rho_1 = \rho_2$. Thus τ is a monomorphism.

Moreover, if ψ is arbitrary in F, then

$$\tau\hat{\varphi}\pi_\psi = \tau\pi_{\varphi\psi} = (\varphi\psi)\sigma\mu = (\varphi\sigma)(\psi\sigma)\mu = (\varphi\sigma)\tau\pi_\psi.$$

Hence $\tau\hat{\varphi} = (\varphi\sigma)\tau$.

Remark 1. One obtains prop. 1 by taking U=X and $\mu = \varepsilon_X$.

If $\Delta(X,F) \neq \emptyset$, one can conversely obtain prop.2 as an immediate result of prop. 1 and §5 prop.3.

Remark 2. It is once more important that the existence of direct joins guarantees the existence of $\check{\sigma}$ as a morphism in K. Hence if K is the category of all topological spaces, $\check{\sigma}$ is always continuous; if K is the category of all groups, $\check{\sigma}$ is a homomorphism, etc.

7. Universal systems in categories.

Definition 1. Let K be a category, and let n be a cardinal number. Let F be a semigroup with unit in K; say F is a subsemigroup of H(A,A), A an object of K.

We call F an n-universal semigroup of morphisms in K if for every semigroup G contained in K - say $G \subset H(B,B)$ - containing ε_B, and such that, as an abstract semigroup, G can be generated by a set of power $\leq n$ of its elements, there exist a map ρ of F onto G, and a morphism $\mu \in K$, with the following properties:

(i) ρ is a homomorphism of the abstract semigroup F onto the abstract semigroup G;

(ii) μ is a monomorphism $B \to A$;

(iii) $(\varphi)\rho \cdot \mu = \mu \cdot \varphi$, for all $\varphi \in F$.

Definition 2. Let K, n, F, A be as in definition 1. We call F an n-universal group of bimorphisms in K if for every group G contained in K-say $G \subset H(B,B)$- such that, as an abstract group, G can be generated by a set of power $\leq n$ of its elements, there exist ρ and μ with the properties (i), (ii), (iii) described in definition 1.

The dual concepts are called dual n-universal semigroups of morphisms in K and dual n-universal groups of bimorphisms in K, respectively- (that is, $F \subset K$ is a dual n-universal semigroup of morphisms in K if it is an n - universal semigroup of morphisms in the dual category of K, etc.)

The following proposition shows that condition (i) is not unreasonable.

Proposition 1. Let F be a semigroup of transformations of a set A, and let G be a semigroup of transformations of a set B (both acting effectively). Let $A_0 \subset A$ be F-invariant, and suppose (G,B) and $(F|A_0, A_0)$ are equivalent by means of μ. Then

$$\varphi \to \mu \circ (\varphi | A_0) \circ \mu^{-1}$$

is a homomorphism of F onto G.

Proof.

It is obvious that $\varphi \to \varphi | A_0$ is a homomorphism of F onto $F|A_0$. On the other hand, one verifies at once that $\psi \to \mu^{-1} \circ \psi \circ \mu$ is an isomorphism of $F|A_0$ onto G.

If we write ρ for the described homomorphism, we see that

$$(\varphi)\rho \circ \mu = \mu \circ \varphi,$$

in accordance with (iii).

The definitions 1 and 2 are further illustrated by proposition 2.

<u>Proposition 2.</u> Let F be a semigroup of transformations of a set A and G a semigroup of transformations of a set B, both acting effectively. Assume moreover that ρ is a homomorphism of F onto G and that μ is a 1-1 map of B into A such that $(\varphi)\rho \circ \mu = \mu \circ \varphi$, for all $\varphi \in F$.

Then $A_o = B\mu$ is an F-invariant subset of A, and (G,B) and $(F(A_o, A_o)$ are equivalent by means of μ.

<u>Proof.</u>

The fact that A_o is F-invariant is immediate from the equality $(\varphi)\rho \circ \mu = \mu \circ \varphi$. As $\mu \circ (\varphi | A_o) \circ \mu^{-1} = (\varphi)\rho$ and as ρ is onto, the assertions follow.

There is of course a close relation between the concepts of universal morphisms and bimorphisms (as defined in [4]) and the concepts defined above. This connection is indicated by the next proposition.

<u>Proposition 3.</u> Let K be a category. A morphism φ: A→A of K is a universal morphism if and only if it generates in H(A,A) a 1-universal semigroup . It is a universal bimorphism if and only if is a bimorphism and generates in H(A,A) a 1-universal group of bimorphisms. For the dual concepts an analogous assertion holds.

<u>Proof:</u> obvious.

Definition 3. Let n be a cardinal number. We will say that a category K has property (D_n) if every family of at most n objects in K admits a direct join in K. We say that K has property (U) if K contains a universal object. We say that K has property (U_n) if K contains a universal object A such that every family of at most n copies of A has a direct join in K. The dual properties are designated by ($\bar{D}_n$), ($\bar{U}$) and ($\bar{U}_n$). Obviously (D_n) & (U) $\Rightarrow$ (U_n) and ($\bar{D}_n$) & (U) $\Rightarrow$ ($\bar{U}_n$).

Theorem 1. Let n be a transfinite cardinal. Every category K with property (U_n) contains an n-universal semigroup of morphisms and an n-universal group of bimorphisms.

Proof.

