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Cocompactness and the Baire category theorem

by

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1. Introduction. The Baire category theorem in its usual form states that a locally compact Hausdorff space, or a completely metrizable space is a Baire space, i.e. is not the union of countable number of nowhere dense closed subsets.

This theorem is fundamental and has important applications in analysis. The unsatisfactory status of this theorem is well known and clear. Firstly, its formulation deals with two important classes of spaces of a totally different nature; secondly, why the countability of the number of subsets? The answer to the second question seems to be easy. Indeed, if one takes the topological product of a segment and a compact Hausdorff space containing as many points as one wants, still the product is always the union of continuously many nowhere dense subsets. So the countability is essential in a way. Nevertheless, one can get rid of the countability (and this starts to be of interest in spaces without a countable base, as one would expect) by generalizing the definition of nowhere dense subset. This has been carried out in section three, where we define, for any cardinal m, an m-thin subset, and m-Baire space. If m is countable, we obtain just the ordinary definition of nowhere dense subset and of Baire space. The first objection, however, is more serious. We propose a solution by introducting the notion of cocompactness. (Perhaps we should have used the term completeness, but this notion is already used in the theory of uniform spaces.) Roughly speaking, cocompactness of a space is a weak form of compactness relative to some open base of the space; cocompactness in completely regular spaces is identical to compactness if the base of all open sets is used. The exact definition in section 2 uses regular filter bases which are easy to handle, and we avoid, in this paper, the dual wording by means of covers (we should say co-covers). While cocompactness is a weaker form of compactness, and of local compactness as well, it appears for metrizable spaces to be identical with topological completeness (i.e. completeness in a suitable metric).

Furthermore, cocompact regular spaces have some nice properties, e.g., they are invariant under the forming of topological products and topological unions.

Cocompact spaces turn out to be m-Baire spaces, and we obtain a unifying Baire category theorem which gives us a wide range of spaces, including the usual ones, in which the Baire category theorem holds (see corollary, section 3).

Since for our purposes only regular spaces are important, we suppose all spaces to be regular (i.e., regular and Hausdorff).

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2. Cocompactness

Definitions. Let $\{U\}$, $U \neq \emptyset$ be an open base of a topological regular Hausdorff space. A non empty subset F of $\{U\}$ is called a regular filter base relative to $\{U\}$ if the following conditions hold:

(i) every $U \in F$ contains a $U' \in F$ with $\overline{U'} \subset U$

(ii) every finite intersection of elements of F contains an element of F.

Observe furthermore that the empty set is not an element of F. (i) and (ii) could be replaced by the single condition:

> every finite intersection of elements of F contains an element of F whose closure is contained in this intersection.

A regular filter base F is called preconvergent if the elements of F, considered as subsets of the space, have a non-empty intersection in this space, and F is called convergent, if this intersection consists of one point (its limit). One readily shows that every F is contained in a maximal one, a regular ultrafilter base.

A convergent regular ultrafilter base apparently consists of all those elements U of the base $\{U\}$ which contain the limit point, that is, all neighborhoods of this point of type U.

A topological space is called cocompact if there exists an open base $\{U\}$ such that either of the following two equivalent conditions holds

(1) every regular filter base relative to $\{U\}$ is preconvergent

(2) every regular ultrafilter base relative to $\{U\}$ is convergent.

The space is then also said to be cocompact relative to $\{U\}$.

A topological space is called countably cocompact if every countable regular filter base (with respect to some suitable open base $\{U\}$) is preconvergent.

A compact Hausdorff space is clearly cocompact relative to any open base of the space. Conversely, if a completely regular space R is cocompact relative to the base of all open sets of the space, it is compact.

Proof. Let $\bar{R}$ be any Hausdorff compactification of R. We will show that $\bar{R} \setminus R$ is empty. If not, take a point $p \in \bar{R} \setminus R$. The family of all sets O open in $\bar{R}$ and containing p is a convergent regular ultrafilter base over the base of open sets of $\bar{R}$. Since R is dense in $\bar{R}$, the intersections $O \cap R$ are non-empty and form a regular filter base of R over the base of all open sets of R. Since R is cocompact the intersection of these filter base elements contains a point of R, but then this must be p, since the intersection of the O is $\{p\}$. This contradicts the fact that $p \in \bar{R} \setminus R$.

