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Properties that are closely related to compactness

by

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Introduction

Let $\mathcal{P}$ be some topological property defined on a suitable class of topological spaces. (e.g. the class of completely regular spaces).

In category theory it is natural to ask, whether there exists a functor γ from the category $\underline{C}$ of spaces in consideration, to the subcategory $\underline{D}$ of spaces satisfying $\mathcal{P}$, which is left-adjoined to the forget-full functor from $\underline{D}$ to $\underline{C}$. In topology this means that we ask whether it is possible to embed an arbitrary space X (of the class in consideration) densely in a space γX satisfying $\mathcal{P}$, such that each continuous map of X into any space Y satisfying $\mathcal{P}$ has a continuous extension which carries γX into Y. Now take for $\mathcal{P}$ the compactnessproperty and for $\underline{C}$ the class of completely regular spaces, then every space X has indeed such a "maximal $\mathcal{P}$-fication" = maximal compactification. Its name is βX, the Čech-Stone compactification of X and the functor in consideration is β.

The question is: how can we characterize all the properties that admit maximal $\mathcal{P}$-fications? It turns out (this is the main result of section 1) that every space has a maximal $\mathcal{P}$-fication if and only if $\mathcal{P}$ is closed-hereditary, productive and almost fitting. (for definitions cf. §1 of these notes).

In section 2 we define for each cardinal number $\underline{m}$ the property $\underline{m}$-ultracompactness which satisfies this maximility condition. If $\underline{m}$ is a finite cardinal number, then $\underline{m}$-ultracompactness coincides with compactness; for $\underline{m} = \aleph_0$ it is closely related to realcompactness (cf. [1] or §3 for the definition of realcompactness).

Section 4 is devoted to the study of a generalized notion of the Lindelöfproperty: A space is called a generalized Lindelöfspace provided that there exists a subbase for its topology such that each open cover of it by members of the subbase has a countable subcover.

§1. Almost-fitting properties, maximal embedding.

Until explicitly stated, all spaces in consideration are completely regular. Bold face letters stand for cardinals. $\aleph_0$ stands for the cardinalnumber of a countable set, $\mathbf{c}$ denotes the cardinal of the continuum. If $\mathcal{U}$ is a family of subsets of a space X, then the symbol $\bar{\mathcal{U}}^X$, or simply $\bar{\mathcal{U}}$ will be used to denote the family of all $\bar{U}^X$ for which $U \in \mathcal{U}$. The union and intersection of a family of sets $\mathcal{U}$ will be denoted by $\cup\mathcal{U}$ or $\cap\mathcal{U}$ respectively.

(1.1) Conventions. Let $\mathcal{P}$ be a topological property defined on the class of completely regular spaces.

$\mathcal{P}$ is called productive or sometimes arbitrary productive if the product of an arbitrary collection of spaces enjoying $\mathcal{P}$, has property $\mathcal{P}$.

$\mathcal{P}$ is called countable productive (respectively finite productive) if the product of a countable (respectively finite) collection of spaces enjoying $\mathcal{P}$ has property $\mathcal{P}$.

$\mathcal{P}$ is called hereditary (respectively closed-hereditary) if every subspace (respectively closed subspace) of a space satisfying $\mathcal{P}$, has property $\mathcal{P}$.

$\mathcal{P}$ is called almost-fitting property, if whenever f is a perfect [1] map of a space X onto a space Y, then X has property $\mathcal{P}$ if Y has property $\mathcal{P}$.

$\mathcal{P}$ is called a fitting property, if whenever f is a perfect map of a space X onto a space Y, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.

Compactness and realcompactness[2] are examples of properties that are almost fitting; closed-hereditary and productive.

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1) A mapping f of a space X into a space Y will be called perfect if f is continuous, closed (the images of closed sets are closed) and the inverses of points are compact.

2) For the definition of realcompactness cf [1] or section 3 of these notes.

Local compactness, σ-compactness, countable compactness, paracompactness, countable paracompactness, Čech-completeness are examples of properties that are fitting and closed-hereditary. (but not productive). Each of these properties at infinity is also a fitting property which is closed-hereditary . For further information we refer to [2].

If a topological space X is densely embedded in a space γX with property $\mathcal{P}$ then we call γX a <u>$\mathcal{P}$-fication</u> of X.
Sometimes γX is of the type that to each continuous mapping f of X into any space Y with property $\mathcal{P}$, we can find a continuous extension of f which carries γX into Y. γX is then said to be a <u>maximal</u> $\mathcal{P}$-fication of X.