Let U be a universal object of K with the property that direct joins of n copies of U exist. Let F be a free semigroup-with-unit with n generators. By assumption there exists a $V \in \Delta(U,F)$; say

$$V = \prod_{\varphi \in F} U_\varphi \quad (\pi_\varphi).$$

Let $G \subset H(X,X)$ be any semigroup of morphisms in K, containing ε_X, that can be generated by one of its subsets of power $\leq n$. Then there exists a homomorphism σ of F onto G such that $(1)\sigma = \varepsilon_X$. As U is a universal object, there exists a monomorphism $\mu : X \to U$. Let the monomorphism τ be as in §6, prop.2; if $\varphi \in F$, let $\hat{\varphi}$ be the corresponding S-map constructed with U as base, $\hat{\varphi} : V \to V$.

By §5 prop.4, $\varphi \to \hat{\varphi}$ is an isomorphism of F onto $\hat{F}$; let ω be the inverse isomorphism, and let ρ be the homomorphism $\omega\sigma : \hat{F} \to G$. We know from §6 prop.2 that

$$\tau\varphi = (\varphi\sigma)\tau = (\hat{\varphi})(\omega\sigma)\tau = (\hat{\varphi})\rho.\tau ,$$

for all $\hat{\varphi} \in \hat{F}$.

It follows that $\hat{F}$ is an $\mathfrak{n}$-universal semigroup of morphisms in K.

To prove the existence of an $\mathfrak{n}$-universal group we proceed in the same way, starting with a free group F (and taking into account the corollary of §5 prop. 4).

It may be worthwile to formulate explicitly the dual theorem.

Theorem 2. Let $\mathfrak{n}$ be a transfinite cardinal. If the category K contains a dual-universal object A, and if there exists a free join in K of $\mathfrak{n}$ copies of A, then K contains a dual $\mathfrak{n}$-universal semigroup of morphisms and a dual $\mathfrak{n}$-universal group of bimorphisms.

8. Examples.

a) The categories $K(S,\mathfrak{m})$ and $K(S,\bar{\mathfrak{m}})$ contain universal and dual-universal objects. They possess property $(D_{\mathfrak{n}})$ (or$(U_{\mathfrak{n}})$) if and only if $\mathfrak{m}^{\mathfrak{n}} = \mathfrak{n}$; the dual property $(\bar{D}_{\mathfrak{n}})$ is satisfied iff $\mathfrak{m}\cdot\mathfrak{n} = \mathfrak{m}$.

Hence we have from §7 theorems 1 and 2 (cf. also §4 theorem 1' and 2'):

Theorem 1. $K(S,\mathfrak{m})$ and $K(S,\bar{\mathfrak{m}})$ contain $\mathfrak{n}$-universal semigroups and groups for all $\mathfrak{n}$ such that $\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$, and dual $\mathfrak{n}$-universal semigroups and groups for all $\mathfrak{n}$ such that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$.

Corollary (cf. [4]). $K(S,\mathfrak{m})$ and $K(S,\bar{\mathfrak{m}})$ contain dual-universal morphisms and bimorphisms for all transfinite $\mathfrak{m}$, and they contain universal morphisms and bimorphisms for all $\mathfrak{m}$ such that $\mathfrak{m}\cdot\mathfrak{n} = \mathfrak{m}$.

As was already remarked in §4 (and proved in [4]) the last assumption can in reality be replaced by "$\mathfrak{m}$ transfinite".

b) Let $K(G, \mathfrak{m})$ be the category the objects of which are all groups that can be generated by at most $\mathfrak{m}$ elements, and whose morphisms are all homomorphisms of one such a group into another.

Clearly $K(G, \mathfrak{m})$ has property $(\overline{U})$: the free group with $\mathfrak{m}$ generators is a dual-universal object. Free joins are equivalent to free products; hence $(\overline{D}_{\mathfrak{n}})$ is satisfied if $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$.

Thus we have:

<u>Theorem 2.</u> If $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$, then $K(G, \mathfrak{m})$ contains dual $\mathfrak{n}$-universal semigroups of morphisms and dual $\mathfrak{n}$-universal groups of bimorphisms.

<u>Corollary.</u> If $\mathfrak{m}$ is transfinite, then $K(G, \mathfrak{m})$ contains dual-universal morphisms and bimorphisms.

If $F_{\mathfrak{m}}$ is a free group with $\mathfrak{m}$ generators, and if $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$, then the free product of $\mathfrak{n}$ copies of $F_{\mathfrak{m}}$ is isomorphic to $F_{\mathfrak{m}}$. Hence it follows from the proofs in §7 that $F_{\mathfrak{m}}$ itself can be taken as the object such that the endomorphism semigroup $E(F_{\mathfrak{m}}) = H(F_{\mathfrak{m}}, F_{\mathfrak{m}})$ contains as a subsemigroup a dual $\mathfrak{n}$-universal semigroup of endomorphism, and a dual $\mathfrak{n}$-universal group of automorphisms.

In particular, there are an endomorphism and an automorphism of $F_{\mathfrak{m}}$ that are dual-universal, as can also be seen in a direct way quite easily.

c). Let $K(AG, \mathfrak{m})$ be the full subcategory of $K(G, \mathfrak{m})$ obtained by admitting only <u>abelian</u> groups as objects. Then $(\overline{U})$ is satisfied, and $(\overline{D}_{\mathfrak{n}})$ again holds if $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$. Hence:

<u>Theorem 3.</u> If $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m}$ then $K(AG, \mathfrak{m})$ contains dual $\mathfrak{n}$-universal semigroups of morphisms and groups of bimorphisms.