Remark. We could have used Cartan's filter bases in the usual sense instead of regular ones. Preconvergence then means that the intersection of the closures of the filter base elements is not empty. The previous result can in this case be improved to: if a regular space is "cocompact" (in this other sense) relative to the base of all open sets, it is compact. Also the main results of this paper remain correct (although the theorems are a little bit weaker). However, to the author it seems for a further development of the notion of cocompactness advisable to adhere to the definitions we have chosen, using regular filter bases.

Every locally compact Hausdorff space H is cocompact relative to a base of open sets with compact closures.

Indeed, the cocompactness follows directly from one of the well known compactness criteria.

The justification of the notion of cocompactness and the reason for its introduction depends partly on the validity of the following theorem and the results in section 3.

Theorem. In a metrizable space the following properties are equivalent:

(1) countable cocompactness

(2) cocompactness

(3) topological completeness (i.e. completeness in some suitable metric compatible with the topology of the space).

Observe that this theorem gives us a topological criterion for topological completeness in metrizable spaces.

We need the following set-theoretic lemma (which may well be known).

Definition. A collection of subsets of a set satisfies the "descending chain condition" if every sequence of elements of the collection which consists of properly decreasing sets, is finite.

Set theoretic lemma. Every cover of a set by a collection of subsets has a subcover satisfying the descending chain condition.

Proof. Well-order the cover $\{C_\alpha\}_{\alpha<\gamma}$.

We define the subset

$$\{C_{\alpha'}\} = \{C_\alpha : C_\alpha \not\subset C_\beta \text{ for all } \beta < \alpha\} . \tag{1}$$

This, indeed, is a subcover, since every element of the set is contained in a C_α with minimal index α, and this C_α obviously belongs to (1).

Proof of theorem.

We will show that for a metrizable space M

$$(3) \Longrightarrow (2) \Longrightarrow (1) \Longrightarrow (3).$$

$(2)\Longrightarrow(1)$ is trivial.

$(3)\Longrightarrow(2)$. Take some metric in which M is complete. The open sets of diameter 1/n (n=1,2,3,...) constitute a base. For each n, the corresponding base elements cover the space and contain, according to the settheoretic lemma, a subcover satisfying the descending chain condition. The collection of the elements of all these subcovers K_n (for all n) constitute again a base K for M, as is easily verified. We show that M is cocompact relative to K.

Suppose some regular filter base in K is not preconvergent. Then there is no minimal element V (a minimal element is one contained in all others) in the filter base. So there is a properly decreasing sequence $\{V_i\}$ of filter base elements. Since the descending chain condition holds for each n^{th} subcover, there are infinitely many V_j no pair of which belongs to the same subcover as mentioned. This means that the diameter of the V_j tend to zero. So, since M is complete, $\bigcap_{j=1}^{\infty} \overline{V}_j$ consists of exactly one point m. Now m belongs to all elements of our filter base, which gives the required contradiction.

Indeed, if $m \notin U$ for some filter base element U, then there is another U' with $\overline{U'} \subset U$ and hence $m \notin \overline{U'}$. Furthermore, there is a V_j sufficiently small with $\overline{U'} \cap V_j = \emptyset$, so $U \cap V_j = \emptyset$ which is impossible in our filter base.

$(1)\Longrightarrow(3)$.

Suppose the metric space M is countably cocompact, relative to an open base $\{U\}$ of M.

Let $\tilde{M}$ be the metric completion of M. Every set U open in M is the intersection with M of a set $\tilde{U}$ open in $\tilde{M}$. Since U is dense in $\tilde{U}$, the diameter of U and $\tilde{U}$ are equal.

Write {U} as the union of a countable number of collections, the i^{th} collection C_i consisting of all those non-empty U whose diameters are $< 1/i$.