If to every space X it is possible to find a maximal $\mathcal{P}$-fication γX of X, then γX is uniquely determined to X.
Indeed, if γX and δX are two maximal $\mathcal{P}$-fications of X, then the identity mappings $i: X \to \gamma X$ and $j: X \to \delta X$ have continuous extensions $\tilde{i}$ and $\tilde{j}$ to all of δX and γX respectively. $\tilde{j} \circ \tilde{i} : \delta X \to \delta X$ takes the dense subset X fixed and is consequently the identity map of δX onto δX (recall that two continuous mappings f and g defined on a Hausdorff-space X with range Y coincide, if they coincide on a dense subset of X). Similarly we show that $\tilde{i} \circ \tilde{j}$ is the identity map of γX onto γX. Consequently γX and δX are topologically equivalent.

Compactness and realcompactness are indisputable the most interesting properties having the property that every (completely regular) space admits a maximal $\mathcal{P}$-fication (respectively called Čech-Stone compactification and Hewitt realcompactification).
With this in mind it seems to be of minor importance to look for all the other properties possessing this feature.
But the following theorem which is the main result of this section actually shows that these properties are most familiar to us.

Main result of §1.

If $\mathcal{P}$ is a topological property which is possessed by some non-empty space, then the following statements are equivalent.

(a) Every space has a maximal $\mathcal{P}$-fication.

(b) $\mathcal{P}$ is almost-fitting, closed-hereditary and productive.

Before we attack the proof, we give some preliminary results which are of interest in itself.

(1.2) Lemma. If ϕ is a continuous map of a space Y into a space Z into a space Z, whose restriction to a dense set X is a homeomorphism, then ϕ carries $Y\setminus X$, into $Z\setminus\phi(X)$.

Proof. Suppose on the contrary $\phi(p)\in\phi X$ and $p \in Y \setminus X$. Let $X' = X\cup\{p\}$. The restriction map $\phi|X$ has an inverse $\psi : \phi X \to X$ which is continuous. Consequently $\psi \circ \phi|X'$ is a continuous mapping from X' into X' whose restriction to the dense set X is the identity on X.

X' is a Hausdorffspace, hence it follows that $\psi \circ \phi|X'$ is the identity map of X' onto X'. In particular we have $\psi(\phi(p)) = p$, contradicting $p \in Y\setminus X$.

(1.3) Lemma. If τ is a perfect map of a space X onto a space Y and $\bar{\tau}$ is the extension of τ which carries βX[1] onto βY, then $\bar{\tau}(\beta X\setminus X) = \beta Y\setminus Y$. For the proof we refer to [2] p. 87 Lemma 1.5.

(1.4) Lemma. Let $\mathcal{P}$ be a topological property which is productive and closed-hereditary.

If Z is a space and $\{X_\alpha|\alpha\in A\}$ is a collection of subspaces with property $\mathcal{P}$ then $X = \cap\{X_\alpha|\alpha\in A\}$ satisfies property $\mathcal{P}$.

An analogous result is obtained for properties that are only countably or even finite productive.

1) βX denotes as usual the Čech-Stone compactification of X.

Proof. Let $Y = \pi\{ X_\alpha | \alpha \in A\}$, and $\Delta \subset Y$ given by $\Delta = \{x=(x_\alpha)\in Y | x_{\alpha_1} = x_{\alpha_2}\ \forall \alpha_1, \alpha_2 \in A\}$.
It is not hard to see that X is homeomorphic to the subspace Δ. Since $\mathcal{P}$ is topological it remains to show that Δ has property $\mathcal{P}$.
Y has property $\mathcal{P}$ since each X_α has property $\mathcal{P}$ and $\mathcal{P}$ is productive.
Δ is a closed subset of Y because each X_α is a Hausdorffspace. Hence Δ has property $\mathcal{P}$ since $\mathcal{P}$ is closed-hereditary.

(1.5) Corollary. If a topological property $\mathcal{P}$ is closed-hereditary, productive and an invariant for the taking of open subsets, then $\mathcal{P}$ is hereditary.
Indeed, if Y is a space having $\mathcal{P}$ and $X \subset Y$ then $X = \cap\{Y\setminus\{p\} | p\in Y\setminus X\}$ i.e. X is intersection of open subsets of Y. By assumption each open subset of Y has $\mathcal{P}$ and the preceding lemma yields that every intersection of spaces enjoying $\mathcal{P}$ has $\mathcal{P}$. Consequently X has property $\mathcal{P}$.
This corollary can serve as a test to decide whether some property is inherited by open subsets, closed subsets or (arbitrary) topological products.
For instance it is easy to see that the property k[1)] is an invariant for the taking of open and closed subsets. Since the property k is not hereditary the above result shows that the property k is not productive.