Corollary.. If $m \geq \aleph_0$, $K(AG, m)$ contains dual-universal morphisms and bimorphisms.

If A_m is a free abelian group with m generators, then A_m is a dual-universal object. Free joins are equivalent to restricted direct products in this category; hence the dual-universal subgroup can be taken as a subsemigroup of the endomorphism semigroup of A_m and the dual-universal group can be taken as a subgroup of the automorphism group of A_m.

Theorems 2 and 3 give sufficient conditions for the possibility of raising simultaneously sets of n endomorphisms (automorphisms) of a group of at most m generators to endomorphisms (automorphisms) of F_m or A_m.

d) Let K(CMAG) be the category of all continuous homomorphisms between compact metrizable abelian groups.

This category has property (U): the infinite-dimensional torus $T_{\aleph_0}$ - product of $\aleph_0$ copies of the circle group T - is a universal object (see e.g.[17] theorem 2.2.6). Furthermore $(D_{\aleph_0})$ clearly holds (and D_n does not hold for $n > \aleph_0$, as metrizability gets lost). Hence §7 theorem 1 implies:

Theorem 4. K(CMAG) contains an $\aleph_0$-universal semigroup of morphisms and an $\aleph_0$-universal group of bimorphisms. These can be taken as a subsemigroup and a subgroup, respectively, of $H(T_{\aleph_0}, T_{\aleph_0})$.

Corollary. K(CMAG) contains universal morphisms and bimorphisms. More explicitly: there exist a topological automorphism Φ of $T_{\aleph_0}$, and a continuous endomorphism Ψ of $T_{\aleph_0}$, with the following properties: if G any compact metrizable abelian group, and if φ is any topological automorphism of G (ψ any continuous endomorphism of G) there exists a topological isomorphism μ of G into $T_{\aleph_0}$ such that $\mu\Phi = \varphi\mu$

($\mu\Psi = \psi\mu$, respectively).

$$\begin{array}{ccc} T_{\aleph_0} & \xrightarrow{\Phi} & T_{\aleph_0} \\ \mu\uparrow & & \uparrow\mu \\ G & \xrightarrow{\varphi} & G \end{array}$$

Remarks. The fact that μ is topological is a consequence of the facts that μ is continuous and 1-1 and that G is compact. Of necessity $G\mu$ is a closed subgroup of $T_{\aleph_0}$ (being compact).

e) Let K(OCM) be the category of all continuous maps between zero-dimensional compact metrizable spaces. Then (U) holds: the Cantor discontinuum $C = \{0,1\}^{\aleph_0}$ is a universal object. Also $(D_{\aleph_0})$ is satisfied. Hence:

Theorem 5. K(OCM) contains an $\aleph_0$-universal semigroup of morphisms and an $\aleph_0$-universal group of bimorphisms. These can be taken as a subsemigroup and a subgroup, respectively, of H(C,C).

Corollary. (J. de Groot and P.C. Baayen; cf. [2]). There exist an autohomeomorphism Φ of C, and a continuous map $\Psi : C \to C$, with the following properties: if φ is any autohomeomorphism of a zero-dimensional compact metrizable space M (ψ any continuous map of such a space into itself) there exists a topological embedding $\mu : M \to C$ such that $\mu\Phi = \varphi\mu$ ($\mu\Phi = \Psi\mu$, respectively).

$$\begin{array}{ccc} C & \xrightarrow{\Phi} & C \\ \mu\uparrow & & \uparrow\mu \\ M & \xrightarrow{\varphi} & M \end{array}$$

Remark. It is an open problem whether K(OCM) also contains dual-universal morphisms or bimorphisms. If they exist, it will be impossible to detect their presence by the methods of §7, as $(\overline{U}_{\aleph_0})$ is definitely not valid in K(OCM). However, using the method of S-maps one can in any case show the following:

Theorem 6. (R.D. Anderson [1]). Every autohomeomorphism φ of a zero-dimensional compact metrizable space M (every continuous map ψ of such a space into itself) can be raised to the cantor set. I.e. given φ (ψ), there exist a homeomorphism Φ of C (a continuous map $\Psi : C \to C$) and a continuous map τ of C onto M such that $\tau\varphi = \Phi\tau$ ($\tau\psi = \Psi\tau$).

$$\begin{array}{ccc} C & \xrightarrow{\ \Phi\ } & C \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ M & \xrightarrow{\ \varphi\ } & M \end{array}$$

Proof:

We restrict ourself to the case of an autohomeomorphism $\varphi : M \to M$, the case of a continuous map being entirely similar.

Let F be the subgroup of H(M,M) generated by φ, and let σ be the identity map $F \to H(M,M)$. Then $\check{\sigma} : M \to M^F$ is topological; let $M' = M\check{\sigma}$, $\varphi' = \hat{\varphi} | M'$, and let $\check{\sigma}^{-1}$ be the inverse of the homeomorphism $\check{\sigma} : M \to M'$.

