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Linearization in Hilbert space

Preliminary note

by

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equivalent either to the group of all rotations around an axis or to the group of all proper orthogonal transformations.

In these theorems groups of transformations of E^3 are linearized in the same space. In general this will not be possible, as can be seen from examples of R.H. Bing [3] and of D. Montgomery - L. Zippin [12] (cf. also [5]). R.H. Bing constructed a "wild reflection", an involutory homeomorphism of S^3 with a wildly imbedded plane as set of fixed points. D. Montgomery and L. Zippin modified Bing's example and obtained a "wild rotation", a sense-preserving involution of E^3 having a wildly imbedded topological line as fixed point set. Such homeomorphisms clearly can only be linearized - if they admit linearization at all - in a higher-dimensional E^n.

They can indeed be linearized. In 1957 G.D. Mostow showed:

Theorem [14]. Let G be a compact Lie group operating faithfully on a separable finite-dimensional metrizable space M. Assume G has only a finite number of inequivalent orbits in M. Then G can be linearized by unitary transformations of a euclidean space E^n.

Theorem [15]. Let G be a compact Lie group of homeomorphisms of a compact manifold M. Then G can be linearized by orthogonal transformations of a euclidean space E^n.

In the case of homeomorphisms of finite prime order of a finite-dimensional separable metrizable space, the minimal dimension of a euclidean space in which linearization is always possible has been determined by A.H. Copeland Jr. and J. de Groot [5] . Their results were extended to the case of compact abelian Lie groups with a finite number of distinct isotropy subgroups by J.M. Kister and L.N. Mann [11] .

All these theorems concern linearization in finite-dimensional euclidean spaces. J. de Groot [8] and A.H. Copeland Jr. and J. de Groot [4] , [5] set off in another direction: they studied the possibility of linearization by bounded linear operators in Hilbert space. In [5] they proved:

Theorem. There exists a bounded linear automorphism Φ of separable Hilbert space H with the following property: if M is any separable metrizable space, and if φ is any autohomeomorphism of M, then (M,φ) admits a linearization by (M, Φ).

This result was extended by J. de Groot in the following way (cf. [2]).

Theorem. There exists a free group Γ of bounded linear operators in a separable Hilbert space H with the following property: if M is any separable metrizable space, and if G is any denumerable group of autohomeomorphisms of M, then (M,G) can be linearized by (H,Γ).

In these theorems, the separability assumption can be removed. Moreover, similar results have been obtained for countable semi-groups of continuous maps; see [2] section 11.

Theorem 1 is best compared with the last two theorems mentioned. There are some important differences.

In the first place the transformation groups in the last two theorems need not be compact; consider e.g. the group generated by one autohomeomorphism of infinite order. They are, of course, locally compact (considered as discrete groups). It would be nice if a theorem could be obtained to the extend that all locally compact groups of autohomeomorphisms of some metrizable space are linearizable, or in any case all separable locally compact groups. (For this class of groups a result somewhat comparable to the theorem of Eilenberg mentioned above, though weaker, is contained in J. de Groot [7]:

Theorem. Let G be a locally compact σ-compact transformation group acting on a metrizable space M. Then M can be metrized in such a way that G acts uniformly on M, i.e. every $\gamma \in G$ is uniformly continuous on M.)

We do not have such a theorem, as yet; but in section 3 of this note a sufficient condition is given for a locally compact transformation group in order that it can be linearized, and this condition is met both in the case of a countable discrete group and in the case of compact groups; moreover, it is satisfied also by all Abelian separable locally compact groups.

Secondly it should be remarked that in the case of countable groups one obtains universal linearization (cf. § 4). This universality was also obtained in another case, by means of the same construction (but independently), by G.-C. Rota [16] , [17] . Where A.H. Copeland Jr. and J. de Groot linearized arbitrary autohomeomorphisms φ of metrizable spaces by a universal bounded operator Φ in Hilbert space H, through a suitable, presumably very "crooked" imbedding of M in H, G.-C. Rota started with bounded operators in a Hilbert space and showed that each such an operator with spectral radius < 1 is equivalent to a restriction to a suitable closed linear subspace of a Hilbert space H of a universal bounded operator in H, by means of a linear imbedding. We will say more about this in section 6.

For the sake of completeness we mention the fact that one can also consider linearization in topological linear spaces that are even more general then Hilbert space. In fact, if one admits such locally convex linear spaces as are obtained through forming the full direct product of an infinite number of copies of the real line, then every group of homeomorphisms of an arbitrary completely regular space can be linearized, and there exist universal linearizations (J. de Groot [9] ; see also [2]).

We want to make some remarks concerning the ideas underlying the proof of theorem 1 . These ideas are essentially the same as those expounded in [2] ; they are inspired by the concept of the graph of a mapping.

Let f be a continuous mapping of a topological space X into a topological space Y. As is well-known, the graph Γ of f, considered as a subspace of the topological product $X \times Y$, is homeomorphic to X. Consequently the quite arbitrary mapping f turns out to be topologically equivalent to the restriction to Γ of such a decent mapping as the canonical projection $X \times Y \longrightarrow Y$ is (e.g. if X is a subspace of a euclidean space E^n and Y a subspace of E^m, then f is equivalent to a restriction of a linear map, a projection, in E^{n+m}).

In particular, if f is a continuous map of a space X <u>into itself</u>, the graph may be used in order to linearize f by a mapping that sends one copy of X into another copy of X. However, it is our aim to linearize f by a map that again sends a suitable copy of X <u>into itself</u>.

This objective is reached by considering not the graph, i.e. the set of all pairs (x,fx), but the set of all <u>orbits</u> $(x, fx, f^2x, \ldots, f^nx, \ldots)$. If f is an autohomeomorphism, and if one desires that the mapping by which f is linearized will again be 1-1 and onto, then one should take the <u>full</u> orbits $(\ldots, f^{-2}x, f^{-1}x, x, fx, \ldots)$. And in case one wants to linearize a transformation group or semigroup G, one has to consider the orbits under G. Returning to the simplest case of one map f, it is clear that as a linearizing map one can take the <u>shift</u>, the map sending the point $(x_1, x_2, x_3, \ldots)$ onto $(x_2, x_3, x_4, \ldots)$, as it transforms the oribt of x into the orbit of the point f(x). Formally this amounts to the same thing as the use of the graph to change f into a projection: one skips the first coordinate. Of course the shift will only do, provided one succeeds in choosing a suitable topology in the set $X \times X \times X \times \ldots = X^{\aleph_0}$ of which all orbits are elements (if a (semi-) group G is considered, the orbits can be taken to be points of the set X^G).