Next we observe that if $\mathcal{P}$ is a property which is closed-hereditary, almost-fitting and is possessed by some nonempty space, then $\mathcal{P}$ is possessed by all compact spaces. For if Y is any nonempty space satisfying $\mathcal{P}$, and C is any compact space, then the image of the topological sum X of C and Y under the perfect mapping f that coincides with the identity on Y and sends C to some fixed point of Y is Y. f is almost-fitting and closed-hereditary so X and hence C has property $\mathcal{P}$.

1) A space X has property k provided that a subset is closed if it has a compact intersection with each compact subspace of X.

Proof of the main result.

(a) $\Rightarrow$(b). Let $\mathcal{P}$ be a property such that every space has a maximal $\mathcal{P}$-fication. We will show that $\mathcal{P}$ satisfies the desired invariance properties.

Let $\{X_\alpha | \alpha \in A\}$ be a collection of spaces enjoying $\mathcal{P}$ and $X = \pi\{X_\alpha | \alpha \in A\}$. Each projection map $\pi_\alpha : X \to X_\alpha$ has a continuous extension $\pi_\alpha^* : \gamma X \to X_\alpha$. Let $i^* : \gamma X \to X$ be defined by the conditions $(i^*(x))_\alpha = \pi_\alpha^*(x)$ $(\alpha \in A)$. i^* is the identity on X, so we have by (1.2) that $\gamma X \setminus X = \phi$ i.e. $\gamma X = X$. Consequently X has property $\mathcal{P}$.

Let X be a closed subset of a space Y satisfying $\mathcal{P}$. The inclusion map of X into Y has a continuous extension i^* of γX into Y. By (1.2) the preimage of the closed set X under i^* is X; hence X is closed in γX i.e. $\gamma X = X$. It follows that X has property $\mathcal{P}$.

Now let τ be any perfect mapping of a space X onto a space Y satisfying $\mathcal{P}$. Consider the extension $\tilde{\tau}$ of τ which carries βX (zie 1) pag 4)) onto βY. Sine Y has $\mathcal{P}$, there is an extension $(\tilde{\tau}|_X)^*$ of $\tilde{\tau}|_X$ which carries γX into Y.

The inclusion map $i: X \to \beta X$ has a continuous extension i^* which carries γX into $\gamma \beta X = \beta X$.

We have $\tilde{\tau}|_X = \tilde{\tau} \circ i$, so $(\tilde{\tau}|_X)^* = \tilde{\tau} \circ i^*$ (by uniqueness of continuation). By (1.2) $i^*(\gamma X \setminus X) \subset \beta X \setminus X$ and by (1.3) $(\tilde{\tau} \circ i^*)(\gamma X \setminus X) \subset \beta Y \setminus Y$. But $(\tilde{\tau}|_X)^*$ $(\gamma X \setminus X) \subset Y$ i.e. $\gamma X \setminus X = \phi$. Consequently X satisfies $\mathcal{P}$.

(b) $\Rightarrow$(a). Let $\mathcal{P}$ possess the already cited invariances; let X be a space and βX its Čech-Stone compactification.

Consider for each continuous mapping f which sends X onto a subset of a space Y satisfying $\mathcal{P}$, the extension $\tilde{f}$ of f which carries βX onto βY, and set $X(Y,f) = \tilde{f}^{-1}(Y)$. For each space X(Y,f) the restriction map $\tilde{f}|_{X(Y,f)}$ is a perfect mapping from X(Y,f) onto Y. (remark that this mapping is the restriction of a perfect mapping to a total inverse). Consequently every space X(Y,f) satisfies $\mathcal{P}$ because $\mathcal{P}$ is an almost-fitting property. Now let $\gamma X = \bigcap\{X(Y,f) | Y$ satisfies $\mathcal{P}$; $f: X \to Y$ continuous; fX dense in Y$\}$.

X is clearly densely embedded in γX moreover (1.4) shows that γX is even a $\mathcal{P}$-fication of X.

We shall prove that γX is a maximal $\mathcal{P}$-fication. If g is any continuous mapping from X into a space Z satisfying $\mathcal{P}$, then let Z' be the closure of gX in Z. Z' satisfies $\mathcal{P}$ since $\mathcal{P}$ is closed-hereditary.
Now we have $\gamma X \subset X(Z',g)$ (g considered a a mapping of X into Z') and $\tilde{g}|\gamma X : \gamma X \to Z' \subset Z$ is a continuous extension of g which carries γX into Z.