The topological product $C' = M' \times C$ is homeomorphic to C; Let π be the projection of $M' \times C$ onto M', and define the autohomeomorphism $\Phi : C' \to C'$ through

$$(\mu, \gamma)\Phi = (\mu\varphi', \gamma)$$

($\mu \in M'$, $\gamma \in C$). Then $\tau = \pi \circ \check{\sigma}^{-1}$ maps C' continuously onto M, and $\Phi\tau = \tau\varphi$.

$$\begin{array}{ccc} C' & \xrightarrow{\Phi} & C' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M' & \xrightarrow{\varphi'} & M' \\ \downarrow{\scriptstyle\check{\sigma}^{-1}} & & \downarrow{\scriptstyle\check{\sigma}^{-1}} \\ M & \xrightarrow{\varphi} & M \end{array}$$

f) The category $K(LO, \mathfrak{m})$ of all order-preserving mappings between linearly ordered spaces of power $\mathfrak{m}$ is an example of a category for which universality results can be obtained, but not by means of the constructions described in this note: these constructions are fundamentally useless for this category.

The fundamental difficulty does not so much concern the validity of (U) and ($\overline{U}$), although here there are problems already. If $\mathfrak{m}$ is finite, (U) and ($\overline{U}$) are evidently satisfied. If $\mathfrak{m} = \aleph_0$, (U) holds too: it is well-known that the set Q of all rational numbers is a universal object (cf. [18]). The same set is a dual-universal object for $\mathfrak{m} = \aleph_0$: if A is any denumerable linearly ordered set, then the cartesian product A x Q , ordered lexicographically, is order-isomorphic to Q (as $\eta\nu = \eta$ for every denumerable order-type ν; cf. [17] p.231). As the map $(\alpha, q) \to \alpha$ is an epimorphism $A \times Q \to A$, there also exists an epimorphism $Q \to A$.

For $\mathfrak{m} > \aleph_0$, the question of existence of a universal object in $K(LO, \mathfrak{m})$ is a vexing open problem of set theory (cf. [18], [14]). The problem of the existence of dual-universal objects is open too for these $\mathfrak{m}$, at least for the author of this note.

More serious trouble arises, however, if one proceeds to (D_n) or (U_n) and their dual properties. In fact, none of these can be valid-even if one takes (U) for granted- as soon as $\mathfrak{m} > 1$, as is seen from the next two propositions.

Proposition 1. In $K(LO, m)$, $m > 1$, no free join of two or more objects exists.

Proof.

Let card $(A) \geq 2$, let X_α be an object of $K(LO, m)$ for each $\alpha \in A$, and suppose

$$X = \sum_{\alpha \in A} X_\alpha \ (\sigma_\alpha).$$

Take $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \neq \alpha_2$. Let Y be any object of $K(LO, m)$; as $m > 1$, there are $\eta_1, \eta_2 \in Y$ with $\eta_1 < \eta_2$. We define $\varphi_\alpha : X_\alpha \to Y$ as follows:

$\xi\varphi_\alpha = \eta_1$, for all $\xi \in X_\alpha$, if $\alpha \neq \alpha_2$; $\xi\varphi_{\alpha_2} = \eta_2$ for all $\xi \in X_{\alpha_2}$.

As all φ_α are morphisms of $K(LO, m)$ there exists a morphism $\varphi : X \to Y$ in $K(LO, m)$ such that $\sigma_\alpha \varphi = \varphi_\alpha$, for all $\alpha \in A$.

It follows that $\xi_1 \sigma_{\alpha_1} < \xi_2 \sigma_{\alpha_2}$ in X, for all $\xi_1 \in X_{\alpha_1}$ and ξ_2 and $\xi_2 \in X_{\alpha_2}$. But in the same way one can show that $\xi_1\sigma_{\alpha_1} > \xi_2\sigma_{\alpha_2}$, for all $\xi_1 \in X_{\alpha_1}$ and $\xi_2 \in X_{\alpha_2}$, which is contradictory.

Proposition 2. Let $m > 1$. For any object X of $K(LO, m)$ and any set A with card $(A) > 1$, $\sum(X, A)$ admits no direct join.

Proof.

Suppose $Y = \prod_{\alpha \in A} X_\alpha \ (\pi_\alpha)$, where $X_\alpha = X$ for all $\alpha \in A$.

Let $\alpha_1, \alpha_2 \in A$, $\alpha_1 \neq \alpha_2$.

First we remark that there exists a $y \in Y$ such that $y\pi_{\alpha_1} < y\pi_{\alpha_2}$. For take $\xi_1, \xi_2 \in X$, $\xi_1 < \xi_2$, and define $\varphi_\alpha : Y \to X_\alpha$ as follows: $\varphi_\alpha = \pi_\alpha$ if $\alpha \neq \alpha_1, \alpha_2$; $y\varphi_{\alpha_1} = \xi_1$ for all $y \in Y$; $y\varphi_{\alpha_2} = \xi_2$, for all $y \in Y$. There exists a morphism $\varphi : Y \to Y$ such that $\varphi\pi_\alpha = \varphi_\alpha$, for all $\alpha \in A$. Then $(y\varphi)\pi_{\alpha_1} = \xi_1 < \xi_2 = (y\varphi)\pi_{\alpha_2}$, for all $y \in Y$.

Let $\psi: A \to A$ be the map that exchanges α_1 and α_2 and leaves all other $\alpha \in A$ fixed. We saw in §5 that $K(LO, m)$ contains a morphism $\psi^*: Y \to Y$ such that $\psi^*\pi_\alpha = \pi_{\alpha\psi}$ for all $\alpha \in A$. Take any $y \in Y$ such that $y\pi_{\alpha_1} < y\pi_{\alpha_2}$; then

$$y\psi^*\pi_{\alpha_2} = y\pi_{\alpha_1} < y\pi_{\alpha_2} ,$$

hence, as π_{α_2} is order-preserving, $y\psi^* < y$. Similarly

$$y\psi^*\pi_{\alpha_1} = y\pi_{\alpha_2} > y\pi_{\alpha_1} ,$$

implying $y\psi^* > y$, which is contradictory.