Now in the case of the graph the product topology in $X \times X = X^2$ turns out to be useful. Hence it is plausible that also in X^G the product topology will suit our needs. The effects of this choice were explored in [2] . It indeed leads to several useful results; however, it is clear that this topology cannot be taken if linearization in Hilbert space is desired. For suppose X to be non-trivial and metrizable; then the topological porudct X^G will be non-metrizable, hence non-imbeddable in any Hilbert space, as soon as G is uncountable.

This is where the hypothesis of compactness or locally compactness of G comes in. We exploit the fact that on such groups Haar measure is defined, and consider, instead of the "direct sum" X^G (or

H^G, where H is a suitable Hilbert space into which the metrizable space X is imbedded beforehand) the "direct integral" $L_2(G,H)$. And this space, of course, is a Hilbert space.

The results of section 2 all follow immediately from those in section 3. Thus, the proofs in section 2 are, strictly speaking, superfluous. They are nevertheless given, as they illustrate the methods used more clearly than the proofs in section 3.

Except in this introduction we use the notation of [2] ; in particular, the argument of a function will be written before the function symbol.

2. Linearization of compact transformation groups

If G is a locally compact group and H is a Hilbert space, then $L_2(G,H)$ denotes the Hilbert space of all square-integrable functions (square-integrable with regard to Haar measure) on G with values in H.

We want to consider $L_2(G,H)$ as a subset of H^G; that is, from every element of $L_2(G,H)$, which is a class of equivalent functions, we choose one representing function. This is done - once for all - in a quite arbitrary manner, except that a continuous representing function is chosen whenever possible. We will use the fact that every equivalence class contains at most one continuous function.

Definition 1. If $\gamma_0 \in G$, then $\tilde{\gamma}_0$ denotes the map $L_2(G,H) \to L_2(G,H)$ such that, for $x \in L_2(G,H)$, $(x)\tilde{\gamma}_0$ is the function y with

$$(\gamma)y = (\gamma_0\gamma)x \qquad (\gamma \in G).$$

Proposition 1. For each $\gamma_0 \in G$ the map $\tilde{\gamma}_0$ is a unitary operator in $L_2(G,H)$; if H is non-degenerate $(\dim H > 0)$, then $\gamma \to \tilde{\gamma}$ is an isomorphism of G into the group of all unitary operators in $L_2(G,H)$.

Proof: evident.

Theorem 1. Let G be a compact transformation group acting on a metric space M of weight $\mathfrak{m}$. Then (G,M) can be linearized by unitary operators in a Hilbert space H of weight $\mathfrak{m} . \aleph_0$.

Proof.

If H_o is any Hilbert space of weight $m \cdot \aleph_o$, the space M can be imbedded topologically in H_o (cf. [19]). For simplicity's sake we will assume that M is already a subspace of H_o.

If $\xi \in M$, then $\xi\tau$ will denote the function $x : G \rightarrow H$ such that

$$\gamma(x) = (\xi)\gamma \, ,$$

for all $\gamma \in G$. As G is a topological transformation group, each $\xi\tau$ is a continuous function; as G is compact, it follows that $\xi\tau \in L_2(G,H_o)$. We will prove that the map $\tau : M \rightarrow L_2(G,H_o)$ is topological.

First we show that τ is continuous. Let $x \in M^{\tau}$ and $\varepsilon > 0$; we must prove the existence of a neighborhood V of $\xi = (\iota)x$ in M such that $\|x-y\| < \varepsilon$ as soon as $y \in M\tau$ and $(\iota)y \in V$.

For each $\gamma \in G$, let U_γ be a neighborhood of γ in G and V_γ a neighborhood of ξ in M such that

$$\|(\xi)\gamma - (\eta)\chi\| < \varepsilon/2$$

for all $\chi \in U_\gamma, \eta \in V_\gamma$. For these χ, η one has

$$\|(\xi)\chi - (\eta)\chi\| < \|(\xi)\chi - (\xi)\gamma\| + \|(\xi)\gamma - (\eta)\chi\| < \varepsilon.$$

Cover G by finitely many U_γ , say by $U_{\gamma_1}, \ldots, U_{\gamma_n}$, and let $V = V_{\gamma_1} \cap \ldots \cap V_{\gamma_n}$. Then

$$\|(\xi)\chi - (\eta)\chi\| < \varepsilon$$

for $\eta \in V \cap M$ and arbitrary $\chi \in G$; i.e. if $y = \eta\tau$, $\eta \in V \cap M$:

$$\|(\chi)x - (\chi)y\| < \varepsilon.$$

Consequently

$$\|\xi\tau - \eta\tau\| = \|x-y\| = \left\{ \int_G \|(\chi)x - (\chi)y\|^2 d\chi \right\}^{\frac{1}{2}} < \varepsilon$$

as soon as $\eta \in V$.

Next we remark that τ is 1-1; this is an immediate consequence of the facts that $(\iota)(\xi\tau) = \xi$, for $\xi \in M$, and that each $\xi\tau$ is continuous.

Our next task is to show that τ^{-1} is continuous. Suppose this were not the case at a point $x_o = \xi_o\tau \in M\tau$. Then there are points $x_n = \xi_n\tau \in M\tau$ and an $\varepsilon > 0$ such that

$$\| x_o - x_n \| < \frac{1}{n} , \quad \| \xi_o - \xi_n \| \geqslant \varepsilon,$$

for all n. We assert that for every $\gamma \in G$ there are a $\delta_\gamma > 0$ and a neighborhood U_γ of γ in G such that, for all but finitely many x_n,

$$\| (\chi)x_o - (\chi)x_n \| \geqslant \delta_\gamma \qquad \text{for all } \chi \in U_\gamma .$$

Assuming for the moment the validity of this assertion, we proceed in the following manner. Cover G by finitely many U_γ, say by $U_{\gamma_1}, \dots, U_{\gamma_n}$. Let $\delta = \min(\delta_{\gamma_1}, \dots, \delta_{\gamma_n})$. Then all but finitely many x_n will satisfy

$$\| (\chi)x_o - (\chi)x_n \| > \delta \quad \text{for all } \chi \in G.$$

Integrating the square of this expression over G, we obtain that $\| x_o - x_n \| \geqslant \delta$ for almost all x_n, which is absurd.

