ZW

Differentiaalmeetkunde Capita Selecta 1950-51

Lie-groepen

1950

A. Theorie der normaalcoördinaten in L_n

§ 1. Geodetische lijnen. Laten ξ^κ ; $\kappa = 1,\dots,n$ de coördinaten zijn in een L_n met overbrenging $\Gamma_{\mu\lambda}^\kappa$. De vergelijking

A.1.1) $$\xi^\kappa = f^\kappa(t)$$

stelt dan een kromme in L_n voor en t is een parameter op die kromme. De kromme heet geodetisch indien de raakvector $d\xi^\kappa/dt$ zich pseudoparallel langs de kromme verplaatst:

A.1.2) $$\frac{\delta}{dt}\frac{d\xi^\kappa}{dt} = \alpha(t)\frac{d\xi^\kappa}{dt}$$

of

A.1.3) $$d^2\xi^\kappa/dt^2 + \Gamma_{\mu\lambda}^\kappa\, d\xi^\mu/dt\; d\xi^\lambda/dt = \alpha(t)\, d\xi^\kappa/dt.$$

De geodetische lijnen hangen dus alleen af van $\Gamma_{(\mu\lambda)}^\kappa$. Wordt een nieuwe parameter τ ingevoerd

A.1.4) $$\tau = \tau(t) \;;\quad t = t(\tau),$$

dan gaat de vergelijking over in

A.1.5) $$d^2\xi^\kappa/d\tau^2 + \Gamma_{\mu\lambda}^\kappa\, d\xi^\mu/d\tau\; d\xi^\lambda/d\tau = -\,d\xi^\kappa/d\tau\left(\frac{d^2\tau}{dt^2} - \alpha\frac{d\tau}{dt}\right)$$

Kiest men dus voor τ een oplossing van

A.1.6) $$d^2\tau/dt^2 - \alpha\, d\tau/dt = 0$$

dan krijgt de vergelijking de eenvoudige gedaante

A.1.7) $$\delta\, d\xi^\kappa/d\tau = 0$$

τ heet dan een affine parameter van de geodetische lijn. De algemene oplossing van (A.1.6) heeft de vorm

A.1.8) $$\tau = C_1\int e^{\int\alpha dt}\, dt + C_2 \;;\; C_1 \neq 0,\, C_2 \text{ constanten}$$

en de affine parameter is dus vastgelegd op de plaats van het nulpunt en een constante factor na. De affine parameter stelt ons dus in staat van segmenten op één geodetische lijn een "lengte"-verhouding invariant vast te leggen.

Men kan vragen de $\Gamma_{\mu\lambda}^\kappa$ zo te transformeren dat geodetische lijnen geodetisch blijven. Stel dat $\Gamma_{\mu\lambda}^\kappa \to \Gamma_{\mu\lambda}^\kappa + P_{\mu\lambda}^{\cdot\cdot\kappa}$ met $P_{[\mu\lambda]}^{\cdot\cdot\kappa} = 0$. Dan moet $P_{\mu\lambda}^{\cdot\cdot\kappa}$ een affinor zijn (volgt uit de transformatiewijze der $\Gamma_{\mu\lambda}^\kappa$ bij coördinatentransformaties) en blijkens (A.1.3) moet gelden

A.1.9) $$P_{\mu\lambda}^{\cdot\cdot\kappa}\frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt} = [\text{illegible}](t)\frac{d\xi^\kappa}{dt}.$$

voor iedere keuze van $d\xi^\kappa/dt$. Het kan worden bewezen dat dit alleen kan als $P_{\mu\lambda}^{\cdot\cdot\kappa}$ de vorm heeft

A.1.10) $$P_{\mu\lambda}^{\cdot\cdot\kappa} = p_\mu A_\lambda^\kappa + p_\lambda A_\mu^\kappa.$$

In dat geval heeft de nieuwe affine parameter de vorm

A.1.11) $$'\tau = C_1\int e^{-2\int p_\mu d\xi^\mu}\, d\tau + C_2 \;,\; C_1 \neq 0$$

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en hieruit volgt dat χ dan en alleen dan een affine parameter voor de nieuwe overbrenging is wanneer $p_\lambda d\xi^\lambda = 0$ in elk punt van de kromme; dat is dus wanneer p_λ de kromme tangeert. Is de affine parameter op alle geodetische lijnen invariant bij de transformatie dan is $p_\lambda = 0$ en omgekeerd. Dus: <u>Een symmetrische overbrenging is vastgelegd door de geodetische lijnen en de affine parameters op deze.</u>

§ 2. <u>Constructie der normaalcoördinaten voor een punt in A_n.</u>

We nemen hier verder aan dat de L_n een A_n is $(\Gamma^\kappa_{[\mu\lambda]} = 0)$. Een A_n is <u>vlak</u> of een E_n indien er coördinatenstelsels bestaan waarvoor $\Gamma^\kappa_{\mu\lambda} = 0$ in <u>ieder</u> punt. Het is gemakkelijk in een A_n een stelsel te construeren waarvoor $\Gamma^\kappa_{\mu\lambda} = 0$ in <u>één enkel punt</u> $\overset{\circ}{\xi}{}^\kappa$. Het stelsel heet dan <u>in</u> $\overset{\circ}{\xi}{}^\kappa$ <u>geodetisch</u>. Een heel eenvoudige manier word aangegeven in D blz. 58[1]. Aan Veblen danken we een algemene methode waarbij ook ten aanzien van de afgeleiden van $\Gamma^\kappa_{\mu\lambda}$ nog aan zekere eisen wordt voldaan.

In $\overset{\circ}{\xi}{}^\kappa$ nemen we een willekeurige vector $\overset{\circ}{t}{}^\kappa \neq 0$ en laten deze van $n-1$ parameters afhangen op zodanige wijze dat aan elke richting in $\overset{\circ}{\xi}{}^\kappa$ één vector is toegevoegd. Nu denken we ons door $\overset{\circ}{\xi}{}^\kappa$ alle geodetische lijnen getrokken en kiezen op elk van deze een affine parameter χ zodat in $\overset{\circ}{\xi}{}^\kappa$:

A.2.1) $$\frac{d\xi^\kappa}{d\chi} = \overset{\circ}{t}{}^\kappa \quad ; \chi = 0$$

De vergelijking van een geodetische lijn is dan

A.2.2) $$\frac{d^2\xi^\kappa}{d\chi^2} + \Gamma^\kappa_{\mu\lambda}\frac{d\xi^\mu}{d\chi}\frac{d\xi^\lambda}{d\chi} = 0$$

en met behulp daarvan kan men $d^2\xi^\kappa/d\chi^2$ in $\overset{\circ}{\xi}{}^\kappa$ berekenen

A.2.3) $$\frac{d^2\xi^\kappa}{d\chi^2} = -\Gamma^\kappa_{\mu\lambda}(\overset{\circ}{\xi}{}^\rho)\,\overset{\circ}{t}{}^\mu\,\overset{\circ}{t}{}^\lambda$$

Door herhaalde differentiatie van (A.2.2) vindt men zo alle afgeleiden in $\overset{\circ}{\xi}{}^\kappa$:

A.2.4) $$\frac{d^{p+1}\xi^\kappa}{d\chi^{p+1}} = -\Gamma^\kappa_{\mu_p\cdots\mu_1\lambda}(\overset{\circ}{\xi}{}^\rho)\,\overset{\circ}{t}{}^{\mu_p}\cdots\overset{\circ}{t}{}^{\mu_1}\overset{\circ}{t}{}^\lambda$$

waarin

A.2.5) $$\Gamma^\kappa_{\mu_p\cdots\mu_1\lambda} \overset{\text{def}}{=} \partial_{(\mu_p}\Gamma^\kappa_{\mu_{p-1}\cdots\mu_1\lambda)} - p\,\Gamma^\tau_{(\mu_p\mu_{p-1}}\Gamma^\kappa_{\mu_{p-2}\cdots\mu_1\lambda)\tau}$$

Met behulp van deze afgeleiden kan nu ξ^κ in een $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ in een reeks worden ontwikkeld

A.2.6) $$\begin{aligned}\xi^\kappa &= \overset{\circ}{\xi}{}^\kappa + \left(\frac{d\xi^\kappa}{d\chi}\right)_{\chi=0}\chi + \tfrac{1}{2!}\left(\frac{d^2\xi^\kappa}{d\chi^2}\right)_{\chi=0}\chi^2 + \cdots \\ &= \overset{\circ}{\xi}{}^\kappa + \chi\overset{\circ}{t}{}^\kappa - \tfrac{1}{2!}\Gamma^\kappa_{\mu\lambda}(\overset{\circ}{\xi}{}^\rho)\chi^2\overset{\circ}{t}{}^\mu\overset{\circ}{t}{}^\lambda - \tfrac{1}{3!}\Gamma^\kappa_{\mu_2\mu_1\lambda}(\overset{\circ}{\xi}{}^\rho)\chi^3\overset{\circ}{t}{}^{\mu_2}\overset{\circ}{t}{}^{\mu_1}\overset{\circ}{t}{}^\lambda \cdots\end{aligned}$$

Indien deze reeks convergeert blijkt een punt van een $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ evenzeer te worden vastgelegd door $\chi\overset{\circ}{t}{}^\kappa$ als door ξ^κ. We introduceren nu <u>een nieuw coördinatenstelsel</u> (h):

1) D = cursus differentiaalmeetkunde 49-50.

A.2.7) $$\xi^h \overset{def}{=} \delta^h_\kappa t^\kappa \tau \qquad (h = 1, \dots n)$$

en nemen voor het gemak $\overset{\circ}{\xi}{}^\kappa = 0$. Dan gaat (A.2.6) over in

A.2.8) $$\xi^\kappa = \delta^\kappa_h \xi^h - \frac{1}{2!}\Gamma^\kappa_{\mu\lambda}(0)\,\delta^{\mu\lambda}_{ji}\,\xi^j \xi^i - \frac{1}{3!}\Gamma^\kappa_{\mu_2\mu_1\lambda}(0)\,\delta^{\mu_2\mu_1\lambda}_{j_2 j_1 i}\,\xi^{j_2}\xi^{j_1}\xi^i - \dots$$

De omkering heeft de vorm

A.2.9) $$\xi^h = \delta^h_\kappa \xi^\kappa + \frac{1}{2!}\delta^h_\kappa \Lambda^\kappa_{\mu\lambda}(0)\,\xi^\mu \xi^\lambda + \frac{1}{3!}\delta^h_\kappa \Lambda^\kappa_{\mu_2\mu_1\lambda}(0)\,\xi^{\mu_2}\xi^{\mu_1}\xi^\lambda + \dots$$

en men berekent gemakkelijk dat

$$\Lambda^\kappa_{\mu\lambda} = \Gamma^\kappa_{\mu\lambda}$$

$$\Lambda^\kappa_{\mu_2\mu_1\lambda} = \Gamma^\kappa_{\mu_2\mu_1\lambda} + 3\,\Gamma^\rho_{(\mu_2\mu_1}\Gamma^\kappa_{\lambda)\rho}$$

A.2.10) $$\Lambda^\kappa_{\mu_3\mu_2\mu_1\lambda} = \Gamma^\kappa_{\mu_3\mu_2\mu_1\lambda} + 4\,\Gamma^\rho_{(\mu_3\mu_2\mu_1}\Gamma^\kappa_{\lambda)\rho} + 6\,\Gamma^\rho_{(\mu_3\mu_2}\Lambda^\kappa_{\mu_1\lambda)\rho} - 3\,\Gamma^\kappa_{\sigma\rho}\Gamma^\sigma_{(\mu_3\mu_2}\Gamma_{\mu_1\lambda)}$$

enz.

Door differentiatie van (A.2.8) en (A.2.9) verkrijgt men <u>in</u> $\overset{\circ}{\xi}{}^\kappa$:

A.2.11)

a) $A^\kappa_i = \delta^\kappa_i$; $A^h_\lambda = \delta^h_\lambda$

b) $\partial_j A^\kappa_i = -\delta^{\mu\lambda}_{ji}\,\Gamma^\kappa_{\mu\lambda}(0)$; $\partial_\mu A^h_\lambda = \delta^h_\kappa\,\Gamma^\kappa_{\mu\lambda}(0)$

c) $\partial_k\partial_j A^\kappa_i = -\delta^{\nu\mu\lambda}_{kji}\,\Gamma^\kappa_{\nu\mu\lambda}(0)$; $\partial_\nu\partial_\mu A^h_\lambda = \delta^h_\kappa\,\Lambda^\kappa_{\nu\mu\lambda}(0)$

en bijgevolg (verg. D. blz 42, (6.6) <u>in</u> $\overset{\circ}{\xi}{}^\kappa$

A.2.12) $$\Gamma^h_{ji}(0) = \delta^{\mu\lambda h}_{j\,i\,\kappa}\,\Gamma^\kappa_{\mu\lambda}(0) - A^{\mu\lambda}_{j\,i}\,\partial_\mu A^h_\lambda = 0.$$

Maar nu is de keuze van het coördinatenstelsel (κ) vrij. Men kan dus, (h) eenmaal verkregen zijnde ook van (h) uitgaan en moet dan natuurlijk in (A.2.8) weer (h) terugvinden. Ook (A.2.5) geldt voor ieder coördinatenstelsel, dus ook voor (h) en uit (A.2.8) vindt men derhalve behalve $\Gamma^h_{ji}(0) = 0$ ook

A.2.13) $$\Gamma^h_{j_p \cdots j_1 i}(0) = 0$$

voor alle waarden van p en daaruit volgt in verband met de definitievergelijking van deze uitdrukkingen ook

A.2.14) $$\partial_{(j_{p+1}}\Gamma^h_{j_p \cdots j_1 i)} = 0 \qquad \text{voor} \quad \xi^h = 0$$

A.2.15) $$\partial_{(j_p}\partial_{j_{p-1}}\cdots\partial_{j_1}\Gamma^h_{ji)} = 0 \qquad \text{voor} \quad \xi^h = 0$$

Niet alleen verdwijnen dus in $\xi^\kappa = 0$ alle Γ^h_{ji} maar bovendien ook alle gesymmetrizeerde afgeleiden van de Γ^h_{ji} naar de ξ^h van iedere orde.

De aldus verkregen coördinaten (h) heten naar Veblen (Proc.Nat. Acad. 8 (1922) 192-197) <u>de normaalcoördinaten in A_n ten opzichte van $\overset{\circ}{\xi}{}^\kappa$ en behorende bij het coördinatenstelsel (κ)</u>. Zij zijn een veralgemening van de door Riemann in een V_n geintroduceerde normaalcoördinaten. (Litteratuur Einf. I blz 100).

Houdt men $\overset{\circ}{\xi}{}^\kappa$ vast maar worden de ξ^κ getransformeerd in $\xi^{\kappa'}$ dan krijgt men een ander stelsel (h') behorende bij (κ'). In plaats van (A.2.7) komt dan:

A.2.16) $$\xi^{h'} \overset{\mathrm{def}}{=} \partial^{h'}_{k'} \underset{0}{t}{}^{k'} x = \partial^{h'}_{k'} A^{k'}_{k}(0)\, \underset{0}{t}{}^{k} x = A^{h'}_{h}\, \xi^{h}$$

waarin de $A^{h'}_{h}$ constanten zijn. Ondergaan dus de $\underset{0}{\xi}{}^{\kappa}$ een willekeurige transformatie dan ondergaan de ξ^{h} een lineaire homogene transformatie met constante coefficienten die numeriek gelijk zijn aan de waarden van $A^{\kappa'}_{\kappa}$ in $\underset{0}{\xi}{}^{\kappa}$.

De normaalcoördinaten ten opzichte van $\underset{0}{\xi}{}^{\kappa}$ beelden de punten der A_n in $\mathcal{N}(\underset{0}{\xi}{}^{\kappa})$ eeneerduidig af op de punten van de locale E_n van $\underset{0}{\xi}{}^{\kappa}$. In deze E_n zijn de ξ^{h} rechtlinige coördinaten en het punt van A_n met de normaalcoördinaten ξ^{h} heeft als beeld het punt met radiusvector ξ^{h} in de E_n.

Uit (A.2.11a) volgt dat de maatvectoren van (κ) in $\underset{0}{\xi}{}^{\kappa}$ samenvallen met de maatvectoren van (h). Dus zijn ook de kentallen van iedere grootheid in $\underset{0}{\xi}{}^{\kappa}$ ten opzichte van (κ) en (h) numeriek gelijk. Daaruit volgt echter:

Is van een stel getallen in $\underset{0}{\xi}{}^{\kappa}$ bekend dat zij zich bij overgang van (h) tot (h') tranformeren als de kentallen van een grootheid (affinor of affinordichtheid) dan zijn deze getallen kentallen van een grootheid waarvan de kentallen t.o. van (κ) en (κ') numeriek gelijk zijn aan die t.o.v. (h) en (h') resp.

Van deze eigenschap hebben Veblen en T.Y.Thomas zeer elegant gebruik gemaakt ter constructie van de z.g. normaalaffinoren. De (nu niet gesymmetrizeerde) afgeleiden der Γ_{ji}^{h}:

A.2.17) $$N_{j_p \cdots j_1 j i}^{\cdot\ \ \ \ \ \ \ \ h} \overset{\mathrm{def}}{=} \left(\partial_{j_p} \cdots \partial_{j_1} \Gamma_{ji}^{h}\right)_{\xi^h = 0}$$

transformeren zich blijkbaar bij de overgang tot (h') lineair homogeen, dus als de kentallen van een affinor. Dus zijn de aldus gedefinieerde grootheden N inderdaad affinoren. Men lette erop dat in het algemeen

$$N_{j_p \cdots j_1 j i}^{\cdot\ \ \ \ \ \ \ \ h} \neq \left(\partial_{j_p} N_{j_{p-1} \cdots j_1 j i}^{\cdot\ \ \ \ \ \ \ \ h}\right)_{\xi^h = 0}.$$

Deze normaalaffinoren behorende tot de overbrenging $\Gamma_{\mu\lambda}^{\kappa}$ treden o.a. op in de reeksontwikkeling

A.2.18) $$\Gamma_{ji}^{h} = \xi^{k} N_{kji}^{\cdot\ \ \ h} + \frac{1}{2!} \xi^{k_2} \xi^{k_1} N_{k_2 k_1 j i}^{\cdot\ \ \ \ \ \ h} + \cdots$$

waarvan de convergentie door T.Y.Thomas (Invariants of generalized spaces, Cambridge 1934 blz. 126) onderzocht is. Uit de definitie volgt dat iedere normaalaffinor symmetrisch is in de eerste p en in de laatste 2 covariante indices, terwijl menging over alle covariante indices tot nul leidt.

Nu hebben we de normaalaffinoren alleen geconstrueerd in $\underset{0}{\xi}{}^{\kappa}$. Maar hetzelfde kan natuurlijk in elk punt gedaan worden. Het bezwaar is alleen dat men dan telkens kentallen t.o.v. een ander coördinatenstelsel krijgt, namelijk het stelsel van normaalcoördinaten t.o.v. dat bepaalde punt. We willen liever de kentallen t.o.v. (κ) hebben. Natuurlijk kan men deze berekenen door opvolgende differentiaties van

A.2.19) $$\Gamma_{ji}^{h} = A_{j\,i\,k}^{\mu\lambda h}\,\Gamma_{\mu\lambda}^{\kappa} - A_{j\,i}^{\mu\lambda}\,\partial_\mu A_\lambda^h$$

naar ξ^h, daarbij telkens gebruik makende van (A.2.11 a,b). Voor het eenvoudigste geval vindt men dan in $\underset{0}{\xi}{}^\kappa$ na enig gereken

A.2.20) $$N_{kji}^{\cdot\cdot\cdot h} = \partial_k \Gamma_{ji}^{h} = \delta_{kji\kappa}^{\nu\mu\lambda h}\left(\partial_\nu \Gamma_{\mu\lambda}^{\kappa} - \Gamma_{\nu\mu\lambda}^{\kappa} - 2\,\Gamma_{\nu(\mu}^{\rho}\Gamma_{\lambda)\rho}^{\kappa}\right)$$

en dus, en nu geldig voor alle punten in $\mathfrak{M}(\underset{0}{\xi}{}^\kappa)$

A.2.21) $$N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = \partial_\nu \Gamma_{\mu\lambda}^{\kappa} - \Gamma_{\nu\mu\lambda}^{\kappa} - 2\,\Gamma_{\nu(\mu}^{\rho}\Gamma_{\lambda)\rho}^{\kappa}$$

Maar deze weg is voor de hogere gevallen zeer lang. Beter gaat het met behulp van de kromtegrootheid $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ en haar covariante afgeleiden.

§ 3. De normaalaffinoren en afgeleiden van de kromtegrootheid.