The union of the $\tilde{U}$ corresponding to those U for which $U \in C_i$ is a set O_i open in $\tilde{M}$.

We shall prove that

$$M = \bigcap_{i=1}^{\infty} \tilde{O}_i . \qquad (1)$$

This means that M is a G_δ in the complete metric space $\tilde{M}$, so M is topologically complete according to a well known theorem by Alexandroff-Hausdorff.

To prove (1), we notice that

$$M \subset \bigcap_{i=1}^{\infty} \tilde{O}_i$$

is obvious. Indeed, $\tilde{O}_i \cap M = M$, i.e. the union of all those U for which $U \in C_i$ equals M, since those U whose diameter is $< 1/i$ also constitute a base for M. So we must prove $\bigcap_{i=1}^{\infty} \tilde{O}_i \subset M$.

Let $p \in \bigcap_{i=1}^{\infty} \tilde{O}_i$. If $p \in \tilde{O}_i$, then $p \in \tilde{U}_i$ for some fixed $U_i \in C_i$ $(i=1,2,\ldots)$. Since the diameter of the $\tilde{U}_i$ converge to zero if $i \to \infty$ the intersection $\cap \tilde{U}_i = \{p\}$. From $p \in \tilde{U}_i$ and the fact that the diameters of the open $\tilde{U}_i$ converge to zero it follows that there is an index j ($j=j_i$, so depending on i), such that

$$p \in \overline{\tilde{U}}_j \subset \tilde{U}_i .$$

Hence

$$\overline{U_j}^M \subset U_i .$$

Repeating this process we arrive at a subsequence of the $\{U_i\}$, say $\{U_k\}$ for which $\overline{U_{k+1}}^M \subset U_k$. This sequence $\{U_k\}$ is therefore certainly a countable regular filter base. Hence the intersection of the $\overline{U_k}^M$ is non-empty in M, since M is countably cocompact relative to {U} . This intersection is contained in

$$\{p\} = \bigcap_{i=1}^{\infty} \tilde{U}_i ,$$

so it must be p itself. Hence $p \in M$.

[Countable] cocompactness is an invariant for the forming of topological products and topological unions.

Proof. Take a base in each factor relative to which this factor is cocompact and take care that in each of them the whole space occurs as a base element (one can just add the whole space as base element). These bases generate a base of the topological product and one readily shows that the product space is cocompact relative to this base.
The assertion for topological unions is obvious; take a base which is a union of the bases relative to which each of the summands is cocompact.

Remark. Observe that a countably compact space is countably cocompact relative to the base of all open sets, so by the preceding result a product of two countably compact spaces is countably cocompact, though not necessarily countably compact.

3. Generalization of the Baire category theorem

Definition. Let $\underline{m}$ be an infinite cardinal. An (e.g. closed) subset S of a topological space T is called $\underline{m}$-thin, if the intersection of any family of less than $\underline{m}$ open subsets of T is not (fully) contained in S, unless this intersection is empty.
Complementary an (open) set O in T is called $\underline{m}$-puffed, if the intersection of any family of less than $\underline{m}$ open subsets of T meets O, unless this intersection is empty. So the complement of a $\underline{m}$-puffed set is $\underline{m}$-thin and conversely.
First let us observe that a closed subset is $\underline{a}$-thin (where $\underline{a}$ is the cardinal of the set of natural numbers), if and only if it is nowhere dense; hence an open set is a-puffed, if and only if it is everywhere dense. Indeed if the closed set S is $\underline{a}$-thin in T, no non-empty open subset O of T is contained in S, so S has empty interior and is there-

fore nowhere dense. Conversely, if the closed S is nowhere dense, any finite non-empty intersection of open sets is non-empty open, so is not contained in S. Hence S is a-thin.

Remark on existence. A compact Hausdorff space of weight m (the weight is the minimal cardinal of a set constituting an open base of the space) always contains an m-thin subset. Indeed, it is not difficult to prove that such a space contains a closed subset which is not the intersection of less than m open sets containing this subset (that is the "weight" of this closed subset is m itself). On the other hand, since every closed subset is the intersection of at most m open suitable chosen sets, the space does not contain k-shin sets for cardinals k greater than the weight m of the space.