In order to prove the assertion we assume it to be false. Then for some $\gamma_o \in G$ there exists a sequence $\gamma_n \to \gamma$ such that, for each n,

$$\| (\gamma_n)x_o - (\gamma_n)x_k \| < \frac{1}{n}$$

for infinitely many x_k. Select a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that

$$\| (\gamma_n)x_o - (\gamma_n)x_{k_n} \| < \frac{1}{n}$$

for all n. As $(\gamma_n)x_o \to (\gamma_o)x_o$, it follows that $(\gamma_n)x_{k_n} \to (\gamma_o)x_o$, i.e.

$$(\xi_{k_n})\gamma_n \to (\xi_o)\gamma_o.$$

Now $\gamma_n \to \gamma_o$ implies that $\gamma_n^{-1} \to \gamma_o^{-1}$, and we find

$$\xi_{k_n} \to \xi_o ,$$

contradicting the fact that $\| \xi_n - \xi_o \| \geqslant \varepsilon$ for all n.

If $\gamma \in G$, then $\tilde{\gamma}$ is a unitary operator in $L_2(G,H_o)$, by prop. 1. It is quite straightforwardly verified that

$$\tilde{\gamma} \,\big|\, M\tau = \tau^{-1}\gamma\tau \;;$$

hence (G,M) can be linearized in $L_2(G,H_o)$ by means of the embedding τ.

Finally let H be the Hilbert subspace of $L_2(G,H_o)$ spanned by M. As H contains a dense subset of power $\mathfrak{m}\cdot\aleph_o$, the weight of H is at most $\mathfrak{m}\cdot\aleph_o$. Each $\tilde{\gamma}$ sends H into itself as it sends $M\tau$ into itself. Hence G is linearized by the unitary operators $\tilde{\gamma}\,|\,H$ in H.

Remark 1. It follows from the proof of theorem 1 that the topology $\mathcal{T}_1$ induced in $M\tau$ by $L_2(G,H_o)$ coincides with the weak topology $\mathcal{T}_2$ induced in $M\tau$ by the product topology of H_o^G (or of M^G).

The fact that $\mathcal{T}_1 \subset \mathcal{T}_2$ follows from the continuity of τ : $\| x-y \| < \varepsilon$ as soon as $(\iota)y$ belongs to a small enough neighborhood of $(\iota)x$.

The fact that $\mathcal{T}_2 \subset \mathcal{T}_1$ is implied by the continuity of τ^{-1}. For the continuity of τ^{-1} is equivalent to the continuity of the map $x \to (\iota)x$ on $M\tau$; as $(\gamma)x = ((\iota)x)\gamma$, and as each $\gamma \in G$ is a continuous (even a topological) map $M \to M$, it follows that each "projection map" $x \to (\gamma)x$ is continuous. And $\mathcal{T}_2$ is the weakest topology with this property.

Remark 2. As soon as one knows that $\mathcal{T}_1 = \mathcal{T}_2$ on $M\tau$, the theorem also follows from [2] § 9 prop. 3.

3. Locally compact groups

The main difficulty in the case of a locally compact G that is not compact lies in the fact that a continuous function $G \to H$ need not be integrable. In particular, if $\xi \in M \subset H$ the function x such that $(\gamma)x = (\xi)\gamma$, for $\gamma \in G$, although continuous and hence measurable, need not belong to $L_2(G,H)$. Consequently the imbedding map τ used in the proof of theorem 1 will not be a topological map $M \to L_2(G,H)$ (it remains, however, a topological map $M \to H^G$).

In order to meet this difficulty we change τ through the use of a scalar "weight function" f; $(\xi)\tau$ will then be the $x \in L_2(G,H)$ with $(\gamma)x = (\gamma)f.(\xi)\gamma$, for $\gamma \in G$.

<u>Definition 2.</u> A <u>weight function</u> on a locally compact group G is a continuous real-valued function f on G with the following properties:

(i) $(\iota)f = 1$; $(\gamma)f \neq 0$ for all $\gamma \in G$;

(ii) $(\gamma)f.(\delta)f \leq (\gamma\delta)f$, for all $\gamma, \delta \in G$;

(iii) f is square-integrable on G;

(iv) there exists a sequence $\{G_n\}$ of compact subsets of G such that

$$\int_{G \setminus G_n} |(\gamma)f|^2 \, d\gamma \longrightarrow 0.$$

<u>Remark.</u> If G is also σ-compact, condition (iv) is a consequence of the other ones.

<u>Definition 3.</u> A locally compact group G is said to belong to the class W (is a W-group) if there exists a weight function on G.

<u>Proposition 2.</u> Every compact group is a W-group. Every discrete countable free group is a W-group.

<u>Proof.</u>

If G is a compact group, the function that is identically 1 is a weight function. For countable free groups the assertion follows from [2] § 11 lemma 1.

<u>Proposition 3.</u> The additive group E of all real numbers, with the usual topology, is a W-group.

<u>Proof.</u>

The function $x \rightarrow e^{-|x|}$, $x \in E$, is a weight function.

<u>Definition 4.</u> A topological group G belongs to the class HW (is an HW-group) if it is the image of a W-group under a continuous homomorphism.

<u>Proposition 4.</u> Every continuous homomorphic image and every subgroup of an HW-group is an HW-group. Finite topological direct products of HW-groups are HW-groups.

Proof.

The first two assertions are evident. Now let G_1 and G_2 be HW-groups. Let F_i be a W-group, f_i a weight function on F_i, and σ_i a continuous homomorphism of F_i onto G_i (i=1,2). Then $\sigma_1 \times \sigma_2$ is a continuous homomorphism of $F_1 \times F_2$ onto $G_1 \times G_2$ (by definition, $(\varphi_1,\varphi_2)\sigma_1 \times \sigma_2 = (\varphi_1\sigma_1,\varphi_2\sigma_2)$, for arbitrary $(\varphi_1,\varphi_2) \in F_1 \times F_2$), and a weight function f on $F_1 \times F_2$ is defined by

$$(\varphi_1,\varphi_2)f = (\varphi_1)f_1 \cdot (\varphi_2)f_2 ,$$

for arbitrary $(\varphi_1,\varphi_2) \in F_1 \times F_2$.

Corollary. Every countable discrete group is an HW-group.

Proof.

Follows from the second assertion of prop.2 and the first assertion of prop.4.

In fact, more can be said:

Proposition 5. Every countable discrete group is a W-group.

Proof.

Let G be an arbitrary countable discrete group. Let σ be a homomorphism of a countable free group F onto G.