Aangezien de Γ_{ji}^{h} nul zijn in $\underset{0}{\xi}{}^\kappa$ is aldaar (verg. D blz. 55)

A.3.1) $$R_{kji}^{\cdot\cdot\cdot h} = 2\,\partial_{[k}\Gamma_{j]i}^{h} = 2\,N_{[kj]i}^{\cdot\cdot\cdot h}$$

en dus overal

A.3.2) $$\boxed{R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2\,N_{[\nu\mu]\lambda}^{\cdot\cdot\cdot\kappa}}$$

Daar echter $N_{(\nu\mu\lambda)}^{\cdot\cdot\cdot\kappa} = 0$ en $N_{\nu[\mu\lambda]}^{\cdot\cdot\cdot\kappa} = 0$ kan $N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ uit deze vergelijking worden opgelost:

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} - N_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}$$

$$R_{\nu\lambda\mu}^{\cdot\cdot\cdot\kappa} = N_{\nu\lambda\mu}^{\cdot\cdot\cdot\kappa} - N_{\lambda\nu\mu}^{\cdot\cdot\cdot\kappa}$$

A.3.3) $$2\,R_{\nu(\mu\lambda)}^{\cdot\cdot\cdot\kappa} = 2\,N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} + N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 3\,N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$$

A.3.4) $$\boxed{N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = \tfrac{2}{3}\,R_{\nu(\mu\lambda)}^{\cdot\cdot\cdot\kappa}}$$

Differentieert men

A.3.5) $$R_{kji}^{\cdot\cdot\cdot h} = 2\,\partial_{[k}\Gamma_{j]i}^{h} + 2\,\Gamma_{[k|l|}^{h}\Gamma_{j]i}^{l}$$

covariant dan ontstaat in $\underset{0}{\xi}{}^\kappa$

A.3.6) $$\nabla_{k_2} R_{k_1ji}^{\cdot\cdot\cdot h} = 2\,\partial_{k_2}\partial_{[k_1}\Gamma_{j]i}^{h} = 2\,N_{k_2[k_1j]i}^{\cdot\cdot\cdot\cdot h}$$

omdat weer alle termen met Γ_{ji}^{h} wegvallen en dus overal

A.3.7) $$\boxed{\nabla_{\nu_2} R_{\nu_1\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2\,N_{\nu_2[\nu_1\mu]\lambda}^{\cdot\cdot\cdot\cdot\kappa}}$$

Gebruik makende van $N_{(\nu_2\nu_1\mu\lambda)}^{\cdot\cdot\cdot\cdot\kappa} = 0$, $N_{[\nu_2\nu_1]\mu\lambda}^{\cdot\cdot\cdot\cdot\kappa} = 0$ en $N_{\nu_2\nu_1[\mu\lambda]}^{\cdot\cdot\cdot\cdot\kappa}$ kan men weer N oplossen

A.3.8) $$\boxed{N_{\nu_2\nu_1\mu\lambda}^{\cdot\cdot\cdot\cdot\kappa} = \tfrac{5}{6}\,\nabla_{(\nu_2} R_{\nu_1)(\mu\lambda)}^{\cdot\cdot\cdot\kappa} - \tfrac{1}{6}\,\nabla_{(\mu} R_{\lambda)(\nu_2\nu_1)}^{\cdot\cdot\cdot\kappa}}$$

Verderop gaat het niet zo gemakkelijk omdat er dan rechts afgeleiden van Γ_{ji}^{h} optreden, die in $\underset{0}{\xi}{}^\kappa$ niet nul behoeven te zijn.

We kunnen echter toch belangrijke gegevens verkrijgen over de vorm van de betrekkingen tussen de N's en de opvolgende afgeleiden van R. Algemeen geldt

A.3.9) $$\nabla_{\nu_p} \ldots \nabla_{\nu_1} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = \lambda\, \partial_{\nu_p} \ldots \partial_{\nu_1} \partial_{[\nu} \Gamma_{\mu]\lambda}^{\kappa} + *$$

waarin $*$ staat voor termen die alleen de $\Gamma_{\mu\lambda}^{\kappa}$ en hun afgeleiden tot de orde p bevatten. Schrijft men deze vergelijking t.o.v. (h) dan ontstaat

A.3.10) $$\nabla_{k_p} \ldots \nabla_{k_1} R_{kji}^{\cdot\cdot\cdot h} = \lambda\, N_{k_p \ldots k_1 [kj] i}^{\cdot\,\cdot\,h} + *$$

en hier staat $*$ nu voor termen die geheel bestaan uit overschuivingen van N's met een valentie $\leq p+3$.

Gaan we nu weer terug naar (κ) dan komt er overal

A.3.11) $$\boxed{\nabla_{\nu_p} \ldots \nabla_{\nu_1} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = \lambda\, N_{\nu_p \ldots \nu_1 [\nu\mu] \lambda}^{\cdot\,\cdot\,\kappa} + *}$$

De vorm van deze vergelijking kan nog preciezer omschreven worden. Laten we $\overset{v}{N}$ schrijven voor de normaalaffinor van de valentie v en $\overset{v}{R}$ voor de $(v-4)$- de covariante afgeleide van $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$. Dan zien wij dat $\overset{4}{R}$ en $\overset{5}{R}$ eenvoudig worden uitgedrukt in $\overset{4}{N}$ en $\overset{5}{N}$ resp. Maar in $\overset{6}{R}$ treedt behalve $\overset{6}{N}$ ook een overschuiving van $\overset{4}{N}$ en $\overset{4}{N}$ op. In $\overset{7}{R}$ treedt op $\overset{7}{N}$ en de overschuiving $\overset{4}{N}\overset{5}{N}$ enz. als volgt:

$$\begin{array}{ll}
\overset{4}{R} & \overset{4}{N} \\
\overset{5}{R} & \overset{5}{N} \\
\overset{6}{R} & \overset{6}{N},\ \overset{4}{N}\overset{4}{N} \\
\overset{7}{R} & \overset{7}{N},\ \overset{5}{N}\overset{4}{N}, \quad , \\
\overset{8}{R} & \overset{8}{N},\ \overset{6}{N}\overset{4}{N},\ \overset{5}{N}\overset{5}{N},\ \overset{4}{N}\overset{4}{N}\overset{4}{N} \\
\overset{9}{R} & \overset{9}{N},\ \overset{7}{N}\overset{4}{N},\ \overset{6}{N}\overset{5}{N},\ \overset{5}{N}\overset{4}{N}\overset{4}{N}
\end{array} \qquad \text{A.3.12)}$$

enz.

De algemene regel is dat in $\overset{v}{R}$ alleen dan een term met $\overset{v_1}{N} \ldots \overset{v_u}{N}$ kan optreden indien

A.3.13) $$v = v_1 + \ldots + v_u - \lambda(u-1)$$

en dat er in zulk een term precies $u-1$ overschuivingen voorkomen en wel zo dat elke N met elke andere N over ten hoogste één index overschoven is, en dat elke N in een product van meer N's in tenminste één overschuiving betrokken is.

Uit (A.3.11) kan $\overset{p+4}{N}$ worden opgelost als functie van $\overset{p+4}{R}$ en lagere N's. Om dit te bewijzen merken we op dat in

A.3.14) $$N_{(\dot{\nu}_p\cdot\cdot\nu_1\nu\mu\lambda)}{}^{\kappa}=0$$

drie soorten termen voorkomen, zulke met $\mu\lambda$ aan het eind

A.3.15) $$N_{\dot{\nu}_p\cdot\cdot\nu_1\nu\mu\dot{\lambda}}{}^{\kappa}$$

zulke met één ν in de laatste twee indices

A.3.16) $$N_{\dot{\nu}_p\cdots\nu_1\mu\nu\dot{\lambda}}{}^{\kappa}$$

en termen met twee ν's aan het eind

A.3.17) $$N_{\dot{\nu}_p\cdot\cdot\nu_2\mu\lambda\nu_1\dot{\nu}}{}^{\kappa}$$

Wegens de symmetrie van $\overset{p+4}{N}$ in de eerste p en in de laatste 2 indices doet de plaats der indices er verder niets toe. Een term van de tweede soort kan met behulp van (A.3.11) worden omgezet in een som van: $\overset{p+4}{R}$, een term van de eerste soort, en termen die alleen lagere N's bevatten. Een term van de derde soort kan evenzo worden omgezet in een som van: een isomeer van $\overset{p+4}{R}$, een term van de tweede soort en termen met lagere N's . Zet men dan weer die term van de tweede soort om, dan verschijnen tenslotte in (A.3.14) de term (A.3.15) met een zekere coefficient, een som van isomeren van $\overset{p+4}{R}$ met zekere coefficienten en termen met lagere N's . Worden die lagere N's net zo omgezet dan zien we dat tenslotte de N's kunnen worden uitgedrukt volgens een tabel die er net eender uitziet als (3.12)

$$\begin{array}{lll}
 & \overset{4}{N} & \overset{4}{R} \\
 & \overset{5}{N} & \overset{5}{R} \\
\text{A.3.18)} & \overset{6}{N} & \overset{6}{R},\ \overset{4}{R}\overset{4}{R} \\
 & \overset{7}{N} & \overset{7}{R},\ \overset{4}{R}\overset{5}{R}
\end{array}$$

enz.

dus in het algemeen in termen met $\overset{v_1}{R}\cdot\cdot\overset{v_u}{R}$ en $u-1$ overschuivingen waarvoor (A.3.13) geldt. De omkering van (A.3.11) heeft dus de vorm

A.3.19) $$\boxed{N_{\dot{\nu}_p\cdot\cdot\nu_1\nu\mu\dot{\lambda}}{}^{\kappa} = \mathcal{L}\left(\nabla_{\nu_p}\cdot\cdot\nabla_{\nu_1}R_{\nu\mu\lambda}{}^{\cdot\cdot\cdot\kappa}\right) + *}$$

waarin $\mathcal{L}(\ldots)$ staat voor een polynoom, lineair homogeen in de isomeren van $\nabla_{\nu_p}\cdot\cdot\nabla_{\nu_1}R_{\nu\mu\lambda}{}^{\kappa}$ en waarin $*$ staat voor een aantal termen die elk door $(u-1)$-voudige overschuiving en het toevoegen van een constante coefficient uit $\overset{v_1}{R}\cdot\cdot\overset{v_u}{R}$ ontstaan, waarbij $v_1,\ldots,v_u$ alle waarden $v\geqslant 4$ doorlopen die aan (A.3.13) met $v=p+4$ voldoen.

§ 4. De symmetrische A_n.

De kentallen van de N in $\underset{a}{\xi}{}^{\kappa}$ ten opzichte van (h) zijn tenslotte niets anders dan de afgeleiden van de Γ_{ji}^{h} (zelf nul in $\overset{\circ}{\xi}{}^{\kappa}$) naar de ξ^{h}. Daaruit volgt dat men uit (A.3.19) zeer waardevolle informaties kan halen over de functie $\Gamma_{ji}^{h}(\xi^{k})$ in $\mathfrak{N}(\overset{\circ}{\xi}{}^{h})$. Indien bijvoorbeeld eens alle covariante afgeleiden van $R_{\nu\mu\lambda}{}^{\kappa}$ met <u>oneven</u> valentie in $\overset{\circ}{\xi}{}^{\kappa}$ nul zijn, dan volgt uit (A.3.19) en (A.3.14) dat alle N's met een <u>oneven</u> valentie aldaar nul zijn. Die met een even valentie kunnen nog termen bevatten die R of twee of meer factoren R bevatten. Maar dan volgt uit (A.2.18) dat de functie Γ_{ji}^{h} in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ van teken verandert indien men ξ^{h} vervangt door $-\xi^{h}$. Dit betekent echter dat de A_n invariant is voor de transformatie $\xi^{h} \to -\xi^{h}$ of zoals men zegt <u>symmetrisch</u> is t.o.v. het punt $\xi^{h} = 0$. Inderdaad, een veld met de veldwaarde v^{h} in ξ^{h} heeft na spiegeling aan het punt $\xi^{h} = 0$ de veldwaarde $-v^{h}$ in $-\xi^{h}$. Aangezien een lijnelement $d\xi^{h}$ in ξ^{h} zich spiegelt in een lijnelement $-d\xi^{h}$ in $-\xi^{h}$ volgt dat δv^{h} in ξ^{h} zich spiegelt in een vector met tegengesteld teken in $-\xi^{h}$.

Omgekeerd, is de A_n in deze zin symmetrisch t.o.v. $\overset{\circ}{\xi}{}^{\kappa}$ dan moeten blijkens (A.2.18) de N's met oneven valentie nul zijn in $\overset{\circ}{\xi}{}^{\kappa}$ en uit (A.3.12) volgt dan dat ook alle covariante afgeleiden van $R_{\nu\mu\lambda}{}^{\kappa}$ met oneven valentie aldaar nul zijn.

Is een A_n symmetrisch t.o.v. <u>alle</u> punten in een $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ dan volgt hieruit dat dit dan en alleen dan mogelijk is indien de afgeleiden van R met oneven valentie in alle punten nul zijn. Maar dit is alleen mogelijk indien alle afgeleiden van R overal nul worden. Een dergelijke A_n heet Symmetrisch. Dus:

<u>Een A_n is dan en alleen dan symmetrisch indien $R_{\nu\mu\lambda}{}^{\kappa}$ covariant constant is.</u>

§ 5. <u>De reductiestellingen in</u> A_n.

Het proces dat geleid heeft van de Γ_{ji}^{h} tot de normaalaffinoren werd door Veblen en T.Y.Thomas (The Geometry of paths, Trans. Am. Math.Soc. 25 (1923) 551 - 608) ook toegepast op een willekeurige grootheid. Stel dat we van de een of andere grootheid (affinor of affinordichtheid) de kentallen hebben gevonden t.o.v. (h). Laat deze gesymbolizeerd worden door Φ, waarbij we alle indices onderdrukken. Dan zullen de afgeleiden

A.5.1) $\quad \partial_{j_1}\Phi \;;\; \partial_{j_2}\partial_{j_1}\Phi \;;\;$ enz.

<u>in het punt</u> $\overset{\circ}{\xi}{}^{\kappa}$ zich bij de overgang van (h) tot (h') transformeren als de kentallen van grootheden. We hebben boven al gezien dat zij dan inderdaad kentallen van grootheden zijn. Wij noemen deze grootheden de eerste, tweede enz. <u>normale afgeleide</u> van Φ (bij V. en Th. <u>extension</u>). De normale afgeleiden vormen velden daar men $\overset{\circ}{\xi}{}^{\kappa}$ willekeurig kan verplaatsen. Men lette er op dat de $(p+1)$-de normale afgeleide <u>niet</u> gelijk is aan de normale afgeleide van de p-de afgeleide. De normale afgeleiden worden genoteerd met $\overset{v}{\nabla}_{\mu}$, $\overset{v}{\nabla}_{\mu_2\mu_1}$, enz. De eerste normale afgeleide is gelijk aan de gewone covariante afgeleide, b.v.

in $\overset{\circ}{\xi}{}^{\kappa}$

$$\text{A.5.2)} \quad \nabla_\mu v^\kappa = A^{j\kappa}_{\mu h} \nabla_j v^h = A^{j\kappa}_{\mu h} \partial_j v^h = A^{j\kappa}_{\mu h} \overset{v}{\nabla}_j v^h = \overset{v}{\nabla}_\mu v^\kappa$$

Maar voor de hogere normale afgeleiden gaat dit niet meer op, b.v. geldt in $\overset{\circ}{\xi}{}^{\kappa}$

$$\overset{v}{\nabla}_{\mu_2\mu_1} v^\kappa = \nabla_{\mu_2} \nabla_{\mu_1} v^\kappa = A^{j_2 j_1 \kappa}_{\mu_2\mu_1 h} \nabla_{j_2} \nabla_{j_1} v^h =$$

$$\text{A.5.3)} \quad = A^{j_2 j_1 \kappa}_{\mu_2\mu_1 h} \left(\partial_{j_2} \partial_{j_1} v^h + v^i \partial_{j_2} \Gamma^h_{j_1 i}\right)$$

$$= \overset{v}{\nabla}_{\mu_2\mu_1} v^\kappa + v^\lambda N_{\mu_2\mu_1\lambda}^{\cdot\;\cdot\;\cdot\;\kappa}$$

en omdat de uitkomst de invariante vorm heeft geldt zij overal.

Op deze wijze kunnen alle covariante afgeleiden worden uitgedrukt in normale afgeleiden van dezelfde en lagere orde en normaalaffinoren met een valentie die ten hoogste gelijk is aan 2 + de orde van differentiatie. Omgekeerd kunnen evenzeer de normale afgeleiden worden uitgedrukt in covariante afgeleiden van dezelfde en lagere orde en normaalaffinoren met een valentie niet hoger dan 2 + de orde van differentiatie.

Natuurlijk kunnen in beide gevallen de N's weer worden uitgedrukt in $R_{\nu\mu\lambda}^{\cdot\;\cdot\;\cdot\;\kappa}$ en zijn covariante afgeleiden. Men kan hiervan hele tabellen ontwerpen en bij Veblen en Thomas vindt men enkele van de tabellen. Zonder die tabellen te kennen kan men nu echter alleen uit het feit dat deze omzettingen mogelijk zijn belangrijke conclusies trekken ten aanzien van de differentiaal comitanten van een overbrenging $\Gamma^\kappa_{\mu\lambda}$.

Is er een overbrenging $\Gamma^\kappa_{\mu\lambda}$ met $\Gamma^\kappa_{[\mu\lambda]} = 0$ gegeven en bestaat er een grootheid (affinor of affinordichtheid) waarvan de kentallen zich laten uitdrukken in de $\Gamma^\kappa_{\mu\lambda}$ en hun afgeleiden tot de orde p dan heet die grootheid een differentiaalcomitante van de orde p van de overbrenging. $R_{\nu\mu\lambda}^{\cdot\;\cdot\;\cdot\;\kappa}$ is b.v. een differentiaalcomitante van de orde 1. We beschouwen meestal alleen die comitanten wier kentallen zich algebraisch in de $\Gamma^\kappa_{\mu\lambda}$ en hun afgeleiden laten uitdrukken. Is een differentiaalcomitante een scalar of scalaire dichtheid dan spreekt men dikwijls van absolute differantiaalinvariant resp. relatieve differantiaalinvariant.

We gebruiken nu het coördinatenstelsel (h) voor enig punt $\overset{\circ}{\xi}{}^{\kappa}$ en drukken de kentallen van een differentiaalcomitante t.o.v. (h) uit in de Γ^h_{ji} en hun afgeleiden. In het punt $\overset{\circ}{\xi}{}^{\kappa}$ is $\Gamma^h_{ji} = 0$ en kan men de opvolgende afgeleiden vervangen door grootheden N. Aangezien dit voor ieder punt kan worden gedaan heeft men nu ook de kentallen van de comitante t.o.v. (κ) uitgedrukt in de (κ)-kentallen van de N's. Deze kunnen echter weer worden uitgedrukt in de (κ)-kentallen van R en zijn covariante afgeleiden. Daaruit volgt de eerste reductiestelling

Alle differentiaalcomitanten van de orde p van een symmetrische overbrenging zijn gewone comitanten van $R_{\nu\mu\lambda}^{\cdot\;\cdot\;\cdot\;\kappa}$ en zijn covariante afgeleiden tot en met de orde $p-1$.

Is er in een A_n een stel grootheden $\phi_1,\ldots,\phi_M$ (indices onderdrukt) gegeven, en is er een grootheid waarvan de kentallen zich laten uitdrukken in

1e. de $\Gamma_{\mu\lambda}^{\kappa}$ en hun afgeleiden tot zekere orde

2e. de kentallen der ϕ's en hun afgeleiden tot zekere orde

dan heet die grootheid een differentiaalcomitante van de overbrenging $\Gamma_{\mu\lambda}^{\kappa}$ en de velden $\phi_1,\ldots,\phi_M$. Meestal beschouwen we weer comitanten wier kentallen zich algebraisch in de aangegeven argumenten laten uitdrukken. We nemen eerst weer het coördinatenstelsel (h) t.o.v. $\overset{\circ}{\xi}{}^{\kappa}$ en drukken de (h)-kentallen van de comitante uit in de Γ_{ji}^{h}, hun afgeleiden, de (h)-kentallen der ϕ's en hun afgeleiden. In het punt $\overset{\circ}{\xi}{}^{\kappa}$ vervangen we nu weer de afgeleiden der Γ_{ji}^{h} door opvolgende N's en de afgeleiden der ϕ's door normale afgeleiden der ϕ's . Daarop gaan we weer over tot het algemene stelsel (κ) en hebben dan de (κ)-kentallen van de comitante uitgedrukt in (κ)-kentallen van de N's de ϕ's en de normale afgeleiden van de ϕ's . De N's kunnen weer worden omgezet in R en zijn afgeleiden en de normale afgeleiden van de ϕ's kunnen worden uitgedrukt in de covariante afgeleiden van de ϕ's en R met zijn afgeleiden. Daarmede is de tweede reductiestelling bewezen:

Alle differentiaalcomitanten van een symmetrische overbrenging en een aantal velden $\phi_1,\ldots,\phi_M$ (indices onderdrukt) zijn gewone comitanten van de velden $R,\phi_1,\ldots\phi_M$ en hun covariante afgeleiden.