Definition. A space is called an m-Baire space, if it is not the union of at most m closed, m-thin subsets. a-Baire spaces coincide with Baire spaces in the usual sense.

Generalized Baire-theorem

A cocompact regular space is an m-Baire space for every m.

A (countably)cocompact regular space is a Baire space.

Proof. We indicate and well-order a set of m closed, m-thin subsets of the space by

$$A_o, A_1, \ldots\ldots A_\alpha, \ldots\ldots \quad \Big| \; \alpha < \mu ,$$

where μ is the first ordinal of potency m.

Take some non-empty open subset O of the space. We shall prove that O contains a point outside all A_α.

Let the space be cocompact relative to thebase $\{U\}$. Since the space is regular and A_o is m-thin, there is an $U = U_o$ with

$$U_o \subset O , \quad \overline{U}_o \cap A_o = \emptyset .$$

Take some ordinal $\alpha < \mu$. Suppose for all ordinals β with $\beta < \alpha$ we have determined $U_\beta \in \{U\}$ with

$$\overline{U}_{\gamma+1} \subset U_\gamma \ (\gamma+1 \leqslant \beta), \quad \bigcap_{\gamma \leqslant \beta} U_\gamma \neq \emptyset , \quad \overline{U}_\beta \cap A_\beta = \emptyset .$$

If α is a non-limit number, we take as U_α an element from $\{U\}$ which satisfies the conditions

$$\overline{U}_{\gamma+1} \subset U_\gamma \ (\gamma+1 \leqslant \alpha), \quad \bigcap_{\gamma \leqslant \alpha} U_\gamma \neq \emptyset, \quad \overline{U}_\alpha \cap A_\alpha = \emptyset . \qquad (1)$$

Indeed, observe that according to the induction hypothesis $\bigcap_{\gamma \leqslant \alpha-1} U_\gamma \neq \emptyset$. This intersection cannot be contained in A_α since the cardinal of α is smaller than $\underline{m}$ and A_α is $\underline{m}$-thin. So there is a point p in this intersection outside A_α . Now take a $U_\alpha \in \{U\}$ with

$$p \in U_\alpha \subset \overline{U}_\alpha \subset U_{\alpha-1} , \quad \overline{U}_\alpha \cap A_\alpha = \emptyset .$$

This is possible, since the space is regular, A_α is closed and $\{U\}$ is a base. Hence U_α is found such that (1) is satisfied. If α is a limit number we observe that

$$\bigcap_{\gamma \leqslant \beta} U_\gamma \neq \emptyset \qquad (\beta < \alpha).$$

So the $\{U_\gamma\}_{\gamma<\alpha}$ form a regular filter base and the cocompactness of the space gives us

$$\bigcap_{\gamma<\alpha} U_\gamma \neq \emptyset .$$

This intersection cannot be contained in A_α for the same reason as before and we are again able to find a U_α satisfying the conditions (1) as stated. So by transfinite induction we have defined sets U_α for all $\alpha < \mu$, satisfying (1).

The intersection

$$\bigcap_{\alpha < \mu} U_\alpha = \bigcap_{\alpha < \mu} \overline{U}_\alpha$$

cannot be empty, since the space is cocompact. But this intersection is contained in O and not contained in any A_α , which gives the first part of the theorem.

The second part is proved in a completely analogous way using simple induction.

Any open, continuous image of an m-Baire space is an m-Baire space.

Proof. Take a family of m closed, m-thin subsets in the image space. A pre-image of such a subset is closed and m-thin in the domain space since the map is open and continuous. Since the domain space is an m-Baire space, it contains a point not in the union of the m pre-images. So the image point is not in the union of the given m subsets of the image space. Hence the image space is an m-Baire space.

From the results obtained we immediately deduce the following statement.

Corollary. Any topological product or union of completely metrizable spaces or locally compact Hausdorff spaces (e.g. real lines) or more generally of cocompact regular spaces or any open continuous image of such a space is an m-Baire space. So certainly a Baire-space.