The following proposition gives us a simple criterium to decide whether some property is closed-hereditary or not:

(1.7) Proposition. Let $\mathcal{P}$ be a topological property which is possessed by all compact spaces. If $\mathcal{P}$ is inherited by intersections of two subspaces one of which is compact and the other satisfying $\mathcal{P}$, then $\mathcal{P}$ is closed-hereditary.

Proof. Let Y be a space satisfying $\mathcal{P}$ and X a closed subset of Y. Let δY be a compact extension of Y. $\bar{X}^{\delta Y}$ and Y are subspaces of δY both satisfying $\mathcal{P}$ while $\bar{X}^{\delta Y}$ is compact. Hence their intersection which equals X has $\mathcal{P}$.
There exists also a criterium to decide whether some property is productive or not. It is a "generalisation" of the Tychonoff product theorem.

(1.8) Proposition. Let $\mathcal{P}$ be an almost-fitting property which is an invariant of $\underline{m}$[1] intersections (i.e. each intersection of a family of cardinal $\leq \underline{m}$ of subspaces satisfying $\mathcal{P}$ has $\mathcal{P}$). Then every product of $\underline{m}$ spaces enjoying $\mathcal{P}$, has $\mathcal{P}$.

Proof. Choose an indexset A with cardinal $\underline{m}$. Let $\{X_\alpha | \alpha \in A\}$ be a collection of spaces satisfying $\mathcal{P}$ and $X = \pi\{X_\alpha | \alpha \in A\}$. Each projection map π_α of X onto X_α has a continuous extension $\bar{\pi}_\alpha$ which carries βX into βX_α. For $\alpha \in A$ set $X(\alpha) = \bar{\pi}_\alpha^{-1}(X_\alpha)$. Each $X(\alpha)$ has property $\mathcal{P}$ since $\bar{\pi}_\alpha|_{X(\alpha)}$ is a perfect mapping of $X(\alpha)$ onto X_α,

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1) $\underline{m}$ denoting a finite or an infinite cardinal number.

By assumption $X' = \cap\{X(\alpha)|\alpha\in A\}$ has property $\mathcal{P}$. But X is densely embedded in X' and the mapping $\tilde{i} : X' \to X$ defined by the conditions $(\tilde{i}(x))_\alpha = \tilde{\pi}_\alpha(x)$ $(\alpha \in A)$ is a continuous mapping which is the identity on X. Consequently it follows from (1.2) that X' = X i.e. X has property $\mathcal{P}$.

From (1.4), (1.7) and (1.8) we obtain:
(1.9) Theorem. For an almost-fitting property $\mathcal{P}$ the following conditions are equivalent.

(I). $\mathcal{P}$ is an invariant for the taking of arbitrary intersections and each compact space has $\mathcal{P}$.

(II). $\mathcal{P}$ is closed-hereditary and arbitrary productive.

The equivalence between (I) and (II) remains satisfied if we replace "arbitrary" by "countable" or "finite".

§2. Examples; the notion m-ultracompact.

We are dealing with the following problem: are there "enough" almost-fitting properties that are closed-hereditary and productive? The theory above would obviously be not succesful if real-compactness and compactness were the only candidates.

Definition. A family of subsets of a topological space X has the m-intersection property (m finite or infinite cardinal number) provided that every subcollection of cardinal $\leq$ m has a nonempty intersection.

Definition. An ultrafilter $\mathcal{F}$ in X is said to be an m-ultrafilter if the closed sets of X that are members of $\mathcal{F}$, satisfy the m-intersection property.

Definition. A space X is called $\underline{m}$-ultracompact provided that every $\underline{m}$-ultrafilter in X is covergent.

Obviously compact implies $\underline{m}$-ultracompact for every $\underline{m}$; if $\underline{n} \leq \underline{m}$ then $\underline{n}$-ultracompact implies $\underline{m}$-ultracompact.
It is also easy to see that if X has the Lindelöfproperty then X is $\aleph_0$-ultracompact. The connection between $\aleph_0$-ultracompactness and real-compactness is considered in the next section.

(2.1) Lemma. Let $\mathfrak{F}$ be an $\underline{m}$-ultrafilter in a space X and $f : X \to Y$ a continuous mapping. The collection $\mathfrak{G} = \{f(F) \mid F \in \mathfrak{F}\}$ constitutes a base for an $\underline{m}$-ultrafilter in Y.