Is $\Gamma_{[\mu\lambda]}^{\kappa} \neq 0$ dan behoeven we alleen $S_{\mu\lambda}^{\cdot\cdot\kappa} = \Gamma_{[\mu\lambda]}^{\kappa}$ te beschouwen als eén van de gegeven velden. Daarmede hebben we dan de derde reductiestelling :

Alle differentiaalcomitanten van een algemene overbrenging en een aantal velden $\phi_1,\ldots,\phi_M$ (indices onderdrukt) zijn gewone comitanten van de velden $\overset{0}{R}_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ (de kromtegrootheid voor de symmetrische overbrenging $\overset{0}{\Gamma}_{\mu\lambda}^{\kappa} = \Gamma_{(\mu\lambda)}^{\kappa}$), $S_{\mu\lambda}^{\cdot\cdot\kappa},\phi_1,\ldots\phi_M$ en hun covariante afgeleiden met betrekking tot de overbrenging $\overset{0}{\Gamma}_{\mu\lambda}^{\kappa}$.

Hieruit leidt men weer gemakkelijk af:

Alle differentiaalcomitanten van een algemene overbrenging en een aantal velden $\phi_1,\ldots\phi_M$ (indices onderdrukt) zijn gewone comitanten van de velden $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}, S_{\mu\lambda}^{\cdot\cdot\kappa},\phi_1,\ldots,\phi_M$ en hun covariante afgeleiden.

B. Meetkunde der eindige continue transformatiegroepen.

§ 1 N. Groepen. Onder een groep wordt een verzameling van elementen $A, B, C, \ldots$ verstaan waarvoor een "vermenigvuldiging" is gedefinieerd zodanig dat

1. het product van twee elementen steeds weer tot de verzameling behoort;
2. er een element ,de eenheid, bestaat zodat

1 N. 1) $$\mathcal{J}A = A\mathcal{J} = A$$

voor iedere A;

3. er tot elk element A een invers element $\overset{-1}{A}$ bestaat;

1 N. 2) $$A\overset{-1}{A} = \overset{-1}{A}A = \mathcal{J}$$

4. de associatieve wet geldt

1 N. 3) $$(AB)C = A(BC)$$

Zijn de elementen transformaties dan is $\mathcal{J}$ de identieke transformatie en (4) is vanzelf vervuld.

Enige definities:

a) Abelsche groep: $AB = BA$;
b) Ondergroep $\mathfrak{U}$ van groep $\mathfrak{G}$ is een groep $\mathfrak{U}$ waarvan alle elementen elementen van $\mathfrak{G}$ zijn, en waar dezelfde "product" definitie geldt;
c) Homoloog heten twee elementen A en A' indien er een element B bestaat zodat $A' = BA\overset{-1}{B}$;
d) Twee ondergroepen heten homoloog wanneer zij door $B \ldots \overset{-1}{B}$ en $\overset{-1}{B} \ldots B$ in elkaar overgevoerd worden. Symbool $\mathfrak{U} \to \mathfrak{U}' = B\mathfrak{U}\overset{-1}{B}$;
e) Een ondergroep $\mathfrak{U}$ is invariant in $\mathfrak{G}$ indien $\mathfrak{U} = A\mathfrak{U}\overset{-1}{A}$ voor iedere A ; (een andere naam is normaaldeler)
f) $\mathfrak{G}$ en $\mathfrak{G}'$ heten isomorph wanneer er een correspondentie tussen $A, B, C, \ldots$ en $A', B', C', \ldots$ bestaat zodat als A' bij A en B' bij B behoort, steeds $A'B'$ bij AB behoort. De isomorphie heet holoëdrisch als de correspondentie één-éénduidig is en anders meroëdrisch. (Dikwijls worden in deze betekenissen ook de termen isomorphie resp. homomorphie gebruikt.)

Enige eigenschappen:

1) Ondergroep van ondergroep van $\mathfrak{G}$ is ondergroep van $\mathfrak{G}$;
2) Doorsnede van twee ondergroepen van $\mathfrak{G}$ is ondergroep van $\mathfrak{G}$;
3) Doorsnede van twee in $\mathfrak{G}$ invariante ondergroepen is in $\mathfrak{G}$ invariante ondergroep;
4) Invariante ondergroep van invariante ondergroep van $\mathfrak{G}$ is in het algemeen geen invariante ondergroep van $\mathfrak{G}$.

Voorbeelden:

eindig

1. Permutaties van 2 dingen (2 elementen)
2. Permutaties van 3 dingen (6 elementen), holoëdrisch isomorph met de groep der 6 transformaties van 1 variabele

$$'x = x \; ; \; 'x = \frac{1}{1+x} \; ; \; 'x = \frac{x-1}{x} \; ; \; 'x = \frac{1}{x} \; ; \; 'x = 1-x \; ; \; 'x = \frac{x}{x-1}$$

eindig continu

3. Draaiïngen om een punt in een vlak. ∞' elementen afhankelijk van 1 parameter;
4. Bewegingen in een vlak, 3 parameters;
5. Draaiïngen om een punt in de ruimte. 3 parameters;
6. Bewegingen in de ruimte, 6 parameters;
7. Projectieve transformaties

in lijn	3 parameters
in vlak	8 parameters
in ruimte	15 parameters

In deze eindige continue groepen laat zich iedere transformatie vanuit de identieke transformatie bereiken door continue verandering der parameters. In de gemengde continue groepen kan dit niet, voorbeeld:

gemengd continu

8. Draaiïngen om een punt in een vlak en spiegelingen aan een lijn door dat punt, 2 definitievergelijkingen elk met 1 parameter:

$$'x = x\cos\alpha + y\sin\alpha$$

$$'y = \mp x\sin\alpha \pm y\cos\alpha$$

De transformaties van een oneindige transformatiegroep laten zich niet met behulp van een eindig aantal parameter vastleggen. Voorb.:

oneindig

9.

$$\left.\begin{aligned} 'x &= f(x,y) \\ 'y &= \varphi(x,y) \end{aligned}\right] \; ; \quad \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial\varphi/\partial x & \partial\varphi/\partial y \end{vmatrix} \neq 0$$

V. Lie groups and linear connections.

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§ 1. Finite continuous groups.

An r - parametrical finite continuous group in the sense of Lie is a set of elements between which a "multiplication" is defined and that satifies the following conditions.

a) The elements are in one-to-one correspondence with the points of an $R_r(\eta^\alpha)$ in an X_r with the coordinates $\eta^\alpha;\ \alpha = 1,\dots r$;

b) If the element T_η belongs to η^α and T_ξ to ξ^α the product $T_\theta = T_\eta T_\xi$ belongs to the set and the θ^α are continuous (or analytic) functions of the η^α and ξ^α .

c) There is an element J called "unity" corresponding to η_0^α such that $J T_\eta = T_\eta J = T_\eta$ for every choice of T_η .

d) To every T_η there belongs an element T_η^{-1} such that $T_\eta T_\eta^{-1} = T_\eta^{-1} T_\eta = J$. Its coordinates are continuous (or analytic) functions of the η^α .

e) $$(T_\eta T_\xi) T_\theta = T_\eta (T_\xi T_\theta) .$$

The elements are often transformations in n variables but this is not necessary. From the definition we see that only elements of the group are considered in a neighbourhood of unity. This neighbourhood is often called the group germ (Gruppenkeim).

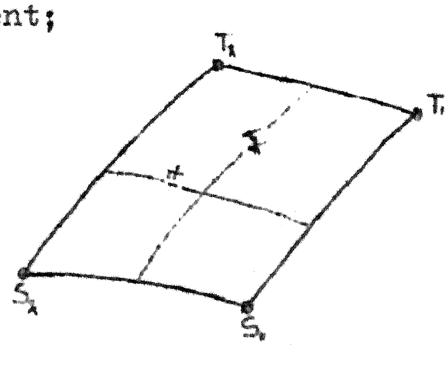
We identify each element with its corresponding point in X_r This X_r is called group space.

To every pair of elements S, T there belong two elements TS^{-1} and $S^{-1}T$. Two pairs S, T and S_1, T_1 are called

(+) - equipollent if $TS^{-1} = T_1 S_1^{-1}$
(-) - equipollent if $S^{-1}T = S_1^{-1} T_1$

From this we get the following deductions:

1. If $S_1, T_1 ; S_2, T_2$ are ($\pm$)-equipollent three of these transformations determine the fourth uniquely;
2. If $S_1, T_1 ; S_2, T_2$ and $S_2, T_2 ; S_3, T_3$ are ($\pm$)-equipollent, the same holds for $S_1, T_1 ; S_3, T_3$;
3. If $S, T ; S_1, T_1$ and $T, U ; T_1, U_1$ are ($\pm$)-equipollent the same holds for $S, U ; S_1, U_1$;
4. If $S_1, T_1 ; S_2, T_2$ are ($\pm$)-equipollent, then $S_1, S_2 ; T_1, T_2$ are ($\mp$)-equipollent;



5. $\mathfrak{M}(T_1)$ can always be mapped on an $\mathfrak{M}(T_2)$ such that T_2 is the image of T_1 and that all pointpairs of $\mathfrak{M}(T_1)$ are mapped $(\pm)$-equipollent on the pointpairs of $\mathfrak{M}(T_2)$;
6. If $\mathfrak{M}(T_1)$ is mapped in the same way on $\mathfrak{M}(T_2)$ $\mathfrak{M}(T_2)$ $\mathfrak{M}(T_3)$ we get the same result as by mapping $\mathfrak{M}(T_1)$ on $\mathfrak{M}(T_3)$ in the same way.

If a pointpair $T_{\eta_0}, T_{\eta_0+d\eta}$ is given there exists for every choice of T_η in an $\mathfrak{M}(\eta_0^\alpha)$ a $(\pm)$-equipollent pointpair $T_\eta, T_{\eta+d\eta}$. Hence, using for instance the (+)-equipollence we get for every choice of a vector $v^\alpha dt$ in η_0^α we get a vectorfield $v^\alpha dt$ in $\mathfrak{M}(\eta_0^\alpha)$. That means that the (+)-equipollence fixes a displacement (III § 2) for contravariant vectors in $\mathfrak{M}(\eta_0^\alpha)$. According to the conditions 1, 2, 3, 5 and 6 this displacement or connection is linear in the sense defined in III § 2 and its parameters $\overset{+}{\Gamma}{}^\alpha_{\gamma\beta}$ can be easily computed by using n linearlyindependent (+)-equipollent vectorfields. In the same way the (-)-equipollence gives rise to another linear connection with parameters $\overset{-}{\Gamma}{}^\alpha_{\gamma\beta}$. The curvature affinors of both connections vanish because in both cases there exist n linearly independent covariant constant vectorfields

1.1) $$\overset{\pm}{R}{}_{\delta\gamma\beta}{}^{\cdot\cdot\cdot\alpha} = 2\,\partial_{[\delta}\overset{\pm}{\Gamma}{}^\alpha_{\gamma]\beta} + 2\,\overset{\pm}{\Gamma}{}^\alpha_{[\delta|\varepsilon|}\overset{\pm}{\Gamma}{}^\varepsilon_{\gamma]\beta} = 0$$

The connections are in general not symmetric.

1.2) $$\overset{\pm}{S}{}_{\gamma\beta}{}^{\cdot\cdot\alpha} \overset{\text{def}}{=} \overset{\pm}{\Gamma}{}^\alpha_{[\gamma\beta]} \neq 0$$

If a curve contains a pointpair R, S and the first point T of a (+)-equipollent pointpair T, U , it may happen that it contains also the point U . If this is the case for every choice of R, S and T the curve is said to be a (+)-geodesic. In the same way (-)-geodesics can be defined. If a (+)-geodesic contains T and $\mathcal{V}T$ it contains also $\mathcal{V}^2T$, $\mathcal{V}^3T$,.. etc. Taking $\mathcal{V}$ infinitesimally different from $\mathcal{J}$ we see that a (+)-geodesic is identical with a geodesic of the connexion $\overset{+}{\Gamma}{}^\alpha_{\gamma\beta}$ and that for every choice of $\mathcal{V}$ the points $T, \mathcal{V}T; \mathcal{V}T, \mathcal{V}^2T$ etc. have the same distance if measured by means of an affine parameter on the geodesic (cf. III § 6). Now if we take $\mathcal{V}T = T\mathcal{W}$ it follows that $\mathcal{V}^2T = T\mathcal{W}^2$ etc. But this implies that a (+)-geodesic is also a (-)-geodesic and that the affine parameters for the (+)-connexion and for the (-)-connexion are the same. Using this parameter the equation of the geodic must have the form

1.3) $$\frac{d^2\eta^\alpha}{dt^2} + \overset{\pm}{\Gamma}{}^\alpha_{\gamma\beta}\frac{d\eta^\gamma}{dt}\frac{d\eta^\beta}{dt} = 0$$

and from this it follows that

1.4) $$\Gamma^\alpha_{\gamma\beta} \overset{\text{def}}{=} \overset{+}{\Gamma}{}^\alpha_{(\gamma\beta)} = \overset{-}{\Gamma}{}^\alpha_{(\gamma\beta)}$$

are the parameters of a symmetric connection that can be obtained by symmetrizing the (+)-connection of the (-)-connexion. This third connection is in general not integrable and its curvation affinor

1.5) $$R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha} \stackrel{\text{def}}{=} 2\,\partial_{[\delta}\Gamma_{\gamma]\beta}^{\alpha} + \Gamma_{[\delta|\varepsilon|}^{\alpha}\Gamma_{\gamma]\beta}^{\varepsilon}$$

plays an important role.

If in η_0^α we fix local measuring vectors $\underset{b}{e}^{\alpha}$; $\overset{a}{e}_{\beta}$; $a, b = 1,\dots n$, these vectors can be displaced (+)-parallel to all points of X_n. Then we get in X_n a new coordinatesystem (a), that is in general non holonomic and for which

1.6) $$A_b^\alpha \stackrel{*}{=} \underset{b}{e}^{\alpha} \quad ; \quad A_\beta^a \stackrel{*}{=} \overset{a}{e}_{\beta}.$$

The $\overset{a}{e}_{\beta}$ are (+)-constant (= covariant constant for the (+)-connection), hence

1.7) $$0 = \overset{+}{\nabla}_\gamma \overset{a}{e}_\beta = \partial_\gamma \overset{a}{e}_\beta - \overset{+}{\Gamma}_{\gamma\beta}^{\alpha}\overset{a}{e}_\alpha$$

and

1.8) $$\boxed{\partial_{[\gamma}\overset{a}{e}_{\beta]} = -\tfrac{1}{2}\, c_{cb}^{a}\,\overset{c}{e}_{\gamma}\,\overset{b}{e}_{\beta}} \qquad ; \; c_{cb}^{a} \stackrel{\text{def}}{=} -2\,\overset{+}{S}_{cb}^{\cdot\cdot a}$$

In the same way a (-)-parallel system $\underset{B}{e}^{\alpha}$; $\overset{A}{e}_{\beta}$; $A, B = 1,\dots n$ can be introduced. Then we get another non holonomic coordinate system (A) with

1.9) $$A_B^\alpha \stackrel{*}{=} \underset{B}{e}^{\alpha} \quad ; \quad A_\beta^A = \overset{A}{e}_{\beta}$$

and

1.10) $$\boxed{\partial_{[\gamma}\overset{A}{e}_{\beta]} = -\tfrac{1}{2}\, {}'c_{CB}^{A}\,\overset{C}{e}_{\gamma}\,\overset{B}{e}_{\beta}} \qquad ; \; {}'c_{CB}^{A} \stackrel{\text{def}}{=} -2\,\overset{-}{S}_{CB}^{\cdot\cdot A}.$$

Now we will prove that the c_{cb}^{a} and ${}'c_{CB}^{A}$ are <u>constants</u>

The equation

1.11) $${}'T = UT$$

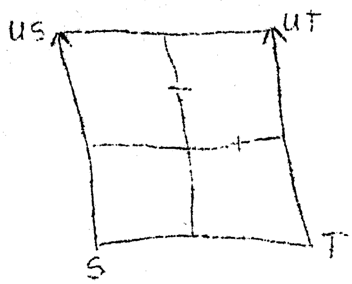
represents a transformation of the elements of the group by which a one-to-one correspondence between the elements and their transforms is fixed. If for U all elements are taken we get a group of element transformations, the <u>first parameter-group</u> and the transformations of this group are in one-to-one correspondence with the elements of the original group. For the transformations of the group the following propositions can be proved easily:

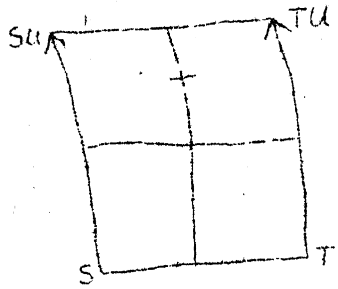
a. If S, T is transformed into ${}'S, {}'T$, the pairs S, T, ${}'S, {}'T$ are (-)-equipollent and the pairs $S, {}'S$; $T, {}'T$ are (+)-equipollent.

b. If S_1, T_1; S_2, T_2 are $(\pm)$-equipollent their transforms are also $(\pm)$-equipollent.

The inversions of these propositions are also true.

In the same way starting from the transformation $T = TU$ the <u>second parameter group</u> can be obtained. For this group the same holds as for the first group if $(\pm)$ is changed into $(\mp)$.





For every choice of b and dt the field $\underset{b}{e}^\alpha dt$ is (+)-constant, hence the pointpairs $\eta^\alpha, \eta^\alpha + \underset{b}{e}^\alpha dt$ are all (+)-equipollent. In the same way the pointpairs $\eta^\alpha, \eta^\alpha + \underset{B}{e}^\alpha dt$ are all (−)-equipollent. Hence the transformation $\eta^\alpha \to \eta^\alpha + \underset{b}{e}^\alpha dt$ is a transformation of the first parameter group (inversion of (a)) that leaves invariant the fields $\underset{B}{e}^\alpha$ (proposition (b)) (and of course $\overset{A}{e}_\beta$) and in the same way $\eta^\alpha \to \eta^\alpha + \underset{B}{e}^\alpha dt$ belongs to the second parameter group and leaves invariant $\underset{b}{e}^\alpha$ and $\overset{a}{e}_\beta$.
That means that the Lie derivatives of $\underset{B}{e}^\alpha$ and $\overset{A}{e}_\beta$ with respect to $\underset{b}{e}^\alpha$ and those of $\underset{b}{e}^\alpha$ and $\overset{a}{e}_\beta$ with respect to $\underset{B}{e}^\alpha$ vanish:

1.12) $$0 = \underset{b}{e}^\gamma \partial_\gamma \underset{B}{e}^\alpha - \underset{B}{e}^\gamma \partial_\gamma \underset{b}{e}^\alpha = -\underset{b}{e}^\gamma \overset{-}{\Gamma}{}^\alpha_{\gamma\beta} \underset{B}{e}^\beta + \underset{B}{e}^\gamma \overset{+}{\Gamma}{}^\alpha_{\gamma\beta} \underset{b}{e}^\beta .$$

or

1.13) $$\overset{+}{\Gamma}{}^\alpha_{\gamma\beta} = \overset{-}{\Gamma}{}^\alpha_{\beta\gamma}$$

and consequently

1.14) $$\overset{+}{S}{}_{\gamma\beta}^{\cdot\cdot\alpha} = -\overset{-}{S}{}_{\gamma\beta}^{\cdot\cdot\alpha}$$

From (1.4) and (1.13) it follows that

1.15) $$2\,\Gamma^\alpha_{\gamma\beta} = \overset{+}{\Gamma}{}^\alpha_{\gamma\beta} + \overset{-}{\Gamma}{}^\alpha_{\gamma\beta}$$

hence, for every quantity Φ (indices suppressed)

1.16) $$2\,\nabla_\gamma \Phi = \overset{+}{\nabla}_\gamma \Phi + \overset{-}{\nabla}_\gamma \Phi$$

For $-2\,\overset{+}{S}{}_{cb}^{\cdot\cdot a}$ equal to $+2\,\overset{-}{S}{}_{cb}^{\cdot\cdot a}$ we write from now on also $C_{cb}^{\cdot\cdot a} \overset{*}{=} C^a_{cb}$

By mixing over $\beta\gamma$ we get once more (1.4)