In [2] § 11 lemma 1 a weight function f_o is constructed for F with the additional property that $(\varphi)f_o = 2^{-n}$ for some natural member n, for each $\varphi \in F$ that is not the unit of F. Consequently, if for $\gamma \in G$ we put

$$(\gamma)f = \sup \left\{(\varphi)f_o \ : \ \varphi \in (\gamma)\sigma^{-1}\right\},$$

then also

$$(\gamma)f = \max \left\{(\varphi)f_o \ : \ \varphi \in (\gamma)\sigma^{-1}\right. .$$

Hence if $\gamma_1,\gamma_2 \in G$, there are $\varphi_1,\varphi_2 \in F$ such that $\varphi_i\sigma = \gamma_i$ and $(\gamma_i)f = (\varphi_i)f_o$ (i=1,2); it follows that

$$(\gamma_1)f.(\gamma_2)f = (\varphi_1)f_o.(\varphi_2)f_o \leq (\varphi_1\varphi_2)f_o \leq (\gamma_1\gamma_2)f,$$

as $(\varphi_1\varphi_2)\sigma = (\varphi_1\sigma).(\varphi_2\sigma) = \gamma_1\gamma_2$. Moreover

$$\sum_{\gamma\in G}((\gamma)f)^2 \leqslant \sum_{\varphi\in F}((\varphi_o)f_o)^2 < \infty \ .$$

Hence f is a weight function for G.

Proposition 6. Every locally compact, compactly generated Abelian group G is a W-group.

Proof.

G is topologically isomorphic to a direct product $E^n \times Z^m \times F$, where n and m are non-negative integers, Z is the infinite cyclic group and F is a compact group (cf. e.g. [10] theorem 9.8). Now use propositions 2,3,4,5.

Once again this proposition can be strengthened (see prop.8). In order to do so we need:

Proposition 7. Let G be a topological group. If G contains a compact normal subgroup G_o such that the factor group G/G_o is discrete and countable, G is a W-group.

Proof.

Let f_o be a weight function for G/G_o. We define, for $\gamma\in G$

$$(\gamma)f = (\gamma G_o)f_o.$$

Then $(\gamma)f \neq 0$ for all $\gamma\in G$; $(\iota)f = 1$; and, for arbitrary $\gamma_1, \gamma_2 \in G$:

$$(\gamma_1)f.(\gamma_2) = (\gamma_1 G_o)f_o.(\gamma_2 G_o)f_o \leqslant$$

$$(\gamma_1 G_o.\gamma_2 G_o)f_o = (\gamma_1\gamma_2)f.$$

As f is constant on cosets and as every coset is open, f is continuous. Moreover, the distinct cosets partition G in countably many measurable sets; therefore

$$\int_G ((\gamma)f)^2 d\gamma = \sum_{A\in G/G_o}\int_A ((\gamma)f)^2 d\gamma =$$

$$= \sum_{A\in G/G_o}((A)f_o)^2 < \infty \ .$$

Proposition 8. Every separable locally compact abelian group G is a W-group.

Proof.

G is isomorphic to a direct product $E^n \times G'$, where G' is a group containing a compact subgroup G_o such that G'/G_o is discrete (see [20] pag.110). As G is separable, G'/G_o must be countable. Now use propositions 3,7 and 4.

The main result of this section is the following

Theorem 2. Let G be a topological transformation group acting on a metrizable space M of weight $\mathfrak{m}$. If G is an HW-group, then G can be linearized by a group of bounded linear operators in a Hilbert space H of weight $\mathfrak{m}.\aleph_o$.

Proof.

Let F be a W-group, f a weight function on F, and σ a continuous homomorphism of F onto G. As in the proof of theorem 1 we suppose M to be imbedded topologically in a suitable Hilbert space H_o. We need more precision then in the proof of theorem 1, however, and must assume M to be imbedded in H_o as a bounded set; say $M \subset S \subset H_o$, where S is the unit sphere in H_o. (This is always possible; it follows from the proofs in [19], but also from the following consideration. If we first only assume $M \subset H_o$, then we can imbed M topologically in the unit sphere of the Hilbert sum $H_o \oplus E^1$ by means of "inverse stereographic projection" with the "north pole" (0,1) as the center of projection).

If $\xi \in M$, we put $\xi\tau = x \in H_o^F$ with

$$(\varphi)x = (\varphi)f.(\xi)(\varphi\tau),$$

for arbitrary $\varphi \in F$. As σ is a continuous map $F \to G$, and as $\xi \to (\xi)\gamma$ (ξ fixed in M) is a continuous map $M \to M$, the map $\varphi \to (\xi)(\varphi\sigma)$ is a continuous map of F into the bounded subset M of H_o. On the other hand, f is square integrable. It follows that $x \in L_2(F,H_o)$. Hence τ maps M into $L_2(F,H_o)$.

Clearly τ is 1-1; for each $\xi\tau$ is continuous, and if ε is the unit element of F, then $(\varepsilon)(\xi\tau) = (\varepsilon)f.(\xi)(\varepsilon\tau) = (\xi)\iota = \xi$. We will prove that τ is topological. The proof is modelled after the proof of theorem 1.

First we show that τ is continuous. Take any $x \in M\tau$, say $x = \xi\tau$, and any $\delta > 0$. If $y \in M\tau$, say $y = \eta\tau$, then

$$\| x-y \|^2 = \int_F |(\varphi)f|^2 \cdot \| (\xi)(\varphi\sigma) - (\eta)(\varphi\sigma) \|^2 d\varphi .$$

Let $\{F_n\}$ be a sequence of compact subsets of F such that

$$\int_{F \setminus F_n} |(\varphi)f|^2 d\varphi \longrightarrow 0$$

and let n_o be a natural number such that

$$\int_{F \setminus F_{n_o}} |(\varphi)f|^2 d\varphi < \frac{\delta^2}{16} .$$

As $M \subset S$, we have

$$\| (\xi)(\varphi\sigma) - (\eta)(\varphi\sigma) \|^2 \leq (\| (\xi)\varphi\sigma \| + \| (\eta)\varphi\sigma \|)^2 \leq 4;$$

hence

$$\| x-y \|^2 \leq \int_{F_{n_o}} |(\varphi)f|^2 \cdot \| (\xi)(\varphi\sigma) - (\eta)(\varphi\sigma) \|^2 d\varphi + \frac{\delta^2}{4} .$$

For each $\varphi \in F$ there exist a neighborhood U_φ of $\varphi\sigma$ in G and a neighborhood V_φ of ξ in M such that

$$\| (\xi)(\gamma) - (\eta)(\gamma) \| < \delta . (4 \int_F |(\varphi)f|^2 d\varphi)^{-\frac{1}{2}},$$

for all $\gamma \in U_\varphi$ and $\eta \in V_\varphi$. The compact set F_{n_o} may be covered by finitely many of the sets $(U_\varphi)\sigma^{-1}$, say by $U_{\varphi_1}\sigma^{-1}, \ldots, U_{\varphi_n}\sigma^{-1}$; put $V = V_{\varphi_1} \cap \ldots \cap V_{\varphi_n}$. Then

$$\int_{F_{n_o}} |(\varphi)f|^2 \| (\xi)(\varphi\sigma) - (\eta)(\varphi\sigma) \|^2 d\varphi < \frac{\delta^2}{4}$$

for all $\eta \in V$. It follows that $\| x-y \| < \delta$ for all $y = \eta\tau$ with $\eta \in V$.