Nevertheless it is possible to obtain results about universality properties of the categories $K(LO, m)$. For instance, in [3] it is shown that $K(LO, m), m \geq \aleph_0$, contains universal morphisms and bimorphisms if and only if it contains universal objects. (For $1 < m < \aleph_0$ there are evidently no universal morphisms or bimorphisms).

9. Categories of topological spaces.

Consider the category $K(T, m)$ of all continuous maps between completely regular spaces of weight m . If $m \geq \aleph_0$, (U) holds for $K(T, m)$, as the Tychonoff cube $I_m = [0,1]^m$ is a universal object. And (D_n) holds for $K(T, m)$ as soon as $m.n = m$ (still supposing m to be transfinite). Hence we have

Theorem 1. If $m \geq \aleph_0$ and $m.n = m$, the category $K(T, m)$ contains n -universal semigroups of morphisms and n -universal groups of bimorphisms.

Corollary. If $m \geq \aleph_0$, then $K(T, m)$ contains universal morphisms and bimorphisms.

However, in this case these results are not quite satisfactory, as our methods in fact give more. The trouble is that in most categories of topological spaces monomorphisms and epimorphisms are not what one wants them to be: topological maps into and continuous maps onto, respectively. This follows from the next proposition, which is easily shown.

Proposition 1. A morphism μ: $X \to Y$ of $K(T, \mathfrak{m})$ is a monomorphism if it is 1-1, and it is an epimorphism if $X\mu$ is dense in Y.

In the case of the categories K(CMAG) and K(OCM) of the previous section there were, afterall, no difficulties: as all objects of these categories are compact, their mono- and epimorphisms are what they ought to be.

In the case of $K(T, \mathfrak{m})$ and other categories of topological spaces, results like theorem 1 can be strengthened even if not every monomorphism and epimorphism in the category is nice, because of the following results.

Proposition 2. Let X be a topological space. If the map $\varphi : A \to B$ is of finite multiplicity, the S-map $\varphi^* : X^B \to X^A$ is open.

Proof: evident.

Remark. We say that $\varphi : A \to B$ is of finite multiplicity if $(\beta)\varphi^{-1}$ is finite for every $\beta \in B$. A map $\tau : X \to Y$ is called open if τ is open as a map $X \to X\tau$.

Proposition 3. Let X be a topological space. For every homomorphism σ of a semigroup F into the semigroup of all continuous maps $X \to X$, the embedding map $\check{\sigma} : X \to X^F$ is topological.

Proof.

We know already that $\check{\sigma}$ is 1-1. As $\check{\sigma}\pi_\varphi = (\varphi)\sigma$ is continuous, for every $\varphi \in F$, $\check{\sigma}$ is continuous. Finally $\check{\sigma}$, considered as a map $X \to X\check{\sigma}$, has a continuous inverse:

$$\check{\sigma}^{-1} = \pi_1 | X\check{\sigma}_2 \quad (\text{as } \check{\sigma}\pi_1 = (1)\sigma = i_X).$$

Thus we obtain from theorem 1, or rather from the construction behind it, the following results of J. de Groot [7]:

Theorem 2^a. Let m, n be cardinal numbers such that $m \cdot n = m \geq \aleph_0$. Then there exists a semigroup F of mappings of the Tychonoff cube I_m into itself with the following properties:

(i) the abstract semigroup F is a free semigroup-with-unit with n generators;

(ii) every $\varphi \in F$ is an S-map $I_m \to I_m$, hence is continuous;

(iii) if X is any completely regular space of weight $\leq m$, and if G is any semigroup of continuous maps $X \to X$ such that the abstract semigroup G can be generated by at most n of its elements, then there exists a topological map $\tau : X \to I_m$ such that $X\tau$ is invariant under F and such that (X,G) and $(X\tau, F|X\tau)$ are equivalent.

Theorem 2^b. If $m \cdot n = m \geq \aleph_0$, there exists a group F of mappings $I_m \to I_m$ with the following properties:

(i) F is a free group with n generators;

(ii) every $\varphi \in F$ is a topological S-map of F onto itself;

(iii) if X is any completely regular space of weight $\leq m$, and if G is any group of autohomeomorphisms of X that can be generated by at most n of its elements, then there exists a topological map $\tau: X \to I_m$ such that $X\tau$ is F-invariant, while (X,G) and $(X\tau, F|X\tau)$ are equivalent.

(As $m \cdot n = m$, I_m^n is homeomorphic to I_m).

Furthermore, if K(OSM)is the category of all continuous maps of one separable metrizable space into another, then §8 theorem 5 can be strengthened to

<u>Theorem 3.</u> K(OSM) contains $\aleph_0$ -universal semigroups of morphisms and $\aleph_0$-universal groups of bimorphism. These can be taken as sub(semi) groups of H(C,C).

Leaving altogether the idée-fixe of categories, propositions 2,3 together with §§2-4 lead to the following results.