<u>Exercise.</u> If $\underset{L}{D}$ symbolizes the Lie derivative with respect to $\underset{b}{e}^\alpha$, prove that:

$$\underset{L}{D} = \underset{b}{e}^\alpha \overset{-}{\nabla}_\alpha$$

If the fields $\overset{a}{e}_\beta$ are invariant for the transformations $\underset{B}{e}^\alpha dt$ the same must hold for the rotations $\partial_{[\gamma} \overset{a}{e}_{\beta]}$. Hence applying the general formula for the Lie derivative {D p.47 (7.16 – 21)} we get

1.17) $\left(\underset{B}{e}^{\delta}\partial_{\delta}c_{cb}^{a}\right)\overset{c}{e}_{\gamma}\overset{b}{e}_{\beta}+c_{c\ell}^{a}\left(\underset{B}{e}^{\delta}\partial_{\delta}\overset{c}{e}_{\gamma}\right)\overset{\ell}{e}_{\beta}+c_{cb}^{a}\overset{c}{e}_{\gamma}\underset{B}{e}^{\delta}\partial_{\delta}\overset{b}{e}_{\beta}+$
$$+c_{c\ell}^{a}\overset{c}{e}_{\delta}\overset{\ell}{e}_{\beta}\partial_{\gamma}\underset{B}{e}^{\delta}+c_{cb}^{a}\overset{c}{e}_{\gamma}\overset{b}{e}_{\delta}\partial_{\beta}\underset{B}{e}^{\delta}=0$$

or, because of the vanishing of the Lie derivatives of $\overset{a}{e}_{\beta}$ with respect to $\underset{B}{e}^{\alpha}$

1.18) $$\partial_{\delta}c_{cb}^{a}=0.$$

In the same way it can be proved that the $'c_{CB}^{A}$ are constants. If for convenience we take the local coordinate systems (a) and (A) in η_0^{α} identical we have in that point

1.19) $$'c_{CB}^{A}\overset{*}{=}-2\bar{S}_{CB}^{\cdot\cdot A}=+2\overset{+}{S}_{CB}^{\cdot\cdot A}\overset{*}{=}2\delta_{CBa}^{cbA}\overset{+}{S}_{cb}^{\cdot\cdot a}=-\delta_{CBa}^{cbA}c_{cb}^{a}.$$

hence

1.20) $$'c_{CB}^{A}=-\delta_{CBa}^{cbA}c_{cb}^{a}.$$

in <u>all</u> points because of the constancy of these parameters. The quantity $\overset{+}{S}_{\gamma\beta}^{\cdot\cdot\alpha}$ has the remarkable property that its components with respect to (a) have the same values in all points of X_n and that the same holds for its components with respect to (A). Because all components $\overset{+}{\Gamma}_{cb}^{a}$ and $\bar{\Gamma}_{CB}^{A}$ vanish it follows that

1.21) $$\overset{+}{\nabla}\overset{+}{S}_{\gamma\beta}^{\cdot\cdot\alpha}=\bar{\nabla}_{\delta}\bar{S}_{\gamma\beta}^{\cdot\cdot\alpha}=\nabla_{\delta}\overset{\pm}{S}_{\gamma\beta}^{\cdot\cdot\alpha}=0$$

Apart from any considerations about groups we ask now whether it is possible to have in an X_n a non-holonomic system (a) with measuring vectors $\underset{b}{e}^{\alpha};\overset{a}{e}_{\beta}$ satisfying equations of the form

1.22) $$2\partial_{[\gamma}\overset{a}{e}_{\beta]}=-c_{cb}^{a}\overset{c}{e}_{\gamma}\overset{b}{e}_{\beta}\qquad,c_{(cb)}^{a}=0.$$

with $\frac{1}{2}n(n-1)$ given (not necessarily constant) coefficients c_{cb}^{a} .
Necessary and sufficient conditions are that the integrability conditions of (1.22) are satisfied:

1.23) $$\begin{aligned}0&=\left(\partial_{[\delta}c_{|cb|}^{a}\right)\overset{c}{e}_{\gamma}\overset{b}{e}_{\beta]}+c_{cb}^{a}\left(\partial_{[\delta}\overset{c}{e}_{\gamma}\right)\overset{b}{e}_{\beta]}+\\&+c_{cb}^{a}\overset{c}{e}_{[\gamma}\partial_{\delta}\overset{b}{e}_{\beta]}=\left(\partial_{[d}c_{cb]}^{a}\right)\overset{d}{e}_{\delta}\overset{c}{e}_{\gamma}\overset{b}{e}_{\beta}-\\&-\tfrac{1}{2}c_{cb}^{a}c_{de}^{c}\overset{d}{e}_{[\delta}\overset{e}{e}_{\gamma}\overset{b}{e}_{\beta]}+c_{cb}^{a}c_{de}^{b}\overset{c}{e}_{[\gamma}\overset{d}{e}_{\delta}\overset{e}{e}_{\beta]}\end{aligned}\qquad\partial_{d}=\underset{d}{e}^{\delta}\partial_{\delta}.$$

or

1.24) $$\partial_{[d}c_{cb]}^{a}+c_{[dc}^{e}c_{b]e}^{a}=0$$

If these conditions are satisfied and if the $c_{c\ell}^{a}$ are regular in a point η_α, to every set of vectors $\overset{\circ}{e}{}_{\ell}^{a}$, $\overset{a}{e}{}_{\beta}$ there exist fields in an $\mathfrak{N}(\eta^\alpha)$ satisfying (1.22).

Secondly we ask, whether there exists an infinitesimal transformation $v^\alpha dt$ leaving the fields $\overset{a}{e}{}_{\beta}$ invariant. Necessary and sufficient conditions are that the Lie derivatives of the $\overset{a}{e}{}_{\beta}$ with respect to v^α vanish:

1.25)
$$0 = v^\gamma \partial_\gamma \overset{a}{e}{}_{\beta} + \overset{a}{e}{}_{\gamma} \partial_\beta v^\alpha = = -v^\gamma c_{c\ell}^{a} \overset{c}{e}{}_{\gamma} \overset{\ell}{e}{}_{\beta} + \partial_\beta (v^\gamma \overset{a}{e}{}_{\gamma})$$

or

1.26)
$$\partial_\beta v^a = c_{\ell a}^{c}\, v^c \overset{\ell}{e}{}_{\beta}$$

The integrability conditions of these equations

1.27)
$$0 = (\partial_{[\gamma} c_{|c\ell|}^{a}) v^c \overset{\ell}{e}{}_{\beta]} + c_{c\ell}^{a} (\partial_{[\gamma} v^c) \overset{\ell}{e}{}_{\beta]} + c_{c\ell}^{a} \partial_{[\gamma} \overset{\ell}{e}{}_{\beta]} =$$
$$= (\partial_d c_{c\ell}^{a}) v^c \overset{d}{e}{}_{[\gamma} \overset{\ell}{e}{}_{\beta]} + c_{c\ell}^{a} c_{de}^{c} v^d \overset{e}{e}{}_{[\gamma} \overset{\ell}{e}{}_{\beta]} - \tfrac{1}{2} c_{c\ell}^{a} c_{de}^{\ell} v^c \overset{d}{e}{}_{[\gamma} \overset{e}{e}{}_{\beta]} =$$
$$= (\partial_{[e} c_{|d|\ell]}^{a} + \tfrac{1}{2} c_{c\ell}^{a} c_{de}^{c} - \tfrac{1}{2} c_{ec}^{a} c_{d\ell}^{e} - \tfrac{1}{2} c_{de}^{a} c_{c\ell}^{e}) v^d \overset{c}{e}{}_{[\gamma} \overset{\ell}{e}{}_{\beta]}.$$

and these conditions must be identically satisfied if n independent solutiona exist. Hence we have as necessary and sufficient conditions from (1.24)

1.28)
$$\partial_d c_{c\ell}^{a} + \partial_c c_{\ell d}^{a} + \partial_\ell c_{dc}^{a} = -3\, c_{[dc}^{e} c_{\ell]e}^{a}$$

and from (1.27)

1.29)
$$\partial_\ell c_{dc}^{a} - \partial_c c_{d\ell}^{a} = -3\, c_{[dc}^{e} c_{\ell]e}^{a}$$

But we see immediately that these conditions are equivalent to

1.30) a) $\boxed{c_{c\ell}^{a} = \text{constant}}$ b) $\boxed{c_{[dc}^{e} c_{\ell]e}^{a} = 0.}$

If these conditions are satisfied, to every set of values v^α in a given point of X_n there exists a solution valid in an $\mathfrak{N}(\eta^\alpha)$. If n independent solutions are chosen as the contravariant measuring vectors $\underset{B}{e}{}^\alpha$; $B = 1, .. n$ of some in general non-holonomic coordinate system $(\overset{B}{A})$, we have from (1.25)

1.31)
$$\overset{a}{e}{}_{\gamma} \partial_\beta \underset{B}{e}{}^\gamma = - \underset{B}{e}{}^\gamma \partial_\gamma \overset{a}{e}{}_{\beta}.$$

and accordingly

1.32) $$\left(\partial_\beta \underset{B}{e}^{\alpha}\right) \overset{A}{e}_{\alpha} = -\underset{B}{e}^{\gamma}\left(\partial_\gamma \overset{a}{e}_{\beta}\right) \underset{a}{e}^{\alpha} \overset{A}{e}_{\alpha}$$

or

1.33) $$\partial_\beta \overset{A}{e}_{\alpha} = \underset{a}{e}^{\gamma} \overset{A}{e}_{\gamma} \partial_\alpha \overset{a}{e}_{\beta}$$

hence

1.34) $$\partial_{[\gamma} \overset{A}{e}_{\beta]} = \tfrac{1}{2}\, c_{cb}^{a}\, \overset{c}{e}_{\gamma}\, \overset{b}{e}_{\beta}\, \underset{a}{e}^{\delta}\, \overset{A}{e}_{\delta}$$

or

1.35) $$\boxed{\partial_{[\gamma} \overset{A}{e}_{\beta]} = \tfrac{1}{2}\, {}'c_{CB}^{A}\, \overset{C}{e}_{\gamma}\, \overset{B}{e}_{\beta}}$$

The fields $\overset{a}{e}_{\beta}$ being invariant for the transformations $\underset{B}{e}^{\alpha} dt$ the fields $\underset{b}{e}^{\alpha}$ are invariant as well. Hence the Lie derivative of $\underset{b}{e}^{\alpha}$ with respect to $\underset{B}{e}^{\alpha}$ vanishes:

1.36) $$\underset{b}{e}^{\gamma} \partial_\gamma \underset{B}{e}^{\alpha} - \underset{B}{e}^{\gamma} \partial_\gamma \underset{b}{e}^{\alpha} = 0$$

and this has as a consequence that the fields $\underset{B}{e}^{\alpha}$ and $\overset{A}{e}_{\beta}$ are invariant for the transformations $\underset{b}{e}^{\alpha} dt$. But we know already that r independent fields $\overset{A}{e}_{\beta}$, satisfying (1.35) and invariant for r independent infinitesimal transformations can only exist if the ${}'c_{CB}^{A}$ are constants satisfying the equation

1.37) $$ {}'c_{[DC}^{E}\; {}'c_{B]E}^{A} = 0$$

Now if we choose the measuring vectors $\underset{B}{e}^{\alpha}$ in such a way that they coincide with the the $\underset{b}{e}^{\alpha}$ in any arbitrarily chosen point of X_n we have in that point $A_a^A = \delta_a^A$ and accordingly

1.38) $$ {}'c_{CB}^{A} = -\delta_{CBa}^{cbA}\; c_{cb}^{a}$$

But then the same equation holds in all points of X_n because the ${}'c_{CB}^{A}$ are constant. Accordingly also (1.37) holds, as a consequence of (1.30 b) and (1.38). Note that there is a difference between the X_n considered here and a group space because in this latter space one point corresponding to the identical transformation is fixed.

After this intermezzo on sets of vectorfields in X_n returning to groupspace and the first and second parameter group, we may ask whether $\lambda^{b} \underset{b}{e}^{\alpha} dt$ represents an infinitesimal transformation of the fir[st] parametric group. Necessary and sufficient is the invariance of the fields $\underset{B}{e}^{\alpha}$.

1.39) $$\lambda^{b} \underset{b}{e}^{\beta} \partial_\beta \underset{B}{e}^{\alpha} - \underset{B}{e}^{\alpha} \partial_\beta \left(\lambda^{b} \underset{b}{e}^{\alpha}\right) = 0$$

But this equation can only be valid if the $\overset{b}{\lambda}$ are constants.

In group theory it is usual to introduce the operator $A_b^x \partial_x$ as a symbol, for instance A_b for the infinitesimal tramsformation $\eta^\alpha \to \eta^\alpha + \underset{b}{e}^\alpha dt$. For the operator $A_c A_b - A_b A_c$ always $(A_c A_b)$ (without a comma) is written. Because of

1.40) $$c_{cb}^a = -2 \underset{c}{e}^\gamma \underset{b}{e}^\beta \partial_{[\gamma} \overset{a}{e}_{\beta]} = 2 \underset{[c}{e}^\gamma \overset{a}{e}_{|\beta|} \partial_{|\gamma|} \underset{b]}{e}^\beta = 2 A_\beta^a A_{[c}^\gamma \partial_{|\gamma|} A_{b]}^\beta$$

it follows that

1.41) $$\boxed{(A_c A_b) = c_{cb}^a A_a}$$

and in the same way it is proved that for the second parametric group

1.42) $$\boxed{(A_C A_B) = {}'c_{CB}^A A_A}$$

The equations (1.41, 42) are called the <u>structural formulae after Lie</u> of the first and second parameter group. <u>Cartan</u> called (1.8, 10) the <u>structural formulae</u> of these groups. These latter formulae were established first by Maurer[1]. Lie starts from infinitesimal transformations and this is the reason why he looks at the matter from the contravariant side. On the contrary Cartan starts from systems of Pfaffians and his point of view is therefore entirely covariant. The rather fine dualism between the two points of view was pointed out by Cartan.

We gather the following results:

An r-parametrical group being given, there exist in group space two anholonomic coordinatesystems (a) and (A), satisfying the equations (1.8), (1.35), (1.41), (1.42) and (1.30). There are two groups of point transformations in X_r, the two parametrical groups, whose transformations are in one-to-one correspondance with the transformations of the given group. The first leaves invariant the fields of measuring vectors of (A) and the second those of (a).

Conversely $\frac{1}{2} r^2(r-1)$ constants $c_{cb}^a = c_{[cb]}^a$ being given satisfying (1.30 b), it is always possible to construct in an arbitrary X_n an anholonomic system (a) satisfying (1.8) and (1.41) and another anholonomic system (A) satisfying (1.35) and (1.42). The $A_a^x \partial_x$ are the symbols of infinitesimal transformations of a group of point transformations in X_r leaving invariant the fields of measuring vectors of (A) and the same holds for the $A_B^\alpha \partial_x$ with respect to the measuring vector of (a). The transformations of these groups can be brought into one-to-one correspondence in an infinite number of ways.

Besides the two parameter-groups there is a third transformation group connected with every group. The equation

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1) Maurer, 1888.

1.43) $$'T = U T U^{-1}$$

represents a transformation of the elements T of the group, If for U all elements are taken we get a group called the adjoint group of the given group. Because of

1.44) $$V U T U^{-1} V^{-1} = (VU) T (VU)^{-1}$$

this group and the given group are isomorph but the isomorphy need not be holoëdric. In fact, two elements U_1 and U_2 may exist such that

1.45) $$U_1 T U_1^{-1} = U_2 T U_2^{-1}$$

This is the case if and only if there exists besides J another element that is commutative with respect to every element of the group. All elements of this kind form an invariant subgroup called the centre of the group .

Let C be such an element, then the trandformation UTU^{-1} of the adjoint group correspondn not only to U but also to CU .

A transformation of the adjoint group transforms every subgroup into a homologous subgroup that is identical with the original one if and only if this subgroup is invariant.

Als voorbeeld nemen we een <u>transformatiegroep</u>, de groep der homogene lineaire punttransformaties in een $\mathcal{E}_n$

1.45 a) $$'x^\kappa = T^\kappa{}_\lambda x^\lambda$$

De $T^{\kappa}_{.\lambda}$ zijn de parameters der transformatie dus is $r = n^2$.
De parametergroepen zijn

eerste: $'T^{\kappa}_{.\lambda} = U^{\kappa}_{.\rho} T^{\rho}_{.\lambda}$ tweede: $'T^{\kappa}_{.\lambda} = T^{\kappa}_{.\rho} U^{\rho}_{.\lambda}$

Zij zijn holoëdrisch isomorph met de groep.
De geadjungeerde groep is

1.45 b) $$'T^{\kappa}_{.\lambda} = U^{\kappa}_{.\rho} T^{\rho}_{.\sigma} \overset{-1}{U}{}^{\sigma}_{.\lambda}$$

aangezien $\alpha A^{\kappa}_{\lambda}$ voor iedere α commutatief is met iedere transformatie van de groep correspondeert de transformatie (1.45 b) zowel met het element $U^{\kappa}_{.\lambda}$ van de groep als ook met alle elementen $\alpha U^{\kappa}_{.\lambda}$. De isomorphie is dus meroëdrisch. Een infinitesimale transformatie van de eerste (tweede) parametergroep is van de vorm

eerste: $$'T^{\kappa}_{.\lambda} = (A^{\kappa}_{\rho} + U^{\kappa}_{.\rho}\, dt) T^{\rho}_{.\lambda} = T^{\kappa}_{.\lambda} + U^{\kappa}_{.\rho} T^{\rho}_{.\lambda}\, dt$$

1.45 c)

tweede: $$'T^{\kappa}_{.\lambda} = T^{\kappa}_{.\lambda} + T^{\kappa}_{.\rho} U^{\rho}_{.\lambda}\, dt$$

en een infinitesimale transformatie van de geadjungeerde groep heeft de vorm

1.45 d) $$\begin{aligned} 'T^{\kappa}_{.\lambda} &= (A^{\kappa}_{\rho} + U^{\kappa}_{\rho}\, dt) T^{\rho}_{.\sigma} (A^{\sigma}_{\lambda} - U^{\sigma}_{.\lambda}\, dt) = \\ &= T^{\kappa}_{.\lambda} + (U^{\kappa}_{.\rho} T^{\rho}_{..\sigma} - T^{\kappa}_{.\rho} U^{\rho}_{.\sigma})\, dt \end{aligned}$$

§ 2. Finite continuous transformations groups.

Till now we did not in any way suppose that the originally given group was a group pf transformations.
Let us take now an r-parametrical group of transformations in n variable ξ^κ, $\kappa = 1,\dots n$

2.1) $$'\xi^\kappa = f^\kappa(\xi^\nu; \eta^\alpha) \qquad \alpha = 1,\dots r$$

First we have to make sure that the ξ^κ can be solved from (2.1) The necessary and sufficient condition is that the matrix of the $\partial_\lambda f^\kappa$ has rank n. Secondly it is necessary that the η^α are essential, that is, that there do not exist equations of the form

2.2) $$f^\kappa(\xi^\nu, \eta^\alpha) = F^\kappa(\xi^\nu, \zeta^{\alpha'}(\eta^\alpha)) \quad , \alpha' = 1,\dots r' ; r' < r$$

From (2.2) follows that

2.3) $$\partial_\beta f^\kappa = (\partial_\beta \zeta^{\alpha'}) \partial_{\alpha'} F^\kappa \qquad ; \partial_{\alpha'} = \frac{\partial}{\partial \zeta^{\alpha'}}$$

and because the rank of $\partial_\beta \zeta^{\alpha'}$ is $\leq r'$ this implies that there exists at least one set of functions $u^\beta(\eta^\alpha)$ such that $u^\beta(\eta^\alpha)\partial_\beta \zeta^{\alpha'} = 0$ and consequently:

2.4) $$u^\beta(\eta^\alpha)\, \partial_\beta f^\kappa(\xi^\nu, \eta^\alpha) = 0$$

Conversely, if an equation of the form (2.4) holds, the $f(\xi^\kappa, \eta^\alpha)$ must be functions of the ξ^κ and of the solutions of the equation

2.5) $$u^\beta(\eta^\alpha)\, \partial_\beta \varphi(\eta^\alpha) = 0$$

But because an equation of this form has at most $r-1$ independent solutions the η^α cannot be essential. Hence the η^α in (2.1) are essential if and only if no equation of the form (2.4) exists. From now on we suppose that the η^α are essential and that η^α_0 corresponds to the identical transformation. Then there is a one-to-one correspondence between the transformations of the group in a neighbourhood of the identical transformation and the points of the X_r of the η^α in an $\mathfrak{N}(\eta^\alpha_0)$.