Next we show that τ^{-1} is continuous. Suppose this where not the case at the point $x = \xi\tau$. Then there must exist a sequence of points $\eta_n \in M$ and a $\delta > 0$ such that for all n

$$\| \xi\tau - \eta_n\tau \| < \frac{1}{n}, \quad \| \xi - \eta_n \| > \delta .$$

We assert that for each $\gamma \in G$ there exist a $\delta_\gamma > 0$ and a neighborhood U_γ of γ in G such that, for all but finitely many n:

$$\| (\xi)\chi - (\eta_n)\chi \| \geq \delta_\gamma \quad \text{for all} \quad \chi \in U_\gamma .$$

This assertion is proved in exactly the same way as in the proof of theorem 1.

Let n be a natural number such that $\int_{F_n} |(\varphi)f|^2 d\varphi \neq 0$. The compact set F_n may be covered by finitely many sets $U_\gamma \sigma^{-1}$, say by $U_{\gamma_1}\sigma^{-1}, \ldots, U_{\gamma_m}\sigma^{-1}$. Let $\delta_n = \min(\delta_{\gamma_1}, \ldots, \delta_{\gamma_m})$. Then for almost all k we have that

$$\| (\xi)(\varphi\sigma) - (\eta_k)(\varphi\sigma) \|^2 \geq \delta_n \quad \text{for all } \varphi \in F_n .$$

It follows that

$$\frac{1}{k^2} > \| \xi\tau - \eta_k\tau \|^2 \geq \int_{F_n} |(\varphi)f|^2 . \quad \| (\xi)(\varphi\sigma) - (\eta_k)(\varphi\sigma) \|^2 d\varphi \geq$$

$$\geq \delta_n^{\;2} \int_{F_n} |(\varphi)f|^2 d\varphi ,$$

for almost all k, which is absurd.

Thus τ is indeed a topological imbedding of M into $L_2(F, H_o)$. We shall now exhibit the linear operators in $L_2(F, H_o)$ by which the $\gamma \in G$ are linearized.

If $\varphi_o \in F$ and $x \in L_2(F, H_o)$, then $(x)\hat{\varphi}_o$ will denote the $y \in H_o^{\,F}$ with

$$(\varphi)y = \frac{(\varphi)f}{(\varphi_o\varphi)f} \cdot (\varphi_o\varphi)x .$$

By def. 2, (ii), we have

$$\| (\varphi)y \| \leq ((\varphi_o)f)^{-1} . \| (\varphi_o\varphi)x \| ;$$

it follows that $y \in L_2(F, H_o)$. Clearly $\hat{\varphi}_o : L_2(F, H_o) \longrightarrow L_2(F, H_o)$ is linear; and $\hat{\varphi}_o$ is also bounded, in fact

$$\| \varphi_o \| \leq ((\varphi_o)f)^{-1}.$$

We remark that $\widehat{\varphi_o \varphi_1} = \hat{\varphi}_o \hat{\varphi}_1$; i.e. $\varphi \to \hat{\varphi}$ is an <u>isomorphism</u> of F into the group of all invertible bounded linear operators in $L_2(F,H_o)$.

<u>Assertion</u>: if $\gamma \in G$, then

$$\hat{\varphi}_o | M\tau = \tau^{-1}\gamma\tau \quad ,$$

for each $\varphi_o \in F$ such that $\varphi_o \sigma = \gamma$.

Indeed, let $\varphi\sigma = \gamma$, $\xi \in M$, and let φ be an arbitrary element of F. Then

$$(\varphi)(\xi\tau\hat{\varphi}_o) = \frac{(\varphi)f}{(\varphi_o\varphi)f} \cdot (\varphi_o\varphi)(\xi\tau) =$$

$$= \frac{(\varphi)f}{(\varphi_o\varphi)f} \cdot (\varphi_o\varphi)f.(\xi)((\varphi_o\varphi)\sigma) =$$

$$= (\varphi)f.(\xi)((\varphi_o\sigma)(\varphi\sigma)) = (\varphi)f.(\xi\gamma)(\varphi\sigma) =$$

$$= (\varphi)(\xi\gamma\tau).$$

Finally we take for H the Hilbert subspace of $L_2(F,H_o)$ spanned by $M\tau$. As each $\hat{\varphi}$, $\varphi \in F$, sends $M\tau$ into itself, it also sends H into itself. Hence all $\hat{\varphi}|H$, $\varphi \in F$ are bounded linear operators in H. The weight of H obviously does not exceed $m . \aleph_o$, where m is the weight of M.

<u>Remark 1.</u> It once again follows from the proof that on $M\tau$ the topology induced by $L_2(F,H_o)$ coincides with the weak topology.

<u>Remark 2.</u> We saw in the proof of theorem 2 that $\| \hat{\varphi} \| \leq ((\varphi)f)^{-1}$. Now if F is a topological transformation group acting on a Hilbert space H and consisting of bounded linear operators in H, the real-valued function f on F, defined by

$$(\varphi)f = \| \varphi \|^{-1}$$

satisfies conditions (i) and (ii) of definition 2. Moreover, f is lower semicontinuous. For let $\varphi_o \in F$ and $\varepsilon > 0$ be chosen arbitrarily. Let $x_o \in H$ such that $\| (x_o)\varphi_o \| > \| \varphi_o \| - \frac{\varepsilon}{2}$, $\| x_o \| = 1$. Let U be a neighborhood of φ_o such that $\|(x_o)\varphi - (x_o)\varphi_o\| < \varepsilon/2$ for all $\varphi \in U$. Then

$$\|\varphi\| = \|x_o\| \cdot \|\varphi\| \geq \|(x_o)\varphi\| \geq \|(x_o)\varphi_o\| - \varepsilon/2 \geq \|\varphi_o\| - \varepsilon ,$$

for all $\varphi \in U$.

Hence if F is also locally compact, f is Haar-measurable. If moreover F is σ-compact, all conditions for a weight function will be satisfied by f as soon as

$$\int_F \frac{d\varphi}{\|\varphi\|^2} < \infty .$$

Even if this is not the case, it may still well be that the condition that a transformation group G be a HW-group is not only sufficient but also necessary in order that G admits linearization in Hilbert space. For the moment this remains an open problem.

As was mentioned already in the introduction, theorem 1 follows at once from theorem 2 (cf. proposition 2). The same holds for part of [2] § 11 theorem 3, stating that every countable group of autohomeomorphisms of a metrizable space admits linearization. Moreover, theorem 2 together with prop.8 gives us:

Corollary. Let G be a separable locally compact abelian transformation group acting on a metrizable space M of weight $\mathfrak{m}$. Then (G,M) can be linearized by a group of bounded linear operators in a Hilbert space H of weight $\mathfrak{m} . \aleph_o$.