<u>Proposition 4^a.</u> Let X be a topological space, A a non-void set, card (A) = $n \geq \aleph_0$. Then $H(X^A, X^A)$ contains a subsemigroup F with the following properties:

(i) the abstract semigroup F is a free semigroup-with-unit with n generators;

(ii) every $\varphi \in F$ is an S-map $X^A \to X^A$, hence is continuous

(iii) for any semigroup G of continuous transformations $X \to Y$ with card $(G) \leq n$ there exists a topological map $\mu : X \to X^A$ such that $X\mu$ is F-invariant while (X,G) and $X\mu$, $F|X\mu$) are equivalent (by means of μ).

<u>Proposition 9^b:</u> obtained from prop. 9^a by substituting "group" for every occurrence of "semigroup" and "topological" for every occurrence of "continuous".

<u>Corollary.</u> If $X^{\aleph_0}$ is homeomorphic to X (which is the case if and only if $X = Y^{\aleph_0}$, for some Y), then there exist an autohomeomorphism Φ of X and a continuous map $\Psi : X \to X$ with the following property: if φ is any other autohomeomorphism of X (if ψ is any other continuous map $X \to X$) there exists a topological map μ: $X \to X$ such that $\mu\Phi = \varphi\mu$($\mu\Psi = \psi\mu$, respectively).

In other words, Φ contains already all other autohomeomorphisms of X, and Ψ contains all continuous maps $X \to X$.

Examples of spaces X for which this result holds are the Cantor set C (for which we established it already in the previous section) , the generalized Cantor discontinua $\{0,1\}^{m}$, $m > \aleph_0$, the Hilbert fundamental cube, all Tychonoff cubes I_m, $m \geq \aleph_0$, the infinite-dimensional torus $T_{\aleph_0}$, etc.

10. Application to compactification.

All results in this section are taken from J. de Groot and R.H. Mc Dowell [11].

Definition 1. ([11] p.251). Let X be a completely regular space, and let G be a set of continuous maps $\gamma : X \to X$. A space $\overline{X}$ is called a G-compactification of X if $\overline{X}$ is a compact space containing X as a dense subset, and if every $\gamma \in G$ can be extended continuously to a $\overline{\gamma} : \overline{X} \to \overline{X}$.

Theorem 1. (J. de Groot and R.H. Mc Dowell [11]). Let X be a completely regular space, and let G be a set of continuous maps $\gamma : X \to X$. Then X admits a G-compactification $\overline{X}$; the weight of $\overline{X}$ can be taken $\leq$ weight (X). card (G). $\aleph_0$.

Proof.

Without loss of generality we may assume G to be a semigroup containing i_X, and we may assume weight (X)= $m \geq \aleph_0$. Then (X,G) is equivalent to (X', $F|X'$), where F is some universal semigroup of continuous mappings $I_m \to I_m$. For $\overline{X}$ we take (a space homeomorphic to) the closure of X' in the compact space I_m.

Corollary 1. If X is a separable metrizable space and G is countable, X has a separable metrizable G-compactification.

If X is a zero-dimensional separable metrizable space, one can embed X topologically in C instead of in $I_{\aleph_0}$, and take for F a universal semigroup in K(OSM). Hence

Corollary 2. If X is zero-dimensional separable metrizable and G is countable, X has a zero-dimensional separable metrizable G-compactification.

If every $\gamma \in G$ is a homeomorphism, then in the proof of theorem 1 we may suppose, without loss of generality, that G is a group, and we may take for F a universal group of autohomeomorphisms. Hence

Corollary 3. If every $\gamma \in G$ is an autohomeomorphism of X, the G-extension X of theorem 1, corollary 1 or corollary 2 can be taken in such a way that all extensions to $\overline{X}$ of $\gamma \in G$ are autohomeomorphisms of $\overline{X}$.

11. Application to linearization.

Definition 1. Let E be a topological vector space, and let S be a set of continuous linear operators $E \to E$. If T is a set of continuous maps of a topological space X into itself we say that (X,T) can be linearized by (E,S) if there exists a topological embedding $\tau : X \to E$ such that $X\tau$ is S-invariant while (X,T) and $(X\tau, S|X\tau)$ are equivalent by means of τ. We also say in this case that T can be linearized in E.

The unit interval I being a subspace of the topological vector space of all real numbers R, we can identify I_{m} with a subset of the locally convex linear space R^{m}. From § 2 props. 4 and 5, § 5 prop. 1 and § 9 theorem 2 we derive:

Theorem 1. Let $m \cdot n = m \geq \aleph_0$. There exists a semigroup F of continuous linear operators of the locally convex space R^{m} into itself with the following properties:

(i) F is a free semigroup-with- unit with n generators;

(ii) (R^{m}, F) linearizes all semigroups (X,G) where X is a completely regular space of weight $\leq m$ and where G is a semigroup of continuous maps $X \to X$ that can be generated by at most n of its elements.

Similarly there exists a group F of invertible continuous linear operators $R^m \to R^m$ which is a free group with n generators and which linearizes all (X,G) with weight (X) $\leq m$ etc. such that all $\gamma \in G$ are autohomeomorphisms of X.

The restriction $m \geq \aleph_0$ is not essential, of course. For instance, the following is true.

Proposition 1. (A.H. Copeland Jr. and J. de Groot [6]). Let M be a separable metric space of finite dimension m, and let G be a semigroup of continuous maps $M \to M$ of finite order n. Then G can be linearized in the finite-dimensional euclidean space R^k, with $k \leq n(2m + 1)$.

Proof.

Embed M topologically into R^{2m+1}. The semigroup $\hat{G}$ of all S-maps $\hat{\gamma}: (R^{2m+1})^G \to (R^{2m+1})^G$, $\gamma \in G$, linearizes G.