We only consider transformations of this group germ. For every definite choice $\eta^\alpha = \eta^\alpha_1$ the equation (2.1)

2.6) $$'\xi^\kappa = f^\kappa(\xi^\nu, \eta^\alpha_1)$$

represents a definite transformation T_{η_1} working on the ξ^κ. Other values $\eta^\alpha = \eta^\alpha_2$ give another transformation T_{η_2}

2.7) $$''\xi^\kappa = f^\kappa(\xi^\nu, \eta^\alpha_2)$$

of the ξ^κ. If η^α_1 is changed into $\eta^\alpha_1 + d\eta^\alpha$ we have

2.8) $$d\,'\xi^\kappa = (\partial_\beta f^\kappa)_{\eta^\alpha = \eta^\alpha_1} d\eta^\beta$$

In the right hand side of this equation the ξ^K can be eliminated by means of (2.6). Then (2.8) represents the transformation of ${}'\xi^K = T_{\eta_1}\xi^k$ into $T_{\eta_1+d\eta}\,\xi^K$, that is the transformation $T_{\eta_1+\underset{1}{d}\eta}T_{\eta_1}^{-1}$ working on the ${}'\xi^K$. This transformation corresponds to the displacement $\underset{1}{d}\eta^\alpha$ in the point $\underset{1}{\eta}^\alpha$ in group space. In the same way

2.9) $$d''\xi^K = (\partial_\beta f^K)_{\eta^\alpha=\underset{2}{\eta}^\alpha}\,\underset{2}{d}\eta^\beta$$

after the elimination of the ξ^K represents the transformation $T_{\eta_2+d\eta}T_{\eta_2}^{-1}$ working on the $''\xi$ and corresponding to the displacement $\underset{2}{d}\eta^\alpha$ in the point $\underset{2}{\eta}^\alpha$ in group space. If now

2.10) $$T_{\eta_1+\underset{1}{d}\eta}T_{\eta_1}^{-1} = T_{\eta_2+\underset{2}{d}\eta}T_{\eta_2}^{-1}$$

then $\underset{1}{d}\eta^\alpha$ in $\underset{1}{\eta}^\alpha$ can be transformed into $\underset{2}{d}\eta^\alpha$ in $\underset{2}{\eta}^\alpha$ by a (+)-parallel displacement. Hence, introducing the coordinate system (a) in group space, we have $\underset{1}{d}\eta^a = \underset{2}{d}\eta^a;\ a = 1,\dots r$. Now

2.11) $$d'\xi^K = (\partial_\beta{}'\xi^K)_{\eta^\alpha=\underset{1}{\eta}^\alpha}\,A^\beta_b(\underset{1}{\eta}^\alpha)\,\underset{1}{d}\eta^b$$

and

2,12) $$d''\xi^K = (\partial_\beta{}''\xi^K)_{\eta^\alpha=\underset{2}{\eta}^\alpha}\,A^\beta_b(\underset{2}{\eta}^\alpha)\,\underset{2}{d}\eta^b$$

after elimination of the ξ^K represent one and the same transformation, working on the ${}'\xi^K$ in (2.11) and on the $''\xi^K$ in (2.12). But <u>this is only possible if the expressions</u>

2.13) $$\Xi^K_b \overset{def}{=} (\partial_\beta{}'\xi^K)A^\beta_b$$

<u>after elimination of the</u> ξ^K, <u>depend only on the</u> ${}'\xi^K$ <u>and are independent of the</u> η^α.

There cannot exist relations of the form

2.14) $$c^b\,\Xi^K_b = 0$$

with coefficients c^b which are independent of the ${}'\xi^K$ because in that case there would exist a relation

2.15) $$c^b A^\beta_b \partial_\beta f^K = 0$$

and we have already proved that this would imply that the parameters were not essential.

Collecting results we have the first part of the first fundamental theorem of Lie

I. 1) <u>If the invertible transformations</u>

2.16) $${}'\xi^K = f^K(\xi^\nu, \eta^\alpha) \qquad \alpha = 1,\dots,r$$

<u>with</u> r <u>essential parameters</u> η^α <u>form a group, there exist</u> r^2 <u>function</u> $A^\alpha_b(\eta^\beta)$; $\mathrm{Det}(A^\alpha_b) \neq 0$ <u>and</u> nr <u>functions</u> $\Xi^K_b({}'\xi^\lambda)$ <u>for which no equations exist of the form</u> $c^b\,\Xi^K_b = 0$ <u>with coefficients</u> c^b independent of ${}'\xi^K$ and for which

2.17)
$$\boxed{\partial_\beta {}'\xi^K = \Xi^K_B(\xi) A^B_\beta(\eta^\alpha)}$$

Starting from the inverse of (2.16)

2.18)
$$\xi^K = F^K({}'\xi^\lambda, \eta^\alpha)$$

instead of (2.1) it can be proved in the same way that

2.19)
$$\partial_\beta {}'\xi^K = {}'\Xi^K_B A^B_\beta(\eta^\alpha)$$

<u>Exercise</u>

Prove that

2.20)
$$'\Xi^K_B(\xi^\lambda) = -\delta^b_B \, \Xi^K_b(\xi^\lambda)$$

<u>Exercise</u>

Prove that the functions $\Xi^K_b(\xi^\lambda)$ and $A^b_\beta(\eta^\alpha)$ can be obtained by algebraic operations and differentiations from the equation (2.16).

Eerste opgave:

De transformatie $'\xi^\kappa \to \xi^\kappa$ voorgesteld door (2.19) is ook een transformatie van de groep en kan dus ook worden geschreven in de vorm

2.20a) $$\xi^\kappa = f^\kappa('\xi^\nu, \underset{2}{\eta}^\alpha)$$

waarin de $\underset{2}{\eta}^\alpha$ nu de coördinaten van dat punt van de groepruimte zijn dat de omkering voorstelt van de door $\underset{1}{\eta}^\alpha$ voorgestelde transformatie. Dan is echter nu ook (na eliminatie der $'\xi^\kappa$)

2.20 b) $$\frac{\partial \xi^\kappa}{\partial \underset{2}{\eta}^\beta} = \Xi^\kappa_\ell(\xi^\lambda)\, A^\ell_\beta(\underset{2}{\eta}^\alpha)$$

en dit geeft in verband met (2.19)

2.20 c) $$'\Xi^\kappa_B(\xi^\lambda)\, A^B_\beta(\underset{1}{\eta}^\alpha)\, \frac{\partial \eta_1{}^\ell}{\partial \eta_2{}^\gamma} = \Xi^\kappa_\ell(\xi^\lambda)\, A^\ell_\gamma(\underset{2}{\eta}^\alpha)$$

Kiest men nu hierin $\underset{1}{\eta}^\alpha = \underset{0}{\eta}^\alpha$, dan wordt ook $\underset{2}{\eta}^\alpha = \underset{0}{\eta}^\alpha$ en $d\underset{1}{\eta}^\alpha = -d\underset{2}{\eta}^\alpha$ en er volgt

2.20 d) $$'\Xi^\kappa_\beta(\xi^\nu) = -\Xi^\kappa_\ell(\xi^\lambda)\, A^\ell_\beta(\underset{0}{\eta}^\alpha)\, A^B_B(\underset{0}{\eta}^\alpha) =$$
$$= -\delta^\ell_B\, \Xi^\kappa_\ell(\xi^\lambda)$$

Tweede opgave

Ga uit van twee op elkander volgende transformaties

2.20 e) $$'\xi^\kappa = f^\kappa(\xi^\nu, \eta^\alpha)$$

en

2.20 f) $$''\xi^\kappa = f^\kappa('\xi^\nu, \zeta^\alpha) = f^\kappa(\xi^\kappa, \theta^\alpha)$$

Houdt men ξ^κ en θ^α constant dan is

2.20 g) $$\frac{\partial''\xi^\kappa}{\partial'\xi^\lambda}\frac{\partial'\xi^\lambda}{\partial\eta^\beta} + \frac{\partial''\xi^\kappa}{\partial\zeta^\alpha}\frac{\partial\zeta^\alpha}{\partial\eta^\beta} = 0$$

en hieruit laat zich $\frac{\partial'\xi^\kappa}{\partial\eta^\beta}$ oplossen omdat $\mathrm{Det}\left(\frac{\partial''\xi^\kappa}{\partial'\xi^\lambda}\right) \neq 0$

2.20 h) $$\frac{\partial'\xi^\kappa}{\partial\eta^\beta} = \Phi^\kappa_\gamma('\xi^\lambda, \zeta^\alpha)\frac{\partial\zeta^\alpha}{\partial\eta^\beta} =$$
$$= \Phi^\kappa_\gamma('\xi^\lambda, \zeta^\alpha)\, \Psi^\gamma_\beta(\eta^\delta, \zeta^\alpha)$$

Geeft men nu aan de ζ^α in (2.20 h) <u>willekeurig vaste waarden</u> dan ontstaat er inderdaad een splitsing van $\partial_\beta\, '\xi^\kappa$ van de gewenste vorm

2.20 i) $$\partial_\beta\, '\xi^\kappa = \varphi^\kappa_\gamma('\xi^\lambda)\, \psi^\gamma_\beta(\eta^\alpha)$$

maar hierin zijn de functies $\varphi^{\kappa}_{\gamma}$ en ψ^{γ}_{β} afhankelijk van de keuze der ζ^{α}. Voert men in elk punt der X_n en anholonoom coördinatenstelsel in met behulp van de formules

2.20 j) $$A^{b}_{\beta}(\eta^{\alpha}) = \delta^{b}_{\beta}\,\psi^{\gamma}_{\beta}$$

dan krijgt men de vorm (2.17)

2.20 k) $$\partial_{\beta}\,{}'\xi^{\kappa} = \delta^{\gamma}_{b}\,\varphi^{\kappa}_{\gamma}({}'\xi^{\lambda})\,A^{b}_{\beta}(\eta^{\alpha}) = \Xi^{\kappa}_{b}({}'\xi^{\lambda})\,A^{b}_{\beta}(\eta^{\alpha})$$

Verandering der gekozen waarden van de ζ^{α} komt neer op een verandering der velden $\underset{b}{e}^{\alpha}$, $\overset{a}{e}_{\beta}$ waarbij deze velden (+)-constant blijven. De bepaling van de Ξ^{κ}_{b} en A^{b}_{β} vergt dus alleen algebraische operaties en differentiaties.

Every infinitesimal transformation of the group can be written in the form $T_{\eta+d\eta}T_{\eta}^{-1}$. According to (2.17) the working of this transformation on the variables ξ^{κ} is given by the equation

2.21) $$d\xi^{\kappa} = \Xi^{\kappa}_{\ell}(\xi^{\lambda})\,(d\eta)^{\ell}$$

Hence, if e^{α} is a (+)-constant vectorfield, e^{a} = constant, the most general infinitesimal transformation is

2.22) $$d\xi^{\kappa} = e^{\ell}\,\Xi^{\kappa}_{\ell}(\xi^{\lambda})\,dt = e^{\ell}\,\Xi^{\lambda}_{\ell}\,\partial_{\lambda}\xi^{\kappa}\,dt$$

The Lie symbol is

2.23) $$X = e^{\ell}X_{\ell} \qquad ; X_{\ell} \overset{def}{=} \Xi^{\lambda}_{\ell}\,\partial_{\lambda}$$

We have proved (cf. 2.14) that there cannot exist a relation of the form $e^{\ell}X_{\ell}=0$ with constant coefficients, but relations of the form $\psi^{\ell}(\xi^{\nu})X_{\ell}=0$ may exist. According to Campbell we call a set of infinitesimal transformations connected if they satisfy a homogeneous linear equation and dependent if the coefficients in this relation are constants.

If the (+)-constant field e^{α} is given, the streamline of this field through η^{α}_{0} arises by (+)-parallel displacement of the vector e^{α} in η^{α}_{0} in its own direction. A parameter t on this curve can be chosen in such a way that in all its points

2.24) $$\frac{d\eta^{\alpha}}{dt} = e^{\alpha} = A^{\alpha}_{a}e^{a}$$

and that $t=0$ in η^{α}_{0}. The transformation corresponding to a general point of this curve can be found in the following way. Leaving ξ^{κ} constant we have in every point of the curve according to (2.17)

2.25) $$\frac{d\,'\xi^{\kappa}}{dt} = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\,e^{\ell}$$

hence for $t=0$

2.26) $$\begin{cases} '\xi^{\kappa} = \xi^{\kappa} \\ \dfrac{d\,'\xi^{\kappa}}{dt} = \Xi^{\kappa}_{\ell}(\xi^{\lambda})\,e^{\ell} = e^{\ell}X_{\ell}\,\xi^{\kappa} = X\xi^{\kappa} \end{cases}$$

By differentiation of (2.25) we get

2.27) $$\frac{d^{2}\,'\xi^{\kappa}}{dt^{2}} = e^{c}\,\Xi^{\mu}_{c}('\xi^{\lambda})\,\frac{\partial\,\Xi^{\kappa}_{\ell}('\xi^{\nu})}{\partial\,'\xi^{\mu}}\,e^{\ell}$$

hence for $t=0$

2.28) $$\frac{d^{2}\,'\xi^{\kappa}}{dt^{2}} = (e^{\ell}X_{\ell})^{2}\,\xi^{\kappa} = X^{2}\xi^{\kappa}$$

Going on in this way we get the series

2.29) $$'\xi^\kappa = \xi^\kappa + \frac{1}{1!} tX\xi^\kappa + \frac{1}{2!} t^2X^2\xi^\kappa + \dots$$

If this series converges we get a set of ∞^1 transformations depending on the parameter t, all belonging to the given group. We use the symbolical notation

2.30) $$e^{tX} = 1 + \frac{1}{1!} tX + \frac{1}{2!} t^2X^2 + \dots$$

and

2.31) $$'\xi^\kappa = e^{tX}\xi^\kappa$$

From

2.32) $$e^{t_2X} e^{t_1X} = e^{(t_2+t_1)X}$$

we see that these transformations form a one-parametric group. This subgroup is said to be generated by the infinitesimal transformation. Now we go back to the X_η that was not a groupspace but possessed a group of point transformations leaving invariant the measuring vectors of the anholonomic system (A). The point transformations in X_η generated by the infinitesimal transformation $e^b A_b = e^b A_b^\alpha \partial_\alpha$ of this group are

2.33) $$'\eta^\alpha = \eta^\alpha + \frac{1}{1!} t(e^bA_b)\eta^\alpha + \frac{1}{2!} t^2(e^bA_b)^2 \eta^\alpha + \dots$$
$$= e^{te^bA_b}\eta^\alpha$$

and by a suitable choice of the e^b and t every transformation of the group can be written in this form. If, however, we keep η^α fixed, say $\eta^\alpha = \eta_0^\alpha$ then (2.33) gives us all points of $\mathfrak{M}(\eta_0^\alpha)$ as transforms of η_0^α and to each such point there belongs one set of numbers te^a. This establishes a one-to-one correspondence between the points of $\mathfrak{M}(\eta_0^\alpha)$ and the transformations of the group germ. That means that the X is now group space and that we have got the same situation as if we had started from an r-parametrical group.

The system of equations

2.34) $$\partial_\beta {'\xi^\kappa} = \Xi_b^\kappa('\xi^\lambda) A_\beta^b(\eta^\alpha)$$

has the solutions

2.35) $$'\xi^\kappa = f^\kappa(\xi^\nu, \eta^\alpha) \quad , \quad \xi^\kappa = f^\kappa(\xi^\nu, \eta_0^\alpha)$$

depending on the n parameters ξ^κ. Accordingly the system is totally integrable and its integrability conditions must be satisfied identically. This means that

2.36) $$0 = \frac{\partial\, \Xi^{\kappa}_{\ell}}{\partial' \xi^{\mu}} \Xi^{\mu}_{c}('\xi^{\lambda}) H^{c}_{[\gamma} H^{\ell}_{\beta]} + {}'\Xi^{\kappa}_{\ell}\, \partial_{[\gamma} H^{\ell}_{\beta]} =$$

$$= H^{c}_{[\gamma} H^{\ell}_{\beta]} \left(\Xi^{\mu}_{c} \frac{\partial}{\partial' \xi^{\mu}} \Xi^{\kappa}_{\ell} - \tfrac{1}{2} c_{c\ell}^{\ a} \Xi^{\kappa}_{a} \right)$$

or

2.37) $$\boxed{(X_c\, X_\ell) = c_{c\ell}^{\ a} X_a}$$

That proves the first part of the second and third fundamental theorem Lie.

II. 1. If X_ℓ are the symbols of r independent infinitesimal transformations of the group mentioned in I. 1.

2.38) $$X_\ell = \Xi^{\lambda}_{\ell}(\xi^{\kappa})\, \partial_\lambda$$

equations of the form (2.37) hold in the constant coefficients

$c_{c\ell}^{\ a}$, $c_{(c\ell)}^{\ a} = 0$

III. 1. The constants $c^{\xi}_{\ell a}$ in (2.37) satisfy the equations (1.30 b).

The second part of the first fundamental theorem is the inverse of the first part:

I. 2. If a set of transformations of the ξ^{κ} is given

2.39) $${}'\xi^{\kappa} = f^{\kappa}(\xi^{\kappa}, \eta^{\alpha}) \quad , \quad \alpha = 1, \dots r,$$

with functions analytic and with $\mathrm{Det}(\partial_\lambda {}'\xi^{\kappa}) \neq 0$ in an $\mathcal{N}(\xi^{\kappa}_{0}, \eta^{\alpha}_{0})$ depending on r essential parameters η^{α} and containing the identical transformation $(\eta^{\alpha} = \eta^{\alpha}_{0})$ and if there exist r² functions $A^{\beta}_{\ell}(\eta^{\alpha})$; $\mathrm{Det}(A^{\beta}_{\lambda}) \neq 0$ analytic in an $\mathcal{N}(\eta^{\alpha}_{0})$ and nr functions $\Xi^{\kappa}_{\ell}(\xi^{\lambda})$ analytic in an $\mathcal{N}(\xi^{\kappa}_{0})$ not satisfying an equation $c^{\ell}\, \Xi^{\kappa}_{\ell}$ with coefficients c^{ℓ}, independent of ξ^{κ} and for which

2.40) $$\partial_\beta {}'\xi^{\kappa} = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\, A^{\ell}_{\beta}(\eta^{\alpha})$$

in an $\mathcal{N}(\xi^{\kappa}_{0}, \eta^{\alpha}_{0})$ these transformations form an r-parametrical group germ.

In order to prove this we remark that just as before according to (2.40) the infinitesimal transformation $T_{\eta+d\eta}\, T^{-1}_{\eta}$ working on the ${}'\xi^{\kappa}$ is given by

2.41) $$d'\xi^{\kappa} = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\, A^{\ell}_{\beta}\, d\eta^{\beta}$$

but it is as yet not sure that this transformation belongs to the set. Moreover, in the X_η of the η^α there do not yet exist any aequipollences or connections.