4. Linearization of semigroups of continuous mappings

In the proof of theorem 2 the left invariance of the Haar integral was only used in showing that $\|\hat{\varphi}\| \leq ((\varphi)f)^{-1}$, for $\varphi \in F$, and clearly it would suffice for this purpose to have a left weakly subinvariant integral on G, i.e. an integral such that

$$\int_G (\gamma_o\gamma)f\, d\gamma \leq \int_G (\gamma)f\, d\gamma$$

for all integrable nonnegative f and for all γ_o.

Another property of the Haar integral, namely the fact that the integral of a nonnegative integrable function which differs from zero on an open set is always different from zero, was also used in an essential way (a.o. in proving that the imbedding map τ is 1-1).

We did not use the existence of inverses in G except in showing that the bounded linear operator $\hat{\varphi}$ is invertible (for this is shown by exhibiting an inverse: $\hat{\varphi}\,\widehat{\varphi^{-1}} = \widehat{\varphi^{-1}}\,\hat{\varphi} = I$). Hence if we do not ask for linearization by invertible operators, we may also start with a semigroup G of transformations.

Definition 5. A topological semigroup G of transformations of a topological space X is called a topological transformation semigroup if it contains the identity transformation of X as its unit and if the map

$$(x,\gamma) \longrightarrow (x)\gamma$$

is a continuous map $X \times G \rightarrow X$.

Definition 6. A weak integral on a locally compact semigroup G is an integral on G which is left weakly subinvariant:

$$\int_G (\gamma_o\gamma)f\ d\gamma \leq \int_G (\gamma)f\ d\gamma$$

for all integrable nonnegative f and all $\gamma_o \in G$, and which has the property that the integral of a nonnegative integrable function which differs from zero on an open set is itself non-zero.

Definition 7. A topological semigroup G is said to belong to the class WW (is a WW-semigroup) if there exists a weak integral on G and a continuous real-valued function f satisfying conditions (i)-(iv) of definition 2 with regard to this integral.
A topological semigroup belongs to the class HWW (is an HWW-semigroup) if it is the image of a WW-semigroup under a continuous homomorphism.

The proof of theorem 2 can immediately be adapted to show:

Theorem 3. Let G be a topological transformation semigroup acting on a metrizable space of weight $\mathfrak{m}$. If G is an HWW-semigroup, then G can be linearized by a semigroup of bounded linear operators in a Hilbert space H of weight $\mathfrak{m} . \aleph_0$.

Topological semigroups with a weak integral are rarely met (at least if they are not groups). They do however exist; examples are all discrete countable semigroups G, with

$$\int_G (\gamma) f \, d\gamma = \sum_{\gamma \in G} (\gamma) f.$$

Hence we have (cf. [2] § 11 theorem 3):

<u>Corollary</u>. Every countable system of continuous maps of a metrizable space M can be linearized by bounded linear operators in a Hilbert space H of weight $\aleph_0 \cdot$weight (M).

5. Remarks on universal linearization

It follows from the proof of theorem 2 that the following theorem is valid:

<u>Theorem 4</u>. Let F be a W-group and let $\mathfrak{m}$ be a transfinite cardinal number. There exists a Hilbert space H and an isomorphism $\varphi \rightarrow \hat{\varphi}$ of F onto a subgroup $\hat{F}$ of the group of invertible bounded linear operators in H, with the following property. If M is any metrizable space of weight $\leq \mathfrak{m}$, and if G is any topological transformation group acting on M such that the topological group G is a continuous homomorphic image of F, then (G,M) can be linearized by $(\hat{F},H)$.

Similarly we have

<u>Theorem 4'</u>. Let F be a WW-semigroup, and let $\mathfrak{m}$ be a transfinite cardinal number. There exists a Hilbert space H and an isomorphism $\varphi \rightarrow \hat{\varphi}$ of F onto a subsemigroup $\hat{F}$ of the semigroup of all bounded linear operators in H, with the following property. If M is any metrizable space of weight $\leq \mathfrak{m}$, and if G is any topological transformation semigroup acting on M such that the topological semigroup G is a continuous homomorphic image of F, then (G,M) admits linearization by $(\hat{F},H)$.

<u>Corollary 1.</u> (cf. [2] § 11 theorem 3). Let $\mathfrak{m}$ be a transfinite cardinal, and let H be a Hilbert space of weight $\mathfrak{m}$. Then there exists a (free) countable group G of invertible bounded linear operators in H (a (free) countable semigroup of bounded linear operators in H) that is universal for all countable groups of autohomeomorphisms (semigroups of continuous self-maps) of metrizable spaces of weight $\leq \mathfrak{m}$.

In particular:

<u>Corollary 2</u> (cf. [5]). Let H be a Hilbert space of transfinite weight $\mathfrak{m}$. There exists an invertible bounded linear operator Φ in H (a bounded linear operator Ψ in H) that is universal for all autohomeomorphisms (for all continuous self-maps) of metrizable spaces of weight $\leq \mathfrak{m}$.

As remarked in the proof of prop.6, every locally compact compactly generated abelian group G is topologically isomorphic to a direct product $E^n \times Z^n \times F$. The integers n,m and the type of F are invariants of G: if G is also topologically isomorphic to $E^{n'} \times Z^{m'} \times F'$, then $n=n'$, $m=m'$ and F is topologically isomorphic to F' (cf. [10] theorem (9.13)).

<u>Corollary 3.</u> Let n be a nonnegative integer, $\mathfrak{m}$ a transfinite cardinal number and F a compact group. There exists a group G of invertible bounded linear operators in some Hilbert space H which is a universal linearization for the class of all those topological transformation groups G_0 such that G_0 acts on a metrizable space of weight $\leq \mathfrak{m}$ while the group G is topologically isomorphic to a direct product $E^{n_0} \times H \times F_0$, with $n_0 \leq n$, H discrete countable, and F_0 and continuous homomorphic image of F.

<u>Proof.</u>

Let $G = \hat{H}$, where H is the topological direct product of E^n, F and a free group with $\aleph_0$ generators, and where $\hat{H}$ consists of the maps $\hat{\chi}$, $\chi \in H$, constructed as in the proof of theorem 2.

<u>Remark.</u> If F is a W-group and G a continuous homomorphic image of F, each $\gamma \in G$ can be linearized by a $\hat{\varphi}$, $\varphi \in F$. A "dual result" would be something of the following sort: let G be a topological transformation

group, and suppose G is a subgroup of a W-group F such that the restriction of the integral in F to G is non-trivial. (Then G is itself a W-group, by means of the restriction to it of a weight function on F). Then each transformation $\gamma \in G$ can be lifted to a $\hat{\varphi}$, $\varphi \in F$.