Proof. A well known argument shows that $\mathfrak{G}$ is base for an ultrafilter $\mathfrak{G}'$ in X. Let $\{S_\alpha \mid \alpha \in A\}$ be a family of closed sets of $\mathfrak{G}'$ with cardinal $\leq \underline{m}$. Clearly every S_α intersects every $f(F)$ $(F \in \mathfrak{F})$. Consequently every $f^{-1}(S_\alpha)$ $(\alpha \in A)$ is a closed subset of X and meets every member of $\mathfrak{F}$. Hence, since $\mathfrak{F}$ is an $\underline{m}$-ultrafilter, $\{f^{-1}(S_\alpha) \mid \alpha \in A\}$ is a subcollection of $\mathfrak{F}$ and $\cap\{f^{-1}(S_\alpha) \mid \alpha \in A\} \neq \phi$. It follows that $\{S_\alpha \mid \alpha \in A\}$ has non-empty intersection.

(2.2) Theorem. The property $\underline{m}$-ultracompact is closed-hereditary and productive for every $\underline{m}$. Moreover $\underline{m}$-ultracompactness is an almost-fitting property (we shall see in the next section that $\underline{m}$-ultracompactness is even a fitting property).

Proof. Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of $\underline{m}$-ultracompact spaces and $X = \pi\{X_\alpha \mid \alpha \in A\}$. Take an $\underline{m}$-ultrafilter $\mathfrak{F}$ in X and let for $\alpha \in A$ $\mathfrak{F}_\alpha = \{\pi_\alpha F \mid F \in \mathfrak{F}\}$. By the previous lemma, each $\mathfrak{F}_\alpha$ is base for an $\underline{m}$-ultrafilter in X_α which is convergent to a point p_α in X_α.
Let p be the point of X whose α'th coordinate is p_α. A well known argument shows that p is limitpoint of $\mathfrak{F}$, i.e. $\mathfrak{F}$ is convergent (since $\mathfrak{F}$ is an ultrafilter).

Now let f be any perfect mapping of X into Y and suppose Y is $\underline{m}$-ultracompact. We will show that X is necessarily $\underline{m}$-ultracompact; hence it follows that $\mathcal{P}$ is almost-fitting and closed-hereditary. (recall that if X is closed in Y, then the inclusion map $f : X \to Y$ is perfect). Take any $\underline{m}$-ultrafilter $\mathcal{F}$ in X. The preceding lemma shows that $\mathcal{G} = \{f(F) | F \in \mathcal{F}\}$ is base for an $\underline{m}$-ultrafilter $\mathcal{G}'$ in Y which is convergent, say to $p \in Y$. f is a closed mapping, so we have $p \in \cap\{\overline{f(F)} | F \in \mathcal{F}\}$ = $= \cap\{f(\bar{F}) | F \in \mathcal{F}\}$. Clearly $\{f^{-1}(p) \cap \bar{F} | F \in \mathcal{F}\}$ satisfies the finite intersection property, and compactness of $f^{-1}(p)$ yields $\cap\{f^{-1}(p) \cap \bar{F} | F \in \mathcal{F}\} \neq \phi$. Consequently $\cap\{\bar{F} | F \in \mathcal{F}\} \neq \phi$ i.e. $\mathcal{F}$ has a limit point in X.

<u>(2.3)Lemma.</u> If X is an $\underline{m}$-ultracompact space and every open cover of it of cardinal $\leq \underline{m}$ has a finite subcover, then X is compact.

<u>Proof.</u> Let $\mathcal{F}$ be an arbitrary ultrafilter in X. Clearly the family of closed subsets of X that are members of $\mathcal{F}$ satisfy the $\underline{m}$-intersection property (otherwise their complements would constitute at least one open cover with cardinal $\leq \underline{m}$ that has no finite subcover). $\underline{m}$-ultracompactness of X now yields that $\mathcal{F}$ is convergent. Consequently each ultrafilter in X is convergent i.e. X is compact.

In particular it follows that a topological space X is compact if X is $\aleph_0$- ultracompact and countably compact. Actually a stronger result is true: X is compact <=> X is pseudocompact and realcompact.

<u>(2.4) Theorem.</u> For each (infinite) cardinal number $\underline{m}$ there exists a normal space X which is $\underline{m}$-ultracompact but not $\underline{n}$-ultracompact for $\underline{n} < \underline{m}$.