Now we try to find curves in X_η with a parameter t such that

2.42) $$d\,'\xi^\kappa = e^\ell\, \Xi^\kappa_\ell('\xi^\lambda)\, dt$$

with constants e^ℓ. Therefore it is necessary that

2.43) $$e^\ell\, \Xi^\kappa_\ell\, dt = A^\ell_\beta\, \Xi^\kappa_\ell\, d\eta^\beta$$

but because there are no equations $c^\ell\, \Xi^\kappa_\ell = 0$ with coefficients c^ℓ independent of ξ^κ this is only possible if

2.44) $$A^\ell_\beta\, d\eta^\beta = e^\ell dt$$

Because A^α_β has rank r in an $\mathfrak{N}(\eta^\alpha_0)$ there exists in that region an inverse A^α_ℓ. Hence in this region η^α is a solution of the equation

2.45) $$\frac{d\eta^\alpha}{dt} = e^\ell\, A^\alpha_\ell(\eta^\beta)$$

For a given boundary condition $\eta^\alpha = \eta^\alpha_*$ for $t = 0$ there is one and only one solution for every set of values e^ℓ

2.46) $$\eta^\alpha = \varphi^\alpha(e^\ell, t) \quad , \quad \varphi^\alpha(e^\ell, 0) = \eta^\alpha_*$$

If this solution is substituted in (2.39) the $'\xi^\kappa$ are expressed for every η^α in an $\mathfrak{N}(\eta^\alpha_*)$ as functions of ξ^κ, t and the

2.47) $$'\xi^\kappa = \psi^\kappa(\xi^\lambda, e^\ell, t) \qquad \underset{*}{\xi}^\kappa = \psi^\kappa(\xi^\lambda, e^\ell, 0) = f(\xi^\lambda, \eta^\alpha_*)$$

If the e^ℓ are fixed, this equation represents a set of transformations belonging to (2.39) and depending on one parameter t. <u>All transformations of the given set in an $\mathfrak{N}(\eta^\alpha_*)$ belong to one and only one of these one-parametrical sets. According to (2.42) and (2.47) we have</u>

2.48) $$\frac{d\,'\xi^\kappa}{dt} = e^\ell\, \Xi^\kappa_\ell('\xi^\lambda) \qquad '\xi^\kappa = \underset{*}{\xi}^\kappa = f^\kappa(\xi^\lambda, \eta^\alpha_*) \quad \text{for } t = 0$$

The solution of this equation can be expanded into a series in a neighbourhood of $t = 0$. For $t = 0$ we have

$$'\xi^\kappa = \underset{*}{\xi}^\kappa = f^\kappa(\xi^\lambda, \eta^\alpha_*) \qquad \underset{*}{X}\,\xi^\kappa = e^\ell\, \Xi^\kappa_\ell(\underset{*}{\xi}^\lambda)\, \partial/\partial \underset{*}{\xi}^\kappa$$

2.49) $$\frac{d\,'\xi^\kappa}{dt} = e^\ell\, \Xi^\kappa_\ell(\underset{*}{\xi}^\lambda) = \underset{*}{X}\, \underset{*}{\xi}^\kappa ,$$

$$\frac{d^2\,'\xi^\kappa}{dt^2} = e^\ell\, \frac{\partial\, \Xi^\kappa_\ell('\xi^\lambda)}{\partial\, '\xi^\nu}\, e^c\, \Xi^\nu_c('\xi^\lambda_*) = \underset{*}{X}^2\, \underset{*}{\xi}^\kappa$$

. . . .

hence

2.50) $$'\xi^{\kappa} = \left(1 + \frac{1}{1!}t\underset{*}{X} + \frac{1}{2!}t^2\underset{*}{X}^2 + \dots\right)\underset{*}{\xi}^{\kappa} = e^{t\underset{*}{X}}\underset{*}{\xi}^{\kappa};$$

$$\underset{*}{X} = e^{\ell}\,\Xi^{\mu}_{\ell}(\underset{*}{\xi}^{\lambda})\,\partial/\partial\underset{*}{\xi}^{\mu}$$

The transformations of the $\underset{*}{\xi}^{\kappa}$

2.51) $$\underset{*}{\xi}^{\kappa} \to \left(1 + \frac{1}{1!}tX + \frac{1}{2!}t^2X^2 + \dots\right)\underset{*}{\xi}^{\kappa};\quad X \overset{\text{def}}{=} e^{\ell}\,\Xi^{\mu}_{\ell}(\xi^{\lambda})\,\partial_{\mu}$$

form for every set of values e^{ℓ} a one-parametrical group generated by the infinitesimal transformation X. Hence, comparing (2.50) and (2.51) we see that every transformation of the given set in $\mathfrak{N}(\eta^{\alpha})$ can be obtained by transforming first ξ^{κ} into $\underset{*}{\xi}^{\kappa}$ by means of the transformation T_{η} and by transforming afterwards $\underset{*}{\xi}$ into $'\xi^{\kappa}$ by means of a transformation of a one-parametrical group (2.51) that can be obtained by giving the e^{ℓ} suitable values. Also we see that, conversely, every transformation obtained in this way belongs to the set. If we take $\underset{*}{\eta}^{\alpha} = \underset{0}{\eta}^{\alpha}$ it follows that every transformation T of the set belongs to one of the one-parametrical groups (2.51) and that every transformation of such a group belongs to the set. As a corollary we get that T^{-1} belongs to the set. Now $T_{\underset{*}{\eta}}$ and T are both arbitrary transformations of the given set. But it being proved that $T\,T_{\underset{*}{\eta}}$ and T^{-1} belong to the set, the set is a group.

The second part of the second fundamental theorem is the inverse of the first part.

II, 2. If $X_{\ell} = \Xi^{\lambda}_{\ell}(\xi^{\lambda})\,\partial_{\lambda}$ <u>are the symbols of r independent infinitesimal transformations of the ξ^{κ}, satisfying the equations (2.37) with constant coefficients c^{a}_{cb} these transformations are the infinitesimal transformations of an r-parametrical group germ.</u>

In order to prove this we remark first that the identities:

2.52) $$(X_{[d}(X_c\,X_{b]})) = 0$$

can easily be verified and that they imply that the c^{a}_{cb} satisfy the identities (1.30 b). Now we have already proved that according to these identities and the constancy of the c^{a}_{cb} there exist in the X_r of the auxiliary variables η^{α} r^2 functions $A^{\ell}_{\alpha}(\eta^{\beta})$ satisfying the equations

2.53) $$\partial_{[\gamma}A^{a}_{\beta]} = -\tfrac{1}{2}c^{a}_{cb}\,A^{c}_{[\gamma}A^{b}_{\beta]}\qquad;\quad \mathrm{Det}\,(A^{a}_{\beta}) \neq 0$$

But from (2.37, 52) it follows that the integrability conditions of the equations

2.54) $$\partial_{\beta}\,'\xi^{\kappa} = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\,A^{\ell}_{\beta}(\eta^{\alpha})$$

are identically satisfied. Hence these equations are totally integrable and they have solutions of the form

2.55) $$'\xi^{\kappa} = f^{\kappa}(\xi^{\nu}, \eta^{\alpha}) \quad ; \quad \xi^{\kappa} = f^{\kappa}('\xi^{\nu}, \overset{0}{\eta}{}^{\alpha})$$

$$\mathrm{Det}\,(\partial_{\lambda}{}'\xi^{\kappa}) \neq 0 \quad \text{for } \eta^{\alpha} = \overset{0}{\eta}{}^{\alpha}$$

valid in an $\mathfrak{N}(\overset{0}{\eta}{}^{\alpha})$ and depending on n parameters $\tilde{\xi}^{\kappa}$. That finishes the proof because according to the fundamental theorem I. 2, (2.55) represents an r-parametrical group, and the X_b are infinitesimal transformations of this group. As we see an r-parametrical group or at least its group germ is wholly determined by r independent infinitesimal transformations, It is said to be <u>generated</u> by them.

The second part of the third fundamental theorem is the inverse of the first part.

III. 2. <u>If $\frac{1}{2} r^2(r-1)$ constants C_{cb}^{a} ; $C_{(cb)}^{a} = 0$ satisfy equations of the form (1.30 b) there always exists an r-parametrical group germ with r infinitesimal transformations X_b satisfying (2.37).</u>

In order to prove this we construct in the X_r of the auxiliary coordinates η^{α} in an $\mathfrak{N}(\eta^{\alpha})$, the fields $\overset{a}{e}_{\beta}$ satisfying (1.22). Then the r infinitesimal point transformations $\eta^{\alpha} \rightarrow \eta^{\alpha} + \overset{\alpha}{\underset{b}{e}}\,dt$ with the symbols A_b satisfy (1.41) and in consequence of II. 2. they generate an r-parametrical group germ[1)2)].

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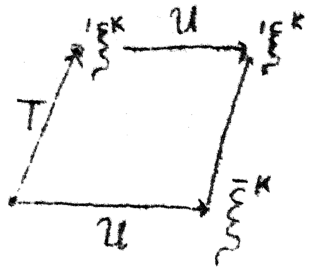
1) In the proofs of the three fundamental theorems given here we made use of the existence theorems of the theory of partial differential equations. For our purposes this is the natural way because sets of partial differential equations are just what we are interested in. But it is not necessary. Freudenthal proved 1938....(Jber. D.Math. Ver. XLIII (1933) p.26) that it is possible to use only the existence theorem from the theory of ordinary differential equations.

2) E.Cartan has shown that the original proofs of the fundamental theorems given by Lie failed if one does not confine oneself to a group germ. Cf. Cartan 1930...., 1937... p.189 (Comptes Rendus 190 (1930) p. 914, 1005; Théorie des groupes finis et continus 1937).

The transformations of the adjoint group show two different aspects in the X_m of the ξ^κ. Let us assume first that T and $\mathcal{U}$ are point transformations

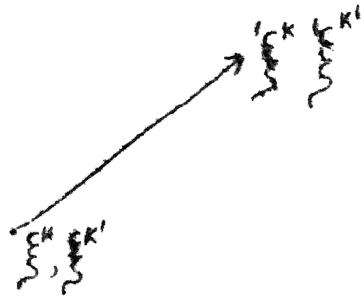


$$2.56)\quad T: \xi^\kappa \to {}'\xi^\kappa \qquad \mathcal{U}: \xi^\kappa \to \bar{\xi}^\kappa,\ {}'\xi^\kappa \to {}'\bar{\xi}^\kappa$$

Then we have

$$2.57)\quad {}'\bar{\xi}^\kappa = \mathcal{U}\,{}'\xi^\kappa = \mathcal{U}T\xi^\kappa = \mathcal{U}T\mathcal{U}^{-1}\bar{\xi}^\kappa$$

If T transforms ξ^κ into ${}'\xi^\kappa$, then $\mathcal{U}\xi^\kappa$ is transformed into $\mathcal{U}\,{}'\xi^\kappa$ by applying $\mathcal{U}T\mathcal{U}^{-1}$ to $\bar{\xi}^\kappa$. In the special case that $\mathcal{U}T\mathcal{U}^{-1} = T$, T and $\mathcal{U}$ are commutative and T is said to be invariant for the transformation $\mathcal{U}$. Now let us assume that T is a point transformation and $\mathcal{U}$ a coordinate transformation



$$2.58)\quad T: \xi^\kappa \to {}'\xi^\kappa \qquad \mathcal{U}: \xi^\kappa \to \xi^{\kappa'}$$

Then we have

$$2.59)\quad {}'\xi^{\kappa'} = \mathcal{U}\,{}'\xi^\kappa = \mathcal{U}T\xi^\kappa = \mathcal{U}T\mathcal{U}^{-1}\xi^{\kappa'}$$

Hence T describes a point transformation with respect to (κ) and $\mathcal{U}T\mathcal{U}^{-1}$ describes the dame transformation with respect to (κ').

The adjoint group is always a group of point transformations in group space. Hence we may ask how this group works on the infinitesimal transformations of the underlying group. Such a transformation corresponds to a point $\overset{\circ}{\eta}{}^\alpha + e^\alpha ds$ in groupspace. Let v^α be some vector given in $\overset{\circ}{\eta}{}^\alpha$ then by $(\pm)$-parallel displacement we can get two vectorfields $\overset{+}{v}{}^\alpha$ and $\bar{v}^\alpha$ with $\overset{+}{v}{}^\alpha = \bar{v}^\alpha = v^\alpha$ in $\overset{\circ}{\eta}{}^\alpha$. $\overset{+}{v}{}^\alpha$ is (+)-constant and $\bar{v}^\alpha$ is (-)-constant. $\eta^\alpha \to \eta^\alpha + \overset{+}{v}{}^\alpha dt$ is an infinitesimal transformation of the first parameter group and $\eta^\alpha \to \eta^\alpha + \bar{v}^\alpha dt$ one of the second parameter group. If we apply the first transformation and after that the inversion of the second $\eta^\alpha \to \eta^\alpha - \bar{v}^\alpha dt$ we get an infinitesimal transformation of the adjoint group. All three transformations correspond to the transformation of the underlying group whose image is the point $\overset{\circ}{\eta}{}^\alpha + v^\alpha dt$.

Now by the first infinitesimal transformation e^α is displaced (-)-parallel from $\overset{\circ}{\eta}{}^\alpha$ to $\overset{\circ}{\eta}{}^\alpha + v^\alpha dt$. Its components in this new point are therefore $e^\alpha - \bar{\Gamma}{}^{\alpha}_{\gamma\beta}\, e^\beta v^\gamma dt$. By the second infinitesimal transformation this new vector is displaced (+)-parallel back to $\overset{\circ}{\eta}{}^\alpha$. The result is a vector in $\overset{\circ}{\eta}{}^\alpha$ with the components

2.60) $$e^{\alpha} - \overset{-}{\Gamma}_{\gamma\beta}^{\alpha} e^{\beta} v^{\gamma} dt + \overset{+}{\Gamma}_{\gamma\beta}^{\alpha} e^{\beta} v^{\gamma} dt = e^{\alpha} + 2 \overset{+}{S}_{\gamma\beta}^{\cdot\cdot\alpha} e^{\beta} v^{\gamma} dt .$$

That means that the transformation of the anholonomic components e^{a} is <u>linear homogeneous</u>:

2.61) $$de^{a} = 2 \overset{+}{S}_{cb}^{\cdot\cdot a} e^{b} v^{c} dt = -c_{cb}^{\cdot\cdot a} e^{b} v^{c} dt$$

and that

2.62) $$Y_{b} = E_{b}^{\cdot a} \partial_{a} \quad ; \quad E_{b}^{\cdot a} \overset{def}{=} e^{c} c_{cb}^{\cdot\cdot a} \quad ; \partial_{a} = \partial / \partial e^{a} .$$

are the symbols of the infinitesimal transformations of this linear homogeneous group. This group is called the <u>linear adjoint group</u>. The infinitesimal transformation $v^{b} Y_{b}$ of the linear adjoint group is said to correspond to the infinitesimal transformations $v^{b} X_{b} = v^{b} \Xi_{b}^{\mu} \partial_{\mu}$; $v^{b} A_{b} = v^{b} A_{b}^{\beta} \partial_{\beta}$ and $v^{b} \delta_{b}^{B} A_{B}^{\beta} \partial_{\beta}$ of the original group and the two parameter-groups.

According to (1.30b) we have

2.63) $$(Y_{c} Y_{b}) = 2 E_{[c}^{\cdot d} \partial_{|d|} E_{b]}^{\cdot e} \partial_{e} = 2 e^{f} c_{f[c}^{\cdot\cdot d} c_{d]b}^{\cdot\cdot e} \partial_{e} = \\ = c_{cb}^{\cdot\cdot a} Y_{a}$$

and this implies that in the case where the linear adjoint group is r-parametrical its structural constants are the same as those of the given group. But as we will see later it is not necessary that the linear adjoint group is r-parametrical.

<u>Voorbeeld:</u> De groep van de draaiingen om een punt in R_3 . Laten η^1, η^2 en η^3 de drie hoeken van Euler zijn en ξ^1, ξ^2, ξ^3 orthogonale cartesische coördinaten in R_3. Dan ontstaat de transformatie door de opvolgende draaiingen

over η^3 om de 3-as van 2 naar 1
over η^1 om de nieuwe 2-as van 1 naar 3
over η^2 om de nieuwe 3-as van 2 naar 1

<u>Overzicht der indices:</u>

In R_3
$\kappa, \lambda, \mu = 1, 2, 3$

In groepruimte
$\alpha, \beta, \gamma = 1, 2, 3$ holonoom
$a, b, c = 1, 2, 3$ } anholonoom
$A, B, C = 1, 2, 3$ } anholonoom

$$
\begin{array}{c|ccc}
 & \xi^1 & \xi^2 & \xi^3 \\
\hline
'\xi^1 & \cos\eta^1\cos\eta^2\cos\eta^3 - \sin\eta^2\sin\eta^3 & -\cos\eta^1\cos\eta^2\sin\eta^3 - \sin\eta^2\cos\eta^3 & \sin\eta^1\cos\eta^2 \\
'\xi^2 & \cos\eta^1\sin\eta^2\cos\eta^3 + \cos\eta^2\sin\eta^3 & -\cos\eta^1\sin\eta^2\sin\eta^3 + \cos\eta^2\cos\eta^3 & \sin\eta^1\sin\eta^2 \\
'\xi^3 & -\sin\eta^1\cos\eta^3 & \sin\eta^1\sin\eta^3 & \cos\eta^1
\end{array}
\tag{2.63a}
$$

of in verkorte notatie

$$
\begin{array}{c|ccc}
 & \xi^1 & \xi^2 & \xi^3 \\
\hline
'\xi^1 & c^1c^2c^3 - s^2s^3 & -c^1c^2s^3 - s^2c^3 & s^1c^2 \\
'\xi^2 & c^1s^2c^3 + c^2s^3 & -c^1s^2s^3 + c^2c^3 & s^1\cancel{c^2}\, s^2 \\
'\xi^3 & -s^1c^3 & s^1s^3 & c^1
\end{array}
\tag{2.63b}
$$

De omkering ontstaat door η^1, η^2, η^3 te vërvangen door $-\eta^1, -\eta^3, -\eta^2$

2,63c)

	$'\xi^1$	$'\xi^2$	$'\xi^3$
ξ^1	$c^1c^2c^3 - s^2s^3$	$c^1s^2c^3 + c^2s^3$	$-s^1c^3$
ξ^2	$-c^1c^2s^3 - s^2c^3$	$-c^1s^2s^3 + c^2c^3$	s^1s^3
ξ^3	s^1c^2	s^1s^2	c^1

(Contrôle: (2.63b) en (2.63c) zijn elkaars spiegelbeeld t.o.v. de hoofddiagonaal)

Tabel van $\partial_\beta {'\xi^\kappa}$.

2.63d)

$\partial_\beta {'\xi^\kappa}$	1	2	3
1	$'\xi^3 c^2$	$'\xi^3 s^2$	$-'\xi^1 c^2 - '\xi^2 s^2$
2	$-'\xi^2$	$'\xi^1$	0
3	$-'\xi^2 c^1 + '\xi^3 s^1 s^2$	$'\xi^1 c^1 - '\xi^3 s^1 c^2$	$-'\xi^1 s^1 s^2 + '\xi^2 s^1 c^2$

[illegible] determinant [illegible] (zie 2.4)

Een mogelijke ontbinding in $\Xi^\kappa_\ell('\xi^\lambda)$ en $A^\ell_\beta(\eta^\alpha)$ is

2.63e)

a)

Ξ^κ_ℓ	1	2	3
1	0	$'\xi^3$	$'\xi^2$
2	$'\xi^3$	0	$-'\xi^1$
3	$-'\xi^2$	$'\xi^1$	0

b)

A^ℓ_β	1	2	3
1	$-s^3$	c^2	0
2	0	0	1
3	s^1c^2	s^1s^2	c^1

(Det $= s^1$)

alleen bruikbaar voor sin $\eta^1 \neq 0$

c)

A^α_ℓ	1	2	3
1	$-s^2$	$-c^1c^2/s^1$	c^2/s^1
2	c^2	$-c^1s^2/s^1$	s^2/s^1
3	0	$+1$	0

(Det $= 1/s^1$

Omkering alleen mogelijk voor sin $\eta^1 \neq 0$

Tabel van $\partial_{\beta}\xi^{\kappa}$

2.63f)

$\partial_{\beta}\xi^{\kappa}$	1	2	3
1	$-\xi^3 c^3$	$\xi^3 s^3$	$\xi^1 c^3 - \xi^2 s^3$
2	$\xi^2 c^1 - \xi^3 s^1 s^3$	$-\xi^1 c^1 - \xi^3 s^1 c^3$	$\xi^1 s^1 s^3 + \xi^2 s^1 c^3$
3	ξ^2	$-\xi^1$	0

met een mogelijke ontbinding

2.63g) a)

Ξ^{κ}_{β}	1	2	3
1	0	ξ^3	$-\xi^2$
2	$-\xi^3$	0	ξ^1
3	ξ^2	$-\xi^1$	0

b)

A^{B}_{β}	1	2	3
1	s^3	c^3	0
2	$-s^1 c^3$	$s^1 s^3$	c^1
3	0	0	1

(Det = s^1)

c)

A^{α}_{B}	1	2	3
1	s^3	c^3/s^1	$c^1 c^3/s^1$
2	c^3	s^3/s^1	$-c^1 s^3/s^1$
3	0	0	1

(Det = $1/s^1$)

alleen voor sin $\eta^1 \neq 0$

Hieruit volgt voor A^{α}_{B} en A^{A}_{ℓ}

2,63h)

A^{α}_{B}	1	2	3
1	$-s^2 s^3 + c^1 c^2 c^3$	$c^2 s^3 + c^1 s^2 c^3$	$-s^1 c^3$
2	$-s^2 c^3 - c^1 c^2 s^3$	$c^2 c^3 - c^1 s^2 s^3$	$s^1 s^3$
3	$s^1 c^2$	$s^1 s^2$	c^1

2.63i)

A^{A}_{ℓ}	1	2	3
1	$-s^2 s^3 + c^1 c^2 c^3$	$-s^2 c^3 - c^1 c^2 s^3$	$s^1 c^2$
2	$c^2 s^3 + c^1 s^2 c^3$	$c^2 c^3 - c^1 s^2 s^3$	$s^1 s^2$
3	$-s^1 c^3$	$s^1 s^3$	c^1

Vergelijkt men dit met (2.63f.e) dan volgt dat de overgang van het stelsel (a) tot het stelsel (A) in ieder punt van de groepruimte de vorm heeft van een draaiing t.o.v. deze locale coördinatenstelsels. Voor $\eta^{\alpha} = 0$ krijgt men de identiteit dus (a) en (A) vallen daar samen.

De infinitesimale transformaties X_{ℓ} zijn blijkens (2.63e.a.)