We do not know whether this is true. Possibly some conditions should be imposed on the space on which G acts. In any case we mention the following.

The map ν such that $(x)\nu = x|G$, for $x \in L_2(F,H)$, is evidently a bounded linear operator of $L_2(F,H)$ onto $L_2(G,H)$. If $\gamma \in G$, let $\hat{\gamma} : L_2(G,H) \to L_2(G,H)$ be defined as in the proof of theorem 2; if $\varphi \in F$, we denote momentarily the corresponding linear operator $L_2(F,H) \to L_2(F,H)$ by $\hat{\hat{\varphi}}$. Then for each $\gamma \in G$ the diagram

$$\begin{array}{ccc} L_2(F,H) & \xrightarrow{\hat{\hat{\gamma}}} & L_2(F,H) \\ \nu \downarrow & & \downarrow \nu \\ L_2(G,H) & \xrightarrow{\hat{\gamma}} & L_2(G,H) \end{array}$$

is commutative. That is, each $\hat{\gamma}$, $\gamma \in G$, may be lifted to a transformation in $\hat{F}$.

6. Linear imbedding

The same basis idea underlying the proofs of theorems 1 and 2 - to make use of the orbits $(x, x\varphi, x\varphi^2, \dots, x\varphi^n, \dots)$ as points of a new space - has been applied by G.-C.Rota ([16] , [17] ; cf. also [18]) in order to obtain universal operators in Hilbert space.

Definition 8. A semigroup G of bounded linear operators in a Hilbert space H is said to be universal for a class K of semigroups G' of bounded linear operators in Hilbert spaces H', if for every (G',H') in K there exists an invertible bounded linear operator τ mapping H' onto a closed linear subspace of H in such a way that the action of G on $H'\tau$ is equivalent to the action of G' on H'. More precisely, $H'\tau$ is invariant under every $\gamma \in G$, and if $G|H'\tau$ is the semigroup of all $\gamma|H'\tau$, $\gamma \in G$, then

$$\gamma \,|\, H'\tau \longrightarrow \tau(\gamma | H'\tau)\tau^{-1}$$

is an isomorphism of $G|H'\tau$ onto G'.

A group G of invertible bounded linear operators in a Hilbert space H is defined to be <u>universal</u> for a class K of such groups in a similar way.

In the terminology of G.-C. Rota, G would be called a universal model for all $G' \in K$.

<u>Theorem 5.</u> Let H be a Hilbert space of infinite weight $\mathfrak{m}$. There exists a semigroup S of bounded linear operators in H that is universal for the class of all countable semigroups G' of bounded linear operators in a Hilbert space H' of weight $\leq \mathfrak{m}$ that are uniformly bounded in the operator norm (i.e. there exists a real constant c such that $\|\gamma'\| \leq c$ for all $\gamma' \in G'$) and contain the identity operator. It is possible to choose S in such a way that the abstract semigroup S is a free semigroup-with-unit with $\aleph_0$ generators.

<u>Proof.</u>

Let F be a free semigroup-with-unit with $\aleph_0$ generators, and let f be a weight function on the discrete group F. Let H_0 be a Hilbert space of weight $\mathfrak{m}$; let $H_\varphi = H_0$ for each $\varphi \in F$, and let H be the Hilbert sum

$$H = \bigoplus_{\varphi \in F} H_\varphi ;$$

H is again a Hilbert space of weight $\mathfrak{m}$. If $\varphi_0 \in F$, $\hat{\varphi}$ will be the bounded linear operator in H such that, for $x = (x_\varphi)_{\varphi \in F} \in H$ (each $x_\varphi \in H_0$), $x\hat{\varphi}_0 = y$, where

$$y_\varphi = \frac{(\varphi)f}{(\varphi_0\varphi)f} \cdot x_{\varphi_0\varphi} .$$

As $\frac{(\varphi)f}{(\varphi_0\varphi)f} \leq ((\varphi_0)f)^{-1}$ and $\sum_{\varphi \in F} \|x_{\varphi_0}\|^2 \leq \sum_{\varphi \in F} \|x\|^2 = \|x\|^2$,

$\hat{\varphi}_0$ is indeed a bounded linear operator in H (with norm $\leq ((\varphi_0)f)^{-1}$). It is immediately verified that the map $\varphi \longrightarrow \hat{\varphi}$ is an isomorphism of F into the semigroup of all bounded linear operators in H. Let S be its image.

Now let H' be any Hilbert space of weight $\leqslant \mathfrak{m}$ and let G' be any countable semigroup of bounded linear operators in H', such that $\|\gamma'\| \leqslant c$ for all $\gamma' \in G$ and a suitable real c. We may assume without loss of generality that H' is a Hilbert subspace of H_o.

Let σ be any homomorphism of F onto G', sending the identity ε of F into the identity operator ι in G'. If $\xi \in H'$ we put $\xi\tau = x \in H$ with

$$x_\varphi = (\varphi)f.(\xi)(\varphi\sigma).$$

The point x indeed belongs to H, as

$$\sum_{\varphi\in F} \|x_\varphi\|^2 = \sum_{\varphi\in F} |(\varphi)f|^2 . \|(\xi)(\varphi\sigma)\|^2 \leqslant$$
$$\leqslant \|\xi\|^2 . c^2 . \sum_{\varphi\in F} |(\varphi)f|^2 < \infty .$$

Moreover, τ is a linear operator $H' \longrightarrow H$, and τ is bounded; in fact

$$\|\tau\| \leqslant c^2 . \sum_{\varphi\in F} |(\varphi)f|^2 .$$

As $(\xi\tau)_\varepsilon = (\varepsilon)f.(\xi)(\varepsilon\sigma) = \xi$, τ is 1-1; and τ^{-1} is bounded, as

$$\|\xi\|^2 = \|(\varepsilon)f.(\xi)(\varepsilon\sigma)\|^2 \leqslant \|\xi\tau\|^2 .$$

Consequently τ is a linear homeomorphism, so $H'\tau$ is complete and thus a closed linear subspace of H.

As was the case in the proof of theorem 3 one easily verifies that

$$\hat{\varphi} \mid H'\tau = \tau^{-1}(\varphi\sigma)\tau ;$$

all assertions of the theorem now readily follow.

