STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

WN 19

On topologies congruent with their classes of dense sets

by

George E. Strecker

June 1966

On topologies congruent with their classes of dense sets

In this note a method is described for constructing topological spaces having the property that each non-empty set is open if and only if it is dense (i.e. the collection of non-empty open sets is precisely the same as the collection of dense sets). Such spaces will be called OD spaces.

As is seen below no OD spaces can be Hausdorff. In fact, in a sense they are all extremely non-Hausdorff. Also it is clear from the definition that no OD space has disjoint dense subsets.

In general it is not difficult to construct examples of OD spaces. Clearly a point with the indiscrete topology and the space $(X,\mathcal{T})$ where $X = \{a,b\}$ and $\mathcal{T} = \{\phi, X, \{a\}\}$ are examples. In fact T_0 OD spaces can easily be described on an arbitrary set X. Let p be some distinguished point of X and let

$$\mathcal{T}_1 = \{\phi\} \cup \{A \subset X \mid p \in A\},$$

$$\mathcal{T}_2 = \{X\} \cup \{A \subset X \mid p \notin A\}.$$

Then both $(X,\mathcal{T}_1)$ and $(X,\mathcal{T}_2)$ are T_0 OD spaces.

However, to construct OD spaces with T_1 separation or indeed to determine the existence of such spaces presents a more difficult problem. To do this we first consider a class of spaces more general than the OD spaces.

The following proposition is easily verified.

Proposition 1. In a topological space the following properties are equivalent.

i) every non-empty open set is connected;

ii) every non-empty open set is dense;

iii) every pair of non-empty open sets has a non-empty intersection.

We call a space super-connected provided it satisfies one of the conditions of proposition 1. The co-finite topology on a given set is a well known example of a super-connected space.

The following three propositions are obvious.

Proposition 2. Every OD space is super-connected.

Proposition 3. Every super-connected space is both connected and locally connected.

Proposition 4. No super-connected space is Hausdorff.

It is known that super-connected spaces exist in profusion. In fact each non-compact Hausdorff k-space has an associated T_1 super-connected space which determines it and is determined by it (cf. Epicompactness and anti-spaces, Report WN 20, Math. Centrum Amsterdam, June 1966). Below, a method is given for modifying any given T_1 super-connected space so that it becomes an OD space.

Notation. If $\mathcal{S}$ is a collection of subsets of X, then $\mathcal{S}^{\nabla}$ will denote the topology generated by $\mathcal{S}$ (considered as an open subbase).

Theorem 1. For every T_1 super-connected space, $(X,\mathcal{T})$, there exists an OD space $(X,\mathcal{T}^*)$ such that $\mathcal{T}^*$ is finer (stronger) than $\mathcal{T}$.

Proof. Suppose that $(X,\mathcal{T})$ is super-connected. Let $\mathcal{D}$ be the collection of all $\mathcal{T}$-dense subsets of X. Well order $\mathcal{D}$; i.e. let $\mathcal{D} = \{D_\alpha \mid \alpha < \gamma\}$ where each α is an ordinal number greater than 0. Let $\mathcal{T}_0 = \mathcal{T}$. Proceeding by induction, suppose that α is an ordinal number such that $0 < \alpha < \gamma$ and such that for all $\beta < \alpha$, $\mathcal{T}_\beta$ has been defined. We define $\mathcal{T}_\alpha$ as follows:

$$\mathcal{T}_\alpha = \begin{cases} \left(\bigcup_{\beta<\alpha} \mathcal{T}_\beta \cup \{D_\alpha\}\right)^{\nabla} & \text{if this topology has no isolated points and if } D_\alpha \text{ is dense in } \mathcal{T}_\beta \text{ for all } \beta < \alpha. \\ \left(\bigcup_{\beta<\alpha} \mathcal{T}_\beta\right)^{\nabla} & \text{otherwise.} \end{cases}$$

In this manner $\mathcal{T}_\alpha$ is determined for all $\alpha < \gamma$. Let $\mathcal{T}^* = \left(\bigcup_{\alpha<\gamma} \mathcal{T}_\alpha\right)^{\nabla}$. We claim that $(X,\mathcal{T}^*)$ is an OD space.