2.63j)
$$X_1 = \Xi^{\mu}_{1}(\xi^{\lambda})\partial_\mu = \xi^2\partial_3 - \xi^3\partial_2$$
$$X_2 = \xi^3\partial_1 - \xi^1\partial_3$$
$$X_3 = \xi^1\partial_2 - \xi^2\partial_1$$

dus

2.63k) $$(X_1, X_2) = -X_3 \qquad (\text{cycl.}\ ^1,{}^2,{}^3)$$

Deze transformaties zijn wel onafhankelijk maar omdat $\xi^1 X_1 + \xi^2 X_2 + \xi^3 X_3 = 0$ niet onverbonden.
De enige kentallen van $c_{cb}^{\cdot\cdot a}$ die niet nul zijn, zijn dus

2.63l) $$c_{12}^{\cdot\cdot 3} = -c_{21}^{\cdot\cdot 3} = -1 \qquad (\text{cycl.})$$

Hieruit volgt voor $E_{b}^{\cdot a}$:

2.63m)

$E_{b}^{\cdot a}$	1	2	3
1	0	$-e^3$	e^2
2	e^3	0	$-e^1$
3	$-e^2$	e^1	0

spiegelen!

§ 3. The geometry of group space.

Gathering results we have in group space

1. the integrable but not symmetrical (+)-connection

2. the anholonomic system (a) whose measuring vectors are (+)-constant.

3.1)
$$\underset{b}{e}^{\alpha} \overset{*}{=} A^{\alpha}_{b} \quad , b = 1,..r$$

3.2)
$$\partial_\gamma \underset{b}{e}^{\alpha} = -\overset{+}{\Gamma}{}^{\alpha}_{\gamma\beta} \underset{b}{e}^{\beta} \quad ; \quad \partial_\gamma \overset{a}{e}_{\beta} = +\overset{+}{\Gamma}{}^{\alpha}_{\gamma\beta} \overset{a}{e}_{\alpha}$$

3.3)
$$\partial_{[\gamma} \overset{a}{e}_{\beta]} = \overset{+}{\Gamma}{}^{\alpha}_{[\gamma\beta]} \overset{a}{e}_{\alpha} = \overset{+}{S}{}_{\gamma\beta}{}^{\cdot\cdot\alpha} \overset{a}{e}_{\alpha} = -\tfrac{1}{2} c_{c}{}^{a}{}_{b} \overset{c}{e}_{\gamma} \overset{b}{e}_{\beta}$$

3.4)
$$\overset{+}{\Gamma}{}^{\alpha}_{\gamma\beta} = -A^{b}_{\beta} \partial_\gamma A^{\alpha}_{b} = A^{\alpha}_{b} \partial_\gamma A^{b}_{\beta}.$$

3.5)
$$\overset{+}{\Gamma}{}^{a}_{cb} = 0$$

3. the integrable but not symmetrical (-)-connection;
4. the anholonomic system (A) whose measuring vectors are (-)-constant:

3.6)
$$\underset{B}{e}^{\alpha} \overset{*}{=} A^{\alpha}_{B} \quad ; \quad B = 1,..r$$

3.7)
$$\partial_\gamma \underset{B}{e}^{\alpha} = -\overset{-}{\Gamma}{}^{\alpha}_{\gamma\beta} \underset{B}{e}^{\beta} \quad ; \quad \partial_\gamma \overset{A}{e}_{\beta} = +\overset{-}{\Gamma}{}^{\alpha}_{\gamma\beta} \overset{A}{e}_{\alpha}.$$

3.8)
$$\partial_{[\gamma} \overset{A}{e}_{\beta]} = \overset{-}{\Gamma}{}^{\alpha}_{[\gamma\beta]} \overset{A}{e}_{\alpha} = \overset{-}{S}{}_{\gamma\beta}{}^{\cdot\cdot\alpha} \overset{A}{e}_{\alpha} = -\tfrac{1}{2}\, c_{C}{}^{A}{}_{B} \overset{C}{e}_{\gamma} \overset{B}{e}_{\beta}$$

3.9)
$$\overset{-}{\Gamma}{}^{\alpha}_{\gamma\beta} = -A^{B}_{\beta} \partial_\gamma A^{\alpha}_{B} = A^{\alpha}_{B} \partial_\gamma A^{B}_{\beta}$$

3.10)
$$\overset{-}{\Gamma}{}^{A}_{CB} = 0$$

5. the relations

3.11) $$\overset{+}{S}_{\gamma\beta}{}^{\cdot\cdot\alpha} = \overset{+}{\Gamma}{}^{\alpha}_{[\gamma\beta]} = -\overset{-}{\Gamma}{}^{\alpha}_{[\gamma\beta]} = -\overset{-}{S}_{\gamma\beta}{}^{\cdot\cdot\alpha}$$

3.12) $$c_{cb}{}^{\cdot\cdot a} = -2\overset{+}{S}_{cb}{}^{\cdot\cdot a} \overset{*}{=} c^{a}_{cb} = -\delta^{CBa}_{\cdot cbA}\, {}'c^{A}_{CB}$$

3.13) $$c_{CB}{}^{\cdot\cdot A} = A^{cbA}_{CBa}\, c_{cb}{}^{\cdot\cdot a} = \delta^{cbA}_{CBa}\, c_{cb}{}^{\cdot\cdot a}$$

$$\left(A^{a}_{\beta} = \delta^{a}_{\beta} \text{ only in } \eta^{\alpha}\,! \right)$$

3.14) $$c_{[dc}{}^{\epsilon}\, c_{b]e}{}^{a} = 0$$

6. the symmetric but not integrable connection

3.15) $$\Gamma_{\gamma\beta}{}^{\alpha} = \overset{+}{\Gamma}{}^{\alpha}_{(\gamma\beta)} = \overset{-}{\Gamma}{}^{\alpha}_{(\gamma\beta)} = \overset{+}{\Gamma}{}^{\alpha}_{\gamma\beta} - \overset{+}{S}_{\gamma\beta}{}^{\cdot\cdot\alpha} = \overset{-}{\Gamma}{}^{\alpha}_{\gamma\beta} - \overset{-}{S}_{\gamma\beta}{}^{\cdot\cdot\alpha}$$

7. the first and the second parameter-group of point transformations with the infinitesimal transformations $A_b = A^{\beta}_{b}\partial_\beta$ and $A_B = A^{\beta}_{B}\partial_\beta$ and the finite transformations $T \to \mathcal{U}T$ and $T \to T\mathcal{U}$.
8. the adjoint group with the finite transformations $T \to \mathcal{U}T\mathcal{U}^{-1}$ and the linear adjoint group working on the infinitesimal transformation $e^{b}X_b$ of the group

3.16) $$de^{a} = E_{b}{}^{a}(d\eta)^{b} = e^{c}\, c_{cb}{}^{a}(d\eta)^{b}$$

From (1.1), (1.21) and (1.30b) we get for the curvature affinor of $\Gamma_{\gamma\beta}^{\alpha}$

$$R_{\delta\gamma\beta}{}^{\cdot\cdot\cdot\alpha} = 2\,\partial_{[\delta}\overset{-}{\Gamma}{}^{\alpha}_{\gamma]\beta} + 2\,\overset{-}{\Gamma}{}^{\alpha}_{[\delta|\epsilon|}\overset{-}{\Gamma}{}^{\epsilon}_{\gamma]\beta} =$$

$$= \overset{+}{R}_{\delta\gamma\beta}{}^{\cdot\cdot\cdot\alpha} - 2\,\partial_{[\delta}\overset{+}{S}_{\gamma]\beta}{}^{\cdot\cdot\alpha} - 2\,\overset{+}{S}_{[\delta|\epsilon|}{}^{\cdot\cdot\alpha}\,\overset{+}{\Gamma}{}^{\epsilon}_{\gamma]\beta} -$$

3.17) $$-2\,\overset{+}{\Gamma}{}^{\alpha}_{[\delta|\epsilon|}\,\overset{+}{S}_{\gamma]\beta}{}^{\cdot\cdot\epsilon} + 2\,\overset{+}{S}_{[\delta|\epsilon|}{}^{\cdot\cdot\alpha}\,\overset{+}{S}_{\gamma]\beta}{}^{\cdot\cdot\epsilon} =$$

$$= -\lambda \overset{+}{\nabla}_{[\delta} \overset{+}{S}_{\gamma]\beta}^{\cdot\cdot\alpha} - \lambda \overset{+}{S}_{\delta\gamma}^{\cdot\cdot\varepsilon} \overset{+}{S}_{\varepsilon\beta}^{\cdot\cdot\alpha} + \lambda \overset{+}{S}_{[\delta|\varepsilon|}^{\cdot\cdot\alpha} \overset{+}{S}_{\gamma]\beta}^{\cdot\cdot\varepsilon} =$$

$$= -\tfrac{1}{2} c_{\delta\gamma}^{\cdot\cdot\varepsilon} c_{\varepsilon\beta}^{\cdot\cdot\alpha} + \tfrac{1}{2} c_{[\delta|\varepsilon|}^{\cdot\cdot\alpha} c_{\gamma]\beta}^{\cdot\cdot\varepsilon} = -\tfrac{1}{4} c_{\delta\gamma}^{\cdot\cdot\varepsilon} c_{\varepsilon\beta}^{\cdot\cdot\alpha}.$$

and from (1.21) it follows that

3.18)
$$\nabla_\varepsilon R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha} = 0$$

Hence (cf. III § 6) all differential comitants of group space are ordinary comitants of $c_{\gamma\beta}^{\cdot\cdot\alpha}$. This has as a consequence that all these comitants have constant components with respect to the anholonomic systems (a) and (A).
Moreover, according to (3.18) group space is symmetric with respect to all its points (cf. III § 6).
Every transformation of the group germ can be generated by an infinitesimal transformation $e^b X_b$ and can accordingly be written in the form

3.19)
$$'\xi^\kappa = \xi^\kappa + \frac{1}{1!} t e^b X_b \xi^\kappa + \frac{1}{2!} t^2 (e^b X_b)^2 \xi^\kappa + \dots$$

Hence the r numbers te^b can be taken as holonomic coordinates in group space. Writing $\eta^{\mathfrak{a}}$ for these new coordinates we get

3.20)
$$\eta^{\mathfrak{a}} = \delta_b^{\mathfrak{a}} t e^b \quad ; \; \mathfrak{a} = \bar{1}, \dots \bar{r} \quad ; \; b = 1, \dots r.$$

The measuring vectors $\underset{b}{e}^\alpha$ and $\underset{B}{e}^\alpha$ coincide in $\overset{0}{\eta}^\alpha$. Now we take them in such a way that they coincide also with the $\underset{\beta}{e}^\alpha$ in that point. Then according to (2.24) we have in every point $\frac{d\eta^\alpha}{dt} = e^\alpha$ and in $\overset{0}{\eta}^\alpha$ we have $e^\alpha = \delta_a^\alpha e^a$. Hence (cf. III § 6) the $\eta^{\mathfrak{a}}$ are the normal coordinates in X_r belonging to the point $\underset{0}{\eta}^\alpha$ and the coordinate system (α).
Because of

3.21)
$$d\eta^{\mathfrak{a}} = \delta_b^{\mathfrak{a}} e^b dt = \delta_\alpha^{\mathfrak{a}} e^\alpha dt = \delta_\alpha^{\mathfrak{a}} d\eta^\alpha \quad \text{for } \eta^\alpha = \underset{0}{\eta}^\alpha.$$

the four coordinate systems (α), (a), (A) and $(\mathfrak{a})$ have in $\underset{0}{\eta}^\alpha$ <u>the same measuring vectors.</u>

If the e^a are subjected to the transformations of the adjoint group and if t remains constant, the $\eta^{\mathfrak{a}}$ transform in the same way as the e^a i.e. linear homogeneous.

<u>The adjoint group works on the finite transformations of the given group just in the same way as on the infinitesimal transformations, provided that normal coordinates with respect to $\overset{\circ}{\eta}{}^{\alpha}$ are used.</u>

In III § 6 we have derived the relations between ordinary coordinates and normal coordinates. Taking $\alpha,\beta,\gamma = 1,\dots r$ instead of $\kappa,\lambda,\mu = 1,\dots n$ and $a,b,c = \bar{1},\dots\bar{r}$ instead of $h,i,j = 1,\dots n$ we get from (6.12, 13)

3.22)

$$\eta^{\alpha} = \delta^{\alpha}_{a}\,\eta^{a} - \frac{1}{2!}\,\Gamma^{\alpha}_{\gamma\beta}(0)\,\delta^{\gamma\beta}_{cb}\,\eta^{c}\eta^{b} - \frac{1}{3!}\,\Gamma^{\alpha}_{\gamma_2\gamma_1\beta}(0)\,\delta^{\gamma_2\gamma_1\beta}_{c_2c_1b}\,\eta^{c_2}\eta^{c_1}\eta^{b}\dots$$

3.23)

$$\eta^{a} = \delta^{a}_{\alpha}\,\eta^{\alpha} + \frac{1}{2!}\,\delta^{a}_{\alpha}\,\Lambda^{\alpha}_{\gamma\beta}(0)\,\eta^{\gamma}\eta^{\beta} + \frac{1}{3!}\,\delta^{a}_{\alpha}\,\Lambda^{\alpha}_{\gamma_2\gamma_1\beta}(0)\,\eta^{\gamma_2}\eta^{\gamma_1}\eta^{\beta}.$$

with (cf. III (6.10a, 14)) {A(2.5, 10)}

3.24)

$$\Gamma^{\alpha}_{\gamma_p\cdots\gamma_1\beta} \overset{\text{def}}{=} \partial_{(\gamma_p}\Gamma^{\alpha}_{\gamma_{p-1}\cdots\gamma_1\beta)} - p\,\Gamma^{\delta}_{(\gamma_p\gamma_{p-1}}\Gamma^{\alpha}_{\gamma_{p-2}\cdots\gamma_1\beta)\delta}$$

$$\Lambda^{\alpha}_{\gamma\beta} = \Gamma^{\alpha}_{\gamma\beta}$$

$$\Lambda^{\alpha}_{\gamma_2\gamma_1\beta} = \Gamma^{\alpha}_{\gamma_2\gamma_1\beta} + 3\,\Gamma^{\delta}_{(\gamma_2\gamma_1}\Gamma^{\alpha}_{\beta)\delta}$$

3.25)

$$\Lambda^{\alpha}_{\gamma_3\gamma_2\gamma_1\beta} = \Gamma^{\alpha}_{\gamma_3\gamma_2\gamma_1\beta} + 4\,\Gamma^{\delta}_{(\gamma_3\gamma_2\gamma_1}\Gamma^{\alpha}_{\beta)\delta} +$$

$$+ 6\,\Gamma^{\delta}_{(\gamma_3\gamma_2}\Lambda^{\alpha}_{\gamma_1\beta)\delta} - 3\,\Gamma^{\alpha}_{\delta\varepsilon}\Gamma^{\delta}_{(\gamma_3\gamma_2}\Gamma^{\varepsilon}_{\gamma_1\beta)}$$

Instead of III (6.18, 19, 19a) {A(2,13, 14, 15)} we have here

3.26)

$$\Gamma^{a}_{c_p\cdots c_1 b} = 0$$

3.27)

$$\partial_{(c_{p+1}}\Gamma^{a}_{c_p\cdots c_1 b)} = 0 \qquad \text{for } \eta^{a} = 0$$

3.28)

$$\partial_{(c_{p+1}}\cdots\partial_{c_1}\Gamma^{\alpha}_{cb)} = 0$$

From (3.22, 23) we may get A_b^α and A_β^a . But in group space we can get very simple expressions for A_b^a and A_b^a and from there we can get A_b^α and A_β^a because A_b^α and A_β^a are already known.

In every point of a geodesic through $\overset{0}{\eta}{}^\alpha$ we have at one hand

3.29)
$$(d\eta)^a = e^a dt = \delta_\alpha^a d\eta^\alpha$$

but at the other hand

3.30)
$$(d\eta)^a = A_\alpha^a d\eta^\alpha$$

From this it follows because of $d\eta^\alpha :: \eta^\alpha$

3.31)
$$\eta^\alpha A_\alpha^a = \eta^\alpha \delta_\alpha^a$$

though $A_\alpha^a = \delta_\alpha^a$ only in $\overset{0}{\eta}{}^\alpha$. By differentiation of (3.31) we get

3.32)
$$\eta^b \partial_c A_b^a + A_c^a = \delta_c^a$$

Now (2.52) is valid for every choice of the coordinate system (κ), hence also for (a):

3.33)
$$\partial_{[c} A_{b]}^a = -\tfrac{1}{2} C_{cb}^{\cdot\cdot a} A_{cb}^{cb}$$

If this is substituted in (3.32) we get

3.34)
$$-\eta^b C_{cb}{}^a A_{cb}^{cb} + \eta^b \partial_b A_c^\alpha + A_c^a = \delta_c^a$$

For differentiation along the geodesic holds

3.35)
$$\frac{d}{dt} = \frac{d\eta^\beta}{dt}\partial_\beta = \frac{d\eta^b}{dt}\partial_b = \delta_b^b e^b \partial_b = \frac{1}{t}\eta^b \partial_b$$

hence, from (3.34) (cf. (3.20) and (2.61))

3.36)
$$\frac{d}{dt} A_c^a + \frac{1}{t}(A_c^a - \delta_c^a) = C_{cb}^{\cdot\cdot a} A_c^c e^b = -A_c^c E_c{}^a$$

We remark that $A^{a}_{c}(0) = \delta^{a}_{c}$ and compare (3.36) with the ordinary differential equation

3.37)
$$\frac{dx}{dt} + \frac{1}{t}\left(x - \underset{0}{x}\right) = \alpha x \qquad ; \quad \underset{0}{x} = x(0) \quad .$$

whose wellknown solution can be written as a series

3.38)
$$x = \underset{0}{x}\,\frac{e^{\alpha t}-1}{\alpha t} = \underset{0}{x}\left(1 + \frac{1}{2!}\alpha t + \frac{1}{3!}\alpha^2 t^2 + \dots\right) \quad .$$

From this it follows that

3.39)
$$A^{a}_{b} = \delta^{b}_{b}\left(A^{a}_{b} - \frac{1}{2!}E_{b}{}^{a}t + \frac{1}{3!}E_{b}{}^{c}E_{c}{}^{a}t^{2} - \dots\right) \quad .$$

is formally a solution of (3.36). It has been proved that the series converges and gives a solution if the absolute values of the C^{c}_{ba} and the e^{a} do not exceed a certain limit. The inversion of (3.38) is

3.40)
$$\underset{0}{x} = x\,\frac{\alpha t}{e^{\alpha t}-1} = x\left(1 + \lambda_1 \alpha t + \lambda_2 \alpha^2 t^2 + \dots\right) \quad .$$

with

3.41)
$$\lambda_q = (-1)^q \frac{B_q}{q!} \quad .$$

where the B_q are the Bernoulli numbers[1)]

3.42)
$$\begin{array}{lll} B_0 = 1, & B_1 = 1/2\ ; & B_2 = 1/6 \\ B_3 = 0, & B_4 = -1/30 & B_5 = 0 \\ B_6 = 1/42 & B_7 = 0 & B_8 = -1/30 \end{array}$$

This gives the inversion of (3.39)

3.43)
$$A^{\alpha}_{b} = \delta^{\alpha}_{a}\left(A^{a}_{b} - \lambda_1 E_{b}{}^{a}t + \lambda_2 E_{b}{}^{c}E_{c}{}^{a}t^{2} - \dots\right)$$

---

1) Cf. Pascal Repertorium I p. 437.