<u>Proof.</u> We may suppose $\underline{m} > \aleph_0$.
Let α be the smallest ordinal number of potency $\underline{m}$. Let $W = \{\xi$ ordinal $|\xi<\alpha\}$ and $W^* = \{\xi | \xi \leq \alpha\}$ be supplied with the usual order topology. W is $\underline{m}$-ultracompact. For, since βW is homeomorphic to W^*, and ultrafilter $\mathcal{F}$ in W that has no limit point in W must contain the $\underline{m}$ sets $F_\beta =$ $= \{\xi \in W | \xi \geq \beta\}$ $(\beta < \alpha)$. Since $\cap\{F_\beta | \beta<\alpha\} = \phi$ $\mathcal{F}$ cannot be an $\underline{m}$-ultrafilter.

If $\underline{n} < \underline{m}$, then W is not $\underline{n}$-ultracompact. Indeed $\underline{n}$-ultracompactness would together with the fact that every open cover of W of cardinal $\leq n$ has a finite subcover, disprove (2.3).

§3. Relationship between $\aleph_0$-ultracompactness and realcompactness

Definition. A space X is called realcompact provided that every maximal centered family of zerosets[1] which satisfies the countable intersection property has non empty intersection.

It is well known (cf. [1]) that a space X is realcompact if and only if X is homeomorphic to a closed subset of a product of real lines. Consequently the property realcompactness is closed-hereditary and productive.

The notion, realcompact is closely related to the notion $\aleph_0$-ultracompact. It appears that these concepts coincide for countably paracompact normal spaces:

(3.1) Theorem. Every realcompact space is $\aleph_0$-ultracompact[2]. Every countably paracompact normal $\aleph_0$-ultracompact space is real-compact[3].

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[1] $Z \subset X$ is a zeroset in X if there exists a real-valued continuous function f on X such that $Z = \{x \in X \mid f(x) = 0\}$.

[2] The Tychonoffplank is an example of a space which is $\aleph_0$-ultracompact but not realcompact.

[3] Added in the proof: Using a result of Frolik (cf. [3]) I can prove that (3.1) remains true for normal spaces. The same holds for (4.3).

Proof. The first statement is clear since we know that $\mathcal{S}_0$-ultracompactness is closed-hereditary and productive whence it follows that a closed subset of a product of real lines is $\mathcal{S}_0$-ultracompact.
The second statement is settled in the following non trivial lemma.

Lemma. Let $\delta = \{\mathcal{U}\}$ be the family of all countable open covers of a space X. For each $\mathcal{U}\in\delta$ and member U of $\mathcal{U}$, set $U^* = \beta X\setminus\overline{X\setminus U}^{\beta X}$ and $\mathcal{U}^* = \{U^* | U\in\mathcal{U}\}$.

(1) If X is a countably paracompact normal space then

$$\bigcap\{\bigcup\mathcal{U}^* | \mathcal{U}\in\delta\} = \bigcap\{\overline{\bigcup\mathcal{U}^*}^{\beta X} | \mathcal{U}\in\delta\}$$

(2) If X is an $\mathcal{S}_0$-ultracompact space then

$$X = \bigcap\{\bigcup\mathcal{U}^* | \mathcal{U}\in\delta\}$$

If in particular X is $\mathcal{S}_0$-ultracompact and countably paracompact normal it follows that X, being intersection of σ-compact subsets of βX, is realcompact.

Proof(1). Let X' denotes the left-hand side of (1) and X" the right side. It is obvious that $X' \subset X''$. To prove $X'' \subset X'$ let $\mathcal{U}$ be an arbitrary fixed countable open cover of X. Choose a countable open cover $\mathcal{V}$ such that for each V in $\mathcal{V}$ there exists $U\in\mathcal{U}$ such that V and $X\setminus U$ are completely separated.
Such a cover exists because X is countably paracompact and normal.
Clearly $\overline{\bigcup\mathcal{V}}^{\beta X} \subset \bigcup\mathcal{U}^*$ and we have proved $X'' \subset X'$.
Proof of (2). Let us suppose that there exists a point $q\in X'\setminus X$. Let $\mathcal{F}$ be the collection of all open neighbourhoods of q in βX. Put $\mathcal{G} =$
$= \{F\cap X | F\in\mathcal{F}\}$.
It is easy to see that $\mathcal{G}$ is a filterbase in X and hence contained in some ultrafilter $\mathcal{G}'$. We will first show that $\mathcal{G}'$ is an $\mathcal{S}_0$-ultrafilter in X. Indeed, if $\mathfrak{S}$ is a countable family of closed members of $\mathcal{G}'$ with empty intersection then the collection $\{X\setminus S | S\in\mathfrak{S}\}$ is a countable open cover of X and hence there exists $S\in\mathfrak{S}$ such that $q\in(X\setminus S)^*$. Consequently $X\setminus S\in\mathcal{G}\subset\mathcal{G}'$.