Remark 1. The number n(2m+1) is far too high; cf. [5] and [12].

Remark 2. S-maps in a euclidean space are particularly nice; they are e.g. orthogonal maps.

For arbitrary separable metric spaces theorem 1 is not the correct generalization of prop. 1, as it gives a linearization in the non-metrizable locally convex space $R^{\aleph_0}$. As any separable metric space can be embedded in separable Hilbert space, one would expect that linearization of countable sets of continuous maps would be possible by bounded linear operators in Hilbert space. In fact, this is indeed possible:

Theorem 2. (J. de Groot). There exists a semigroup F of bounded linear operators in the separable Hilbert space H with the following properties:

(i) F is a free semigroup-with-unit with denumerably many generators;

(ii) every denumerable semigroup G of continuous maps of a separable metric space M into itself can be linearized by (H,F).

Similarly there exists a group F of invertible bounded linear operators in H that linearizes every denumerable group G of autohomeomorphisms of any separable metric space M.

Proof.

First we remark the following. Let L be the Tychonoff cube $I_{\aleph_0}$ and let F be either a free semigroup-with-unit or a free group, with denumerably many generators. Then it suffices to show that the (semi) group $\hat{F}$ of S-maps $L^F \longrightarrow L^F$ can be linearized in H, as this (semi) group is universal for all countable (semigroups of continuous maps of a separable metric space into itself. (In the case of a free group F, we must also take care to linearize by invertible operators).

For each $\varphi \in F$, let H_φ be the separable Hilbert space l_2; let K_1 be the fundamental cube in l_2, and let $K_\varphi = c_\varphi K_1$, where the c_φ are real non-zero constants, to be fixed later, such that $c_1=1$, $\sum_{\varphi \in F} c_\varphi^2 < \infty$.

As F is denumerable, the Hilbert sum H of all H_φ is again separable, and its subset K, consisting of all $x = (x_\varphi)_{\varphi \in F}$ such that $x_\varphi \in K_\varphi$, is precisely the topological product of all K_φ.

Let ρ be any homeomorphism of L onto K_1, and define $\tau : L^F \to K$ as follows: if $y = (y_\varphi)_{\varphi \in F} \in L^F$, we put

$$(1) \qquad y\tau = (c_\varphi \cdot y_\varphi \rho)_{\varphi \in F}.$$

Then τ clearly is a homeomorphism of L^F onto K.

If $\varphi \in F$, $\tau^{-1}\hat{\varphi}\tau$ is a continuous map $K \to K$ (a topological map of K onto itself if φ is invertible in F). As K spans H, there is at most one linear operator $\bar{\varphi}: K \to K$ with $\bar{\varphi}|K = \tau^{-1}\hat{\varphi}\tau$. And this linear operator $\hat{\varphi}$, if it indeed exists, must evidently have the form

$$(2) \qquad (x_\psi)_{\psi \in F}\,\varphi = (d_\psi\, x_{\varphi\psi})_{\psi \in F},$$

with $d_\psi = \frac{c_\psi}{c_{\varphi\psi}}$. Hence the theorem will be proved if we succeed in choosing the constants c_ψ in such a way that all linear operators (2) exist and are bounded in H. The possibility of doing this is an immediate consequence of the lemma below.

<u>Lemma 1.</u> (<u>J. de Groot</u>) Let F be a free group or a free semi-group-with-unit with at most denumerably many generators. There exist non-zero constants $c_\varphi, \varphi \in F$, such that

(i) $c_1 = 1, \quad \sum_{\varphi \in F} c_\varphi^2 < \infty$;

(ii) $c_\varphi\, c_\psi \leq c_{\varphi\psi}$, for all $\varphi, \psi \in F$.

<u>Proof.</u>

Let $\alpha_1, \alpha_2, \ldots$ be the free generators of F. We treat the case of a free group F ; the case of a free semigroup is processed along exactly the same lines.

Every $\varphi \in F$, $\varphi \neq 1$, can be uniquely written as a reduced word

$$(3) \qquad \varphi = \alpha_{\nu_1}^{\kappa_1} \ldots \alpha_{\nu_s}^{\kappa_s} ;$$

we take

$$c_\varphi = 2^{-\sum_{\sigma=1}^{s} |\kappa_\sigma|\, \nu_\sigma}$$

and put

$$c_1 = 1.$$

Then the c_φ are non-zero real numbers. If one defines the length of the element φ given by (3) as $\sum_{\sigma=1}^{s} |\kappa_\sigma|$, then one easily shows inductively that

$\sum \{ c_\varphi^2 : \varphi \in F$ and length $(\varphi) = n \} \leq (\frac{2}{3})^n$; hence $\sum_{\varphi \in F} c_\varphi^2 < \infty$.

To conclude the proof, we show that (ii) holds for this system $(c_\varphi)_{\varphi \in F}$.

First suppose φ arbitrary and ψ of length 1. Then ψ is of the form α_n^ε, $\varepsilon = \pm 1$. If $\varphi = 1$, then $c_\varphi c_\psi = c_\psi = 2^{-n} = c_{\varphi\psi}$. If φ is given by (3), then

$$c_{\varphi\psi} = \begin{cases} 2^{-n} c_\varphi & \text{if } \nu_s \neq n \text{ or } \nu_s = n \text{ and } \varepsilon.\kappa_s > 0; \\ 2^n c_\varphi & \text{if } \nu_s = n \text{ and } \varepsilon\kappa_s < 0. \end{cases}$$

Hence for all $\varphi \in F$ we have: $c_\varphi c_\psi = 2^{-n} c_\varphi \leq c_{\varphi\psi}$. For <u>arbitrary</u> $\psi \in F$ (ii) now follows readily (using induction on the length of ψ).