If one considers semigroups G' with a certain finite number of generators, one can take for S a free semigroup with the same number of generators. In particular, considering semigroups G' generated by one bounded linear operator:

Corollary. Let $\mathfrak{m}$ be any transfinite cardinal number. There exists a bounded linear operator U in Hilbert space of weight $\mathfrak{m}$ that is universal for all bounded linear operators T in a Hilbert space of weight $\leqslant \mathfrak{m}$ such that the norms of the iterates T^n are uniformly bounded ($\|T^n\| \leqslant c$, $n=1,2,\ldots$, for some real constant c). (Hence in any case U is universal for all T with $\|T\| \leqslant 1$;cf. [16]).

The free semigroup-with-unit S with one generator may be identified with the additive semigroup of all natural numbers. If a is any real number > 1, $(n)f = a^{-n}$ is a weight function on S. It follows that U can be choosen in such a way that $\|U\| \leqslant a$. I.e. for each $\varepsilon > 0$ there exists a universal operator U of norm $\leqslant 1+\varepsilon$.

Both the hypothesis that G' is uniformly bounded and the use of a weight function serve to obtain that $\xi\tau \in H$ for all $\xi \in H'$. If G' is such that there exists a homomorphism σ of F onto G' with the property

$$\sum_{\varphi \in F} \|(\varphi)\sigma\|^2 < \infty$$

then no other hypothesis on G' is needed, and weight functions are superfluous: just put $\xi\tau = x$, where

$$x_\varphi = (\xi)\varphi \,,$$

and let $\hat{\varphi}_o$, $\varphi_o \in F$, be the bounded linear operator in H such that $x\,\hat{\varphi}_o = y$, where

$$y_\varphi = x_{\varphi_o \varphi} \,.$$

In the case of semigroups G' generated by one bounded linear operator T, this means that we want that

$$\sum_{n=1}^{\infty} \|T^n\|^2 < \infty \,.$$

The latter is certainly the case if T has spectral radius < 1: see [17] p.470. In this way G.-C. Rota obtained the following result, which is stronger than the above corollary ([17] theorem 1):

Theorem. There exists a bounded linear operator U of norm 1 in Hilbert space of weight $\mathfrak{m}$ that is universal for all bounded linear operators T in a Hilbert space of weight $\mathfrak{m}$ that have spectral radius < 1.

A result similar to theorem 7 is valid for linear homeomorphisms; the proof is a copy of the proof of theorem 7.

Theorem 5'. Let H be a Hilbert space of infinite weight $\mathfrak{m}$. There exists a group F of linear autohomeomorphisms (invertible bounded linear operators) of H that is universal for the class of all countable groups G' of invertible bounded linear operators in a Hilbert space H' of weight $\leq \mathfrak{m}$ that are uniformly bounded in the operator norm. It is possible to choose G in such a way that the abstract group G is a free group with $\aleph_0$ generators.

If one restricts oneself to groups G' with a certain finite number of generators, then G can be taken to be a free group with that same number of generators. In particular:

Corollary. Let $\mathfrak{m}$ be any transfinite cardinal number. There exists a linear homeomorphism U in Hilbert space of weight $\mathfrak{m}$ that is universal for all invertible bounded linear operators T in a Hilbert space of weight $\leq \mathfrak{m}$ such that the norms of the iterates T^n are uniformly bounded.

The universal operator U can again be chosen such that $\|U\| < 1+\varepsilon$. Moreover, a stronger result can be proved:

Proposition 9. There exists a linear homeomorphism U (of norm 1) in Hilbert space of weight $\mathfrak{m}$ that is universal for all invertible bounded linear operators T in a Hilbert space of weight $\leq \mathfrak{m}$ that have spectral radius ≤ 1.

The proof is similar to Rota's proof of [17] theorem 1, as indicated above, except that one uses a Hilbert sum $\bigoplus_{n=-\infty}^{+\infty} H_n$, while ξ is mapped on the full orbit

$$(\ldots\ldots, \xi T^{-2}, \xi T^{-1}, \xi, \xi T, \ldots) .$$

References

1. P.C. BAAYEN, Toepassingen van een lineariseringsprincipe. Report ZW 1962-011, Mathematical Centre, Amsterdam, 1962.

2. P.C. BAAYEN, S-maps. Preliminary note WN3, Mathematical Centre, Amsterdam, 1963.

3. R.H. BING, A homeomorphism between the 3-sphere and the sum of two solid normed spheres. Annals of Math. 56 (1952), 354-362.

4. A.H. COPELAND JR. and J. DE GROOT, Linearization of autohomeomorphisms. Notices Am. Math. Soc. 6 (1959), 844.

5. A.H. COPELAND JR. and J. DE GROOT, Linearization of a Homeomorphism. Math. Annalen 144 (1961), 80-92.

6. S. EILENBERG, Sur les groupes compacts d'homéomorphics. Fund. Math. 28 (1936), 75-80.

7. J. DE GROOT, The action of a locally compact group on a metric space. Nieuw Archief voor Wiskunde (3) 7 (1959), 70-74.

8. J. DE GROOT, Every continuous map is linear. Notices Am. Math. Soc. 6 (1959), 754.

9. J. DE GROOT, Linearization of mappings. In: General Topology and its Relations to Modern Analysis and Algebra, Proceedings 1961 Prague Symposium. Prague, 1962.

10. E. HEWITT and K.A. ROSS, Abstract harmonic analysis. Berlin, 1963.

11. J.M. KISTER and L.N. MANN, Equivariant imbeddings of compact abelian Lie groups of transformations. Math. Annalen 148 (1962), 89-93.

12. D. MONTGOMERY and L. ZIPPIN, Examples of transformation groups. Proc. Amer.Math. Soc. 5 (1954), 460-465.

13. D. MONTGOMERY and L. ZIPPIN, Topological transformation groups. New York, 1955.

14. G.D. MOSTOW, Equivariant embeddings in Euclidean space. Ann. Math. 65 (1957), 432-446.

15. G.D. MOSTOW, On a conjecture of Montgomery. Ann. Math. 65 (1957), 505-512.

16. G.-C. ROTA, Note on the invariant subspaces of linear operators. Rend.Circ.Mat. Palermo Ser.II, 7 (1959), 182-184.

17. G.-C. ROTA, On models for linear operators. Comm. Pure Appl. Math. 13 (1960), 469-472.

18. G.-C. ROTA, On the classification of periodic flows. Proc. Amer. Math. Soc. 13 (1962), 659-662.

19. Yu. M. SMIRNOV, On metrization of topological spaces. Uspehi Mat. Nauk 6, 6 (1951), 100-111.

20. A. WEIL, L'integration dans les groupes topologiques et ses applications. Paris, 1940.

21. L. ZIPPIN, Transformation groups. In: Lectures in topology, edited by R.L. Wilder and W.L. Ayres, Ann Arbor, 1941.

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