Suppose that B is dense in $(X,\mathcal{T}^*)$. Since $\mathcal{T}^*$ is finer than $\mathcal{T}$, B is dense in $(X,\mathcal{T})$. Thus $B \in \mathcal{D}$, so that $B = D_\alpha$ for some $\alpha < \gamma$. Suppose that $D_\alpha \in \mathcal{T}_\alpha$. Then since $\mathcal{T}_\alpha$ is coarser than $\mathcal{T}^*$, we have $B \in \mathcal{T}^*$; i.e. B is $\mathcal{T}^*$-open.

If $D_\alpha \notin \mathcal{T}_\alpha$, then either

a) D_α is not dense in $\mathcal{T}_\beta$ for some $\beta < \alpha$,

or

b) $(\bigcup_{\beta<\alpha} \mathcal{T}_\beta \cup \{D_\alpha\})^\nabla$ has an isolated point.

<u>case a)</u> Since D_α is $\mathcal{T}^*$-dense it must be $\mathcal{T}_\beta$-dense for all $\beta < \gamma$, since each $\mathcal{T}_\beta$ is coarser than $\mathcal{T}^*$. Thus this case cannot occur.

<u>case b)</u> In this case there must exist some $\beta' < \alpha$ and some $U \in \mathcal{T}_{\beta'}$ such that $U \cap D_\alpha = \{p\}$. But since $\mathcal{T}$ is T_1 and $\mathcal{T}_{\beta'}$ is finer than $\mathcal{T}$, $\mathcal{T}_{\beta'}$ will be T_1. Thus $(U - \{p\}) \in \mathcal{T}_{\beta'}$. But $(U - \{p\}) \cap D_\alpha = \emptyset$, so that D_α is not dense in $\mathcal{T}_{\beta'}$ which, as shown in case a), is not possible. Hence we have shown that every $\mathcal{T}^*$-dense set is $\mathcal{T}^*$-open.

To show that every $\mathcal{T}^*$-open set is $\mathcal{T}^*$-dense, we assume that

$$G = U \cap D_{\alpha_1} \cap \dots \cap D_{\alpha_n} \quad \text{and}$$

$$H = V \cap D_{\beta_1} \cap \dots \cap D_{\beta_m}$$

are non-empty basic open members of $\mathcal{T}^*$; i.e. $U,V \in \mathcal{T}$ and D_{α_i}, $D_{\beta_j} \in \mathcal{D}$, $1 \le \alpha_i \le n$, $1 \le \beta_j \le m$. Then $G \cap H = U \cap V \cap D_{\delta_1} \cap D_{\delta_2} \cap \dots \cap D_{\delta_k}$, where the subscripts of the D's are arranged such that $\delta_i < \delta_j$ if $i < j$, $1 \le i,j \le k$.

Since $\mathcal{T}$ is super-connected, $U,V \in \mathcal{T}$ implies that $U \cap V$ is $\mathcal{T}$-dense and open. But $D_{\delta_1} \in \mathcal{D}$ implies that D_{δ_1} is $\mathcal{T}$-dense. Thus $U \cap V \cap D_{\delta_1}$ is $\mathcal{T}$-dense and $\mathcal{T}_{\delta_1}$-open. Now consider D_{δ_2}. Since $\delta_1 < \delta_2$, by the definition of $\mathcal{T}_{\delta_2}$, D_{δ_2} is $\mathcal{T}_{\delta_1}$-dense. Thus $(U \cap V \cap D_{\delta_1}) \cap D_{\delta_2} \neq \emptyset$ and is, in fact, a member of $\mathcal{T}_{\delta_2}$. Continuing in this manner for a finite number of steps, we find that $G \cap H \neq \emptyset$. Hence by proposition 1 every $\mathcal{T}^*$-open set is $\mathcal{T}^*$-dense.

Corollary 1. There exist T_1 OD spaces.

Corollary 2. There exist connected, locally connected T_1 spaces which do not possess disjoint dense sets.