<u>Remark 1.</u> In the proof of theorem 2 we have: $d_\psi = \frac{c_\psi}{c_{\varphi\psi}} \leq c_\varphi^{-1}$. Hence:

$$\| \tilde{\varphi} \| \leq c_\varphi^{-1} .$$

<u>Remark 2.</u> In the particular case of a free group with one generator we can represent F as the additive group of all integers. Then $c_\nu = 2^{-|\nu|}$, for $\nu = 0, \pm 1, \pm 2, \dots$; H is represented as the Hilbert sum of separable Hilbert spaces H_ν, $\nu = 0, \pm 1, \pm 2, \dots$, and the generator of F is linearized by the invertible continuous operator $\Lambda : H \rightarrow H$ such that, for $x = (x_\nu) \in H$, $x\Lambda = y = (y_\nu)$, with

$$y_\nu = 2^{\operatorname{sign}(\nu)} x_{\nu+1}.$$

This is exactly the operator of which <u>A.H. Copeland Jr.</u> and <u>J. de Groot</u> proved in [6] that it is universal for all autohomeomorphisms of separable metrizable spaces.

The assumption of separability is not essential for theorem 2: by means of lemma 1 we easily prove:

<u>Theorem 3.</u> Let H be a Hilbert space of infinite (Schauder) dimension $\mathfrak{m}$. There exists a semigroup F of bounded linear operators in H (a group F of invertible bounded linear operators in H) with the following properties:

(i) the abstract semigroup (group) F is free, with $\aleph_0$ generators;

(ii) (H,F) linearizes all pairs (M,G) where M is a metrizable space of weight $\leq \mathfrak{m}$ and G is a countable semigroup of continuous maps $M \to M$ (a countable group of autohomeomorphisms of M, respectively).

<u>Proof.</u>

Let F be the free semigroup (group) with $\aleph_0$ generators; choose a system of real non-zero constants $(c_\varphi)_{\varphi \in F}$ satisfying the requirements of lemma 1. For each $\varphi \in F$, let H_φ be a copy of H; in the Hilbert sum of all H_φ, we define, for each $\psi \in F$, a linear operator $\bar{\psi}$ by

$$(x_\varphi)_{\varphi \in F}\bar{\psi} = (d_\varphi \, x_{\psi\varphi})_{\varphi \in F} ,$$

with $d_\varphi = \frac{c_\varphi}{c_{\psi\varphi}}$.
Then each $\bar{\psi}$ is bounded (in fact $\|\bar{\psi}\| \leq c_\psi^{1}$), and $\bar{\psi}$ is invertible iff ψ is invertible in F.

Now let M be any metrizable space of weight $\leq \mathfrak{m}$, and let G be any countable semigroup (group) of continuous maps $M \to M$. Then M can be topologically embedded in H (cf. [19]). It then follows that M can also be embedded topologically in the unit sphere of H (take $M \subset H \subset H \oplus R$; project M into the upper hemisphere of the unit sphere of $H \oplus R$ with, say, $1 \oplus 1$ as center of projection).

Let ρ be a homeomorphism of M into the unit sphere S of H, and let σ be a homomorphism of F onto G. We define $\tau : M \to \bigoplus_{\varphi \in F} H_\varphi$ in the following manner: if $\mu \in M$, then

$$\mu\tau = (c_\varphi \cdot (\mu)(\varphi\sigma)\rho)_{\varphi \in F} .$$

Indeed τ maps into $\bigoplus_{\varphi\in F} H_\varphi$, as $\sum_{\varphi\in F} c_\varphi^2<\infty$ and as $\|(\mu)(\varphi\sigma)\rho\| \leq 1$, for all $\mu\in M$ and $\varphi\in F$. Clearly τ is continuous; as $1\sigma = i_M$, τ is 1-1 and has a continuous inverse. Hence τ is a topological embedding of M into $\bigoplus_{\varphi\in F} H_\varphi$.

We finish the proof by showing that $\tau\bar{\varphi} = (\varphi\sigma)\tau$, for all $\varphi\in F$. Indeed, if $\mu\in M$ then

$$\mu\tau\bar{\varphi} = (c_\psi\cdot(\mu)\ (\psi\sigma)\rho)_{\psi\in F}\bar{\varphi} = (c_\psi\cdot(\mu)\ ((\varphi\psi)\sigma)\rho)_{\psi\in F} =$$

$$= (c_\psi\cdot((\mu)\ (\varphi\sigma))\ (\psi\sigma)\rho)_{\psi\in F} = \mu(\varphi\sigma)\tau\ .$$

Though theorem 3 is an improvement on theorem 1 for the case of <u>metrizable</u> spaces of weight $m \geq \aleph_0$ - as it asserts the existence of linearization in <u>Hilbert</u> spaces - it is evidently deficient in one respect: it only guarantees linearization of <u>countable</u> systems.

This defect can be done away with if one restricts oneself to <u>compact</u> groups of autohomeomorphisms of metrizable spaces [9] . An exposition of proofs of this fact and related ones will be given in a forthcoming note of J. de Groot and P.C. Baayen [10]

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