But $X\setminus S$ and S are two members of the filter $\mathcal{G}'$ with empty intersection. This is impossible and we conclude that $\mathcal{G}'$ is an $\mathfrak{H}_o$-ultrafilter in X. Now the $\mathfrak{H}_o$-ultracompactness of X yields $\cap\overline{\mathcal{G}'}^X \neq \phi$. In particular it is possible to choose a point $r\in X$ such that $r\in\cap\overline{\mathcal{G}}^X$.
But $\{q\} = \cap\overline{\mathcal{G}}^{\beta X}$ and clearly $\cap\overline{\mathcal{G}}^X = \cap\overline{\mathcal{G}}^{\beta X}\cap X = \phi$. This is a contradiction and it completes the proof of (2).

<u>(3.2) Theorem.</u> For a countably paracompact normal space X, the Hewitt-realcompactification νX satisfies:

(α) If a countable family of closed subsets of X has empty intersection in X then their closures in νX have empty intersection in νX.

<u>Proof.</u> Use the same notation as in the proof of the preceding lemma. Take any countable collection $\mathcal{S}$ of closed subsets of X with empty intersection. $\{X\setminus S \mid S\in\mathcal{S}\}$ is a countable open cover of X and consequently $\cup\{(X\setminus S)^* \mid S\in\mathcal{S}\}\supset X'$. By the very definition of the $*$ operator we have for $S\in\mathcal{S}$ $(X\setminus S)^* = \beta X\setminus\overline{S}^{\beta X}$. Hence $\cap\overline{\mathcal{S}}^{\beta X}\cap X' = \phi$ i.e. $\cap\overline{\mathcal{S}}^{X'} = \phi$.
The proof of the theorem is complete when we have shown that X' is the Hewittrealcompactification of X.
Indeed, it is evident that X' is a realcompactification of X since by (3.1) X' is the intersection of σ-compact subspaces of βX and X is densely embedded in X'.
Moreover we have proved that X' is a realcompactification of X with the following property.
(β) If a countable family of zerosets of X has empty intersection then their closures in X' have empty intersection.
Hence by the characterization of the Hewitt-realcompactification νX (cf. [1]) we actually have $\nu X = X'$.

<u>(3.3) Theorem.</u> The image under a perfect map of a countably paracompact normal realcompact space is countably paracompact normal and realcompact.

Since the perfect image of a countably paracompact normal space is countably paracompact normal this theorem is by virtue of (3.1) a direct consequence of the following more general result.

Every space Y which is the perfect f-image of some $\mathfrak{S}_0$-ultracompact space X is $\mathfrak{S}_0$-ultracompact.

<u>Proof</u>. Let $\mathfrak{F}$ be an arbitrary $\mathfrak{S}_0$-ultrafilter in Y and $\mathfrak{G}$ an ultrafilter in X which contains the family $f^{-1}(\mathfrak{F}) = \{f^{-1}(F) \mid F \in \mathfrak{F}\}$.
We shall first prove that $\mathfrak{G}$ is an $\mathfrak{S}_0$-ultrafilter in X. Let us suppose that there exists a countable family $\mathfrak{S}$ of closed members of $\mathfrak{G}$ with empty intersection. Without lost of generality we may suppose that $\mathfrak{S}$ is closed under finite intersections. The members of $f(\mathfrak{S}) = \{f(S) \mid S \in \mathfrak{S}\}$ are closed subsets of Y and they intersect each member of $\mathfrak{F}$. Consequently $f(\mathfrak{S}) \subset \mathfrak{F}$ and we are able to choose $p \in \cap f(\mathfrak{S})$ since $\mathfrak{F}$ is an $\mathfrak{S}_0$-ultrafilter in Y. Now $\{f^{-1}(p) \cap S \mid S \in \mathfrak{S}\}$ is a centered system in X and compactness of $f^{-1}(p)$ yields $\cap\{f^{-1}(p) \cap S \mid S \in \mathfrak{S}\} \neq \phi$. Hence $\cap\mathfrak{S} \neq \phi$. The space X being $\mathfrak{S}_0$-ultracompact, we have $\cap \overline{\mathfrak{G}}^X \neq \phi$, and in consequence $\cap \overline{\mathfrak{F}}^Y \neq \phi$.

<u>Note</u>. (3.3) is not a new result. Frolik proved this theorem in [3] using the notion almost-realcompactness which is somewhat weaker than $\mathfrak{S}_0$-ultracompactness.
Frolik showed that each almost realcompact normal space is realcompact. (However there is an incorrect proof in [3]).

<u>§4. Generalized Lindelöfspace.</u>

<u>Definition</u>. A family $\mathfrak{B}$ of subsets of a topological space satisfies the Lindelöfproperty provided that every cover of it by members of $\mathfrak{B}$ has a countable subcover.

A space X will be called a generalized Lindelöfspace if there exists a subbase for its topology with the Lindelöfproperty.

Obviously a Lindelöfspace is a generalized Lindelöfspace.
The converse is false because every discrete space D of cardinal $\leq c$ is a generalized Lindelöfspace.
(One can suppose that D is a subset of the real line. Let $\mathfrak{B}$ be the collection of all subsets of D of the form $\{x\in D \mid x<a\}$, $\{x\in D \mid x\leq a\}$, $\{x\in D \mid x>a\}$, $\{x\in D \mid x\geq a\}$ $(a\in \mathbb{R})$. $\mathfrak{B}$ is a subbase for the discrete topology for D which satisfies the Lindelöfproperty).

The following proposition is obvious. It is a dual formulation of the notion of generalized Lindelöfspace.

(4.1) Proposition. A space X is a generalized Lindelöfspace if and only if there exists a subbase $\mathcal{S}$ for the closed sets such that each subcollection of $\mathcal{S}$ with the countable intersection property, has non empty intersection.

(4.2) Theorem. Every topological product of generalized Lindelöfspaces is a generalized Lindelöfspace.

Proof. Let $X = \pi\{X_\alpha \mid \alpha\in A\}$ and $\mathcal{S}_\alpha$ a subbase for the closed sets of X_α with the property that each subcollection with the countable intersection property has nonempty intersection.
Let $\mathcal{S}$ be the subbase for the product topology consisting of all sets of the form $\pi_\alpha^{-1}(C)$ where π_α is the projection into the α'th coördinate-space and C a member of $\mathcal{S}_\alpha$.
Let $\mathcal{S}'$ be a subcollection of $\mathcal{S}$ with the countable intersection property, we will show that $\mathcal{S}$ has a nonempty intersection in X.
For $\alpha\in A$ let $\mathcal{S}'_\alpha$ be the subcollection of $\mathcal{S}_\alpha$ consisting of the sets $\pi_\alpha S$ for which $S\in\mathcal{S}'$. For each α $\mathcal{S}'_\alpha$ has the countable intersection property and it is therefore possible to choose a point p_α in $\cap\{\pi_\alpha(S) \mid S\in\mathcal{S}'\}$.

The point p of X whose α'th coordinate equals p_α is in the intersection of $\mathcal{S}'$.

Note. The property "being a generalized Lindelöfspace" is not an invariant for the taking of open subsets (this is immediate from the next theorem and the examples given on page 9).
I don't know whether it is true that every closed subset of a generalized Lindelöfspace is a generalized Lindelöfspace or what comes to the same (because of (4.1)) that every realcompact space is a generalized Lindelöfspace.

(4.2) Theorem. Every generalized Lindelöfspace is $\mathcal{Z}_0$-ultracompact.

Proof. Choose a subbase $\mathcal{B}$ for the topology of X with the Lindelöfproperty. Suppose on the contrary X not $\mathcal{Z}_0$-ultracompact and let $\mathcal{F}$ be an $\mathcal{Z}_0$-ultrafilter which has no limitpoint in X.
For each $p \in X$ choose a subbasic neighbourhood $U_p \in \mathcal{B}$ of p which is not a member of $\mathcal{F}$. $\mathcal{B}$ satisfies the Lindelöfproperty, so the family $\{U_p \mid p \in X\}$ contains a countable subcollection $\{U_{p_i} \mid i=1,2,\dots\}$ which covers X.
There exists a natural member l such that $U_{p_l} \in \mathcal{F}$. For otherwise, $\{X \setminus U_{p_i} \mid i=1,2,\dots\}$ is a countable family of closed members of $\mathcal{F}$ with empty intersection which is impossible since $\mathcal{F}$ is an $\mathcal{Z}_0$-ultrafilter in X. However $U_{p_l} \in \mathcal{F}$ contradicts the fact that $U_p \notin \mathcal{F}$ for $p \in X$.

(4.3) Theorem. Every countably paracompact normal generalized Lindelöfspace is realcompact.

Proof. This follows at once from (3.1) and the foregoing theorem.

References

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