In the same way series for A_{ℓ}^{A} and A_{B}^{α} can be found. They have the same form as (3.39) and (3.43) with $A, B, C, ..$ instead of $a, b, c, ..$ and $'E$ instead of E , $'E$ being defined by

3.44)
$$'E_{B}^{\cdot A} \overset{def}{=} e^{C}\, 'c_{CB}^{\cdot\cdot A}$$

Now in γ^{α} we have $'c_{CB}^{\cdot\cdot A} = -\delta_{CBa}^{c\ell A}\, c_{c\ell}^{\cdot\cdot a}$ hence

3.45)
$$'E_{B}^{\cdot A} = -\delta_{Ba}^{\ell A}\, E_{\ell}^{\cdot a}$$

Accordingly we get for A_{ℓ}^{A} and A_{B}^{α}

3.46)
$$A_{\ell}^{A} = \delta_{\ell a}^{bA}\left(A_{b}^{a} + \frac{1}{2!} E_{b}^{\cdot a} t + \frac{1}{3!} E_{b}^{\cdot c} E_{c}^{\cdot a} t^{2} + \dots\right)$$
$$A_{B}^{\alpha} = \delta_{aB}^{\alpha\ell}\left(A_{\ell}^{a} + \lambda_{1} E_{\ell}^{\cdot a} t + \lambda_{2} E_{\ell}^{\cdot c} E_{c}^{\cdot a} t^{2} + \dots\right)$$

Gathering results in a symbolic way we have

3.47)
$$\text{a)}\quad A_{\ell}^{a} = \delta_{\ell}^{b}\left(\frac{e^{-Et} - A}{-Et}\right)_{b}^{a} \; ; \qquad \text{c)}\quad A_{\ell}^{A} = \delta_{\ell a}^{bA}\left(\frac{e^{Et} - A}{Et}\right)_{b}^{a}.$$
$$\text{b)}\quad A_{\ell}^{\alpha} = \delta_{a}^{\alpha}\left(\frac{-Et}{e^{-Et} - A}\right)_{\ell}^{a} \; ; \qquad \text{d)}\quad A_{B}^{\alpha} = \delta_{aB}^{\alpha\ell}\left(\frac{Et}{e^{Et} - A}\right)_{\ell}^{a}.$$

From these equation a remarkable conclusion may be drawn

3.48)
$$\text{a)}\quad A_{B}^{a} = \delta_{B}^{\ell}\left(\frac{e^{-Et} - A}{-Et} \cdot \frac{Et}{e^{Et} - A}\right)_{\ell}^{a} = \delta_{B}^{\ell}\left(e^{-Et}\right)_{\ell}^{a}$$
$$\text{b)}\quad A_{\ell}^{A} = \delta_{a}^{A}\left(e^{Et}\right)_{\ell}^{a} = \delta_{a}^{A}\left(A_{\ell}^{a} + E_{\ell}^{\cdot a} t + \frac{1}{2!} E_{\ell}^{\cdot c} E_{c}^{\cdot a} t^{2} + \dots\right)$$

Now e^{Et} and e^{-Et} are linear transformations belonging to the group generated by the infinitesimal transformation $E\xi^a \partial_a$,that is the linear adjoint group. Hence

<u>In every point of X_n the measuring vectors of (a) transform into the measuring vectors of (A) by a finite transformation of the linear adjoint group.</u>

The matrix notation can be used for the computation of the Γ_{cb}^{a} . From (III 6.20a) {A(2.12)} we get here

3.49)
$$\Gamma_{cb}^{\alpha} = \eta^{\partial} N_{\partial cb}^{\cdots\alpha} + \tfrac{1}{2}\eta^{\partial_2}\eta^{\partial_1} N_{\partial_2\partial_1 cb}^{\cdots\cdot\alpha} + \dots \quad .$$

and according to (3.17) and (III 6.28) {A(3.4)} the first term of the right hand side takes the form

3.50)
$$\tfrac{2}{3}\,\eta^{\partial} R_{\partial(cb)}^{\cdots\alpha} = \tfrac{1}{6}\,\eta^{\partial} c_{\partial(c}^{\cdot\cdot n} c_{b)n}^{\cdot\cdot\alpha} \quad .$$

The normal affinor of valence 5 is zero and the normal affinor valence 6 can be written as the transvection of two factors $R_{\partial cb}^{\cdots\alpha}$ as was discussed in III § 6 {A § 3} , and these factors give four factors C_{cb}^{α} . But this result can be obtained here much easier by means of a series. Because the $\overset{a}{e}_{\beta}$ are (+)-constant we have

3.51)
$$\overset{+}{\Gamma}{}_{cb}^{\alpha} = A_a^{\alpha}\,\partial_c A_b^a \quad .$$

E stands for the matrix $\xi^c C_{cb}^{\cdot\cdot a}$. Hence if we write P for the matrix $\xi^c C_{cb}^{\cdot\cdot a} t = \eta^c \delta_c^c C_{cb}^{\cdot\cdot a}$ and C_c for the matrix $C_{cb}^{\cdot\cdot a}$, we have

3.52)
$$\partial_c P = \delta_c^c C_c \quad .$$

From (3.39) we have

3.53)
$$A_b^a = \delta_b^b\left(A - \tfrac{1}{2!}P + \tfrac{1}{3!}P^2 - \tfrac{1}{4!}P^3 + \dots\right)_b^a \quad .$$

hence (taking into account the non-commutativity of C_c and P !)

3.54)
$$\partial_c A_b^a = \delta_b^c\left(-\tfrac{1}{2!}C_c + \tfrac{1}{3!}C_c P + \tfrac{1}{3!}P C_c - \right.$$

$$-\tfrac{1}{4!}\,C_c P^2 - \tfrac{1}{4!}\,P C_c P - \tfrac{1}{4!}\,P^2 C_c\Big)^a_b$$

and (note that the factor A^{α}_{a} in (3.51) comes at the right hand side!)

3.55)
$$\left(\partial_c A^a_b\right) A^{\alpha}_{a} =$$

$$= \delta^{b\alpha c}_{bac}\Big\{\big(-\tfrac{1}{2}C_c + \tfrac{1}{6}C_c P + \tfrac{1}{6}P C_c - \tfrac{1}{24}C_c P^2 - \tfrac{1}{24}P C_c P -$$

$$- \tfrac{1}{24}P^2 C_c\big)\big(A + \tfrac{1}{2}P + \tfrac{1}{6}P^2 + \dots\big)\Big\}^a_b$$

$$= \delta^{b\alpha c}_{bac}\Big\{-\tfrac{1}{2}C_c + \tfrac{1}{6}C_c P + \tfrac{1}{6}P C_c - \tfrac{1}{24}C_c P^2 - \tfrac{1}{24}P C_c P -$$

$$- \tfrac{1}{24}P^2 C_c - \tfrac{1}{4}C_c P + \tfrac{1}{12}C_c P^2 + \tfrac{1}{12}P C_c P - \tfrac{1}{12}C_c P^2 + \dots\Big\}^a_b$$

$$= \delta^{b\alpha c}_{bac}\Big\{-\tfrac{1}{2}C_c - \tfrac{1}{12}C_c P + \tfrac{1}{6}P C_c - \tfrac{1}{24}C_c P^2 + \tfrac{1}{24}P C_c P - \tfrac{1}{24}P^2 C_c + \dots\Big\}^a_b$$

$$= \delta^{b\alpha c}_{bac}\Big\{-\tfrac{1}{2}C_{cb}{}^{a} - \tfrac{1}{12}C_{cb}{}^{d}\,\eta^{n}\delta^{e}_{n}\,C_{ed}{}^{a} + \tfrac{1}{6}\eta^{n}\delta^{e}_{n}\,C_{eb}{}^{d}\,C_{cd}{}^{a} -$$

$$- \tfrac{1}{24}C_{cb}{}^{d_1}\,\eta^{n_1}\delta^{e_1}_{n_1}\,C_{e_1 d_1}{}^{d_2}\,\eta^{n_2}\delta^{e_2}_{n_2}\,C_{e_2 d_2}{}^{a} +$$

$$+ \tfrac{1}{24}\eta^{n_1}\delta^{e_1}_{n_1}\,C_{e_1 b}{}^{d_1}\,C_{c d_1}{}^{d_2}\,\eta^{n_2}\delta^{e_2}_{n_2}\,C_{e_2 d_2}{}^{a} -$$

$$- \tfrac{1}{24}\eta^{n_1}\delta^{e_1}_{n_1}\,C_{e_1 b}{}^{d_1}\,\eta^{n_2}\delta^{e_2}_{n_2}\,C_{e_2 d_1}{}^{d_2}\,C_{c d_2}{}^{a} + \dots\Big\}$$

Hence, making use of the fact that (a) and (α) have the same measuring vectors in $\underset{\circ}{\eta}^{\alpha}$ we get

3.56)
$$\overset{\pm}{\Gamma}{}^{\alpha}_{cb} = -\tfrac{1}{2}\underset{\circ}{C}{}_{cb}^{\alpha} - \tfrac{1}{12}\eta^{n}\,\underset{\circ}{C}{}_{cb}{}^{\vartheta}\,\underset{\circ}{C}{}_{n\vartheta}{}^{\alpha} + \tfrac{1}{6}\eta^{n}\,\underset{\circ}{C}{}_{nb}{}^{\vartheta}\,\underset{\circ}{C}{}_{c\vartheta}{}^{\alpha} -$$

$$-\tfrac{1}{24}\,\eta^{n_1}\eta^{n_2}\,\overset{\circ}{S}_{cb}{}^{\partial_1}\,\overset{\circ}{S}_{n_1\partial_1}{}^{\partial_2}\,\overset{\circ}{S}_{n_2\partial_2}{}^{a}\;+$$

$$+\tfrac{1}{24}\,\eta^{n_1}\eta^{n_2}\,\overset{\circ}{S}_{n_1 b}{}^{\partial_1}\,\overset{\circ}{S}_{c\partial_1}{}^{\partial_2}\,\overset{\circ}{S}_{n_2\partial_2}{}^{a}\;-$$

$$-\tfrac{1}{24}\,\eta^{n_1}\eta^{n_2}\,\overset{\circ}{S}_{n_1 b}{}^{\partial_1}\,\overset{\circ}{S}_{n_2\partial_1}{}^{\partial_2}\,\overset{\circ}{S}_{c\partial_2}{}^{a}\;+\cdots$$

From these written terms all vanish by mixing over cb except the third one, hence (cf. (1.30b)).

3.57)
$$\Gamma_{cb}^{\;a}=\overset{+}{\Gamma}_{(cb)}{}^{a}=\tfrac{1}{6}\,\eta^{n}\,\overset{\circ}{S}_{n(b}{}^{\partial}\,\overset{\circ}{S}_{c)\partial}{}^{a}+\text{terms of an odd degree}\geq 3\text{ in }\eta^{a}$$

in accordance with (3.50) and

3.58)
$$C_{cb}^{\;\cdot\cdot a}=-2\,\overset{+}{\Gamma}_{[cb]}{}^{a}=$$

$$=\overset{\circ}{S}_{cb}{}^{a}-\tfrac{1}{24}\,\eta^{n_1}\eta^{n_2}\,\overset{\circ}{S}_{cb}{}^{\partial_1}\,\overset{\circ}{S}_{n_1\partial_1}{}^{\partial_2}\,\overset{\circ}{S}_{n_2\partial_2}{}^{a}+$$

$$+\tfrac{1}{24}\,\eta^{n_1}\eta^{n_2}\,\overset{\circ}{S}_{n_1[c}{}^{\partial_1}\,\overset{\circ}{S}_{|n_2|b]}{}^{\partial_2}\,\overset{\circ}{S}_{\partial_1\partial_2}{}^{a}+\text{terms of an even degree}\geq 4\text{ in }\overset{\circ}{\eta}^{a}$$

By a reflection at the point $\overset{\circ}{\eta}^{a}$ a field with the value v^{a} in η^{a} is changed into a field with the value $u^{a}=-v^{a}$ in $-\eta^{a}$. If v^{a} are the (a)-components of the field in η^{a}, it follows from (3.39) that the u^{a} in $-\eta^{a}$ are not equal to $-v^{a}$ but that $u^{A}=-\delta^{A}_{a}v^{a}$. The reflection of a (+)-parallel displacement of v^{a} over $d\eta^{a}$ in η^{a} is a (−)-parallel displacement of u^{a} over $-d\eta^{a}$ in $-\eta^{a}$. Hence the reflection of a (0)-parallel displacement of v^{a} over $d\eta^{a}$ in η^{a} is a (0)-parallel displacement of u^{a} over $-d\eta^{a}$ in $-\eta^{a}$. Hence, taking a (0)-displacement we get

3.59)
$$dv^{a}=-d\eta^{c}\,\Gamma_{cb}^{\;a}(\eta^{\partial})\,v^{b}=-du^{a}=-d\eta^{c}\,\Gamma_{cb}^{\;a}(-\eta^{\partial})(-v^{b})$$

or

3.60)
$$\Gamma_{cb}^{\;a}(\eta^{\partial})=-\Gamma_{cb}^{\;a}(-\eta^{\partial}).$$

and this proves that in the series of $\Gamma_{cb}^{\;\;\,a}$ only terms of an odd degree in η^a occur. Taking a(+)-displacement in η^a we get

$$3.61)\qquad dv^a = -d\eta^c\,\Gamma_{cb}^{\;\;\,a}(\eta^d)\,v^b = -du^a = -d\eta^c\,\Gamma_{cb}^{\;\;\,a}(-\eta^d)(-v^b)$$

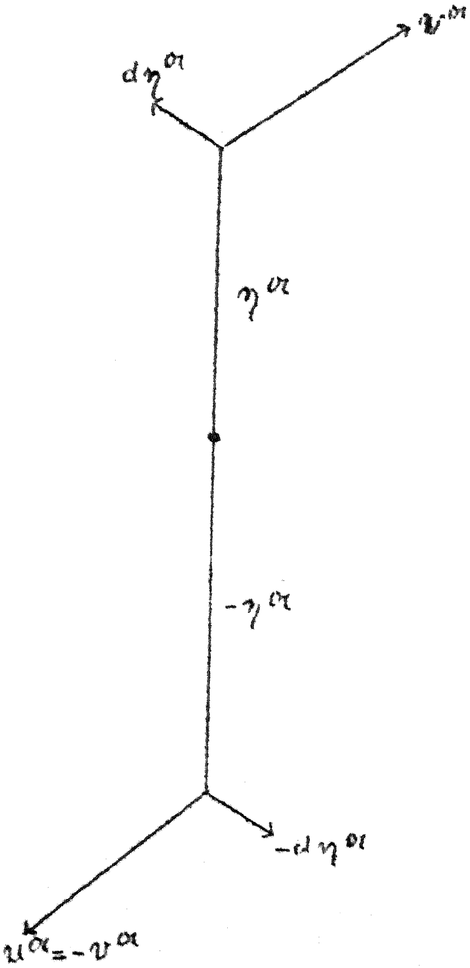
or

$$3.62)\qquad \Gamma_{cb}^{\;\;\,a}(\eta^d) = -\Gamma_{cb}^{\;\;\,a}(-\eta^d)$$

hence (cf. (3.11, 12))

$$3.63)\qquad C_{cb}^{\;\cdot\cdot\;a}(\eta^d) = C_{cb}^{\;\cdot\cdot\;a}(-\eta^d)$$

and this proves that in the series of $C_{cb}^{\;\cdot\cdot\;a}$ only terms of an degree in η^a occur.



$u^A = -\delta^A_a v^a$

Voor $\overset{+}{\Gamma}{}^{\alpha}_{\gamma\beta} = A^{\alpha}_{b}\, \partial_{\gamma} A^{b}_{\beta}$ vindt men

$\overset{+}{\Gamma}{}^{1}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	0	0
2	0	0	0
3	0	s^1	0

$\Gamma^{1}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	0	0
2	0	0	$s^1/2$
3	0	$s^1/2$	0

3.63a)

$\overset{+}{\Gamma}{}^{2}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	c^1/s^1	0
2	0	0	0
3	$-1/s^1$	0	0

$\Gamma^{2}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	$c^1/2s^1$	$-1/2s^1$
2	$c^1/2s^1$	0	0
3	$-1/2s^1$	0	0

$\overset{+}{\Gamma}{}^{3}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	$-1/s^1$	0
2	0	0	0
3	c^1/s^1	0	0

$\Gamma^{3}_{\gamma\beta}$

$\beta \backslash \gamma$	1	2	3
1	0	$-1/2s^1$	$c^1/2s^1$
2	$-1/2s^1$	0	0
3	$c^1/2s^1$	0	0

Men lette er op dat er kentallen zijn, die voor $\eta^1 = 0$ oneindig worden. De reeksontwikkeling van Veblen (A (2.6)) is dus in dit geval niet bruikbaar.

De vergelijkingen der geodetische lijnen worden

3.63b)

$$\frac{d^2\eta^1}{dt^2} + \sin\eta^1 \frac{d\eta^2}{dt}\frac{d\eta^3}{dt} = 0$$

$$\frac{d^2\eta^2}{dt^2} + \cot\eta^1 \frac{d\eta^1}{dt}\frac{d\eta^2}{dt} - \frac{1}{\sin\eta^1}\frac{d\eta^1}{dt}\frac{d\eta^3}{dt} = 0$$

$$\frac{d^2\eta^3}{dt^2} - \frac{1}{\sin\eta^1}\frac{d\eta^1}{dt}\frac{d\eta^2}{dt} + \cot\eta^1 \frac{d\eta^1}{dt}\frac{d\eta^3}{dt} = 0$$

Deze vergelijkingen zijn in de mechanica bekend als de bewegingsvergelijkingen van een homogene bol die om de oorsprong vrij kan draaien.

Langs een geodetische lijn is

3.63c)

$$\frac{d\eta^\alpha}{dt} = A^\alpha_a e^a \quad , e^a = \text{constanten.}$$

en dit wordt hier

3.63d)

$$\frac{d\eta^1}{dt} = -\sin\eta^2 e^1 + \cos\eta^2 e^2$$

$$\frac{d\eta^2}{dt} = -\cot\eta^1 \cot\eta^2 e^1 - \cot\eta^1 \sin\eta^2 e^2 + e^3$$

$$\frac{d\eta^3}{dt} = \frac{\cos\eta^2}{\sin\eta^1} e^1 + \frac{\sin\eta^2}{\sin\eta^1} e^2$$

en dit zijn de eerste integralen van (3.63c)

Formule (2.63m) is bij vergissing geschreven met verwisseling van rijen en kolommen. Men vindt:

3.63e)

$E_{\ell}^{\cdot a}$

$a \backslash \ell$	1	2	3
1	0	e^3	$-e^2$
2	$-e^3$	0	e^1
3	e^2	$-e^1$	0

$(E^3)_{\ell}^{\cdot a}$

$a \backslash \ell$	1	2	3
1	$-e^2e^2-e^3e^3$	e^1e^2	e^1e^3
2	e^1e^2	$-e^3e^3-e^1e^1$	e^2e^3
3	e^3e^1	e^2e^3	$-e^1e^1-e^2e^2$

$$(E^3)_{\ell}^{\cdot a} = -(e^1e^1+e^2e^2+e^3e^3)\,E_{\ell}^{\cdot a}$$

Schrijft men

3.63f)
$$\rho^2 \overset{\text{def}}{=\!=} (e^1e^1+e^2e^2+e^3e^3)\,t^2$$

dan volgt (verg. (3.48))

$$A^a_B = \delta^\ell_B\,(e^{-Et})^a_\ell =$$

3.63g)
$$= \delta^\ell_B\left\{A - Et\left(1-\tfrac{1}{3!}\rho^2+\tfrac{1}{5!}\rho^4-\dots\right)+E^2t^2\left(\tfrac{1}{2!}-\tfrac{1}{4!}\rho^2+\dots\right)\right\}^a_\ell$$

$$= \delta^\ell_B\left(A - Et\,\frac{\sin\rho}{\rho} + E^2t^2\,\frac{1-\cos\rho}{\rho^2}\right)^a_\ell.$$

3.63h)
$$A^A_\ell = \delta^A_a\,(e^{Et})^a_\ell = \delta^A_a\left(A + Et\,\frac{\sin\rho}{\rho} + E^2t^2\,\frac{1-\cos\rho}{\rho^2}\right)^a_\ell$$

$$A^a_b = \delta^\ell_b\left(\frac{e^{-Et}-A}{-Et}\right)^a_\ell = \delta^\ell_b\left(A-\tfrac{1}{2!}Et+\tfrac{1}{3!}E^2t^2-\tfrac{1}{4!}E^3t^3+\dots\right)^a_\ell =$$

3.63i)
$$= \delta^\ell_b\left(A - Et\,\frac{1-\cos\rho}{\rho^2} + E^2t^2\,\frac{\rho-\sin\rho}{\rho^3}\right)^a_\ell$$

3.63j)
$$A^A_b = \delta^{\ell A}_{ba}\left(\frac{e^{Et}-A}{Et}\right)^a_\ell = \delta^{\ell A}_{ba}\left(A + Et\,\frac{1-\cos\rho}{\rho^2} + E^2t^2\,\frac{\rho-\sin\rho}{\rho^3}\right)^a_\ell$$

$$A^\alpha_\ell = \delta^\alpha_a\left(\frac{-Et}{e^{-Et}-A}\right)^a_\ell = \delta^\alpha_a\left(A-\lambda_1Et+\lambda_2E^2t^2-\lambda_3E^3t^3+\dots\right)^a_\ell =$$

3.63k)
$$= \delta^\alpha_a\left\{A - Et(\lambda_1-\lambda_3\rho^2+\lambda_5\rho^4-\dots)+E^2t^2(\lambda_2-\lambda_4\rho^2+\lambda_6\rho^4-\dots)\right\}^a_\ell.$$

Uit (3.40) vindt men gemakkelijk:

$$1+\lambda_2\alpha^2+\lambda_4\alpha^4+\dots = \tfrac{1}{2}\left(\frac{\alpha}{e^\alpha-1}+\frac{-\alpha}{e^{-\alpha}-1}\right) = \frac{\alpha}{2}\,\frac{e^\alpha+1}{e^\alpha-1} = \frac{\alpha}{2}\coth\frac{\alpha}{2}$$

3.63 l)

$$\lambda_1\alpha+\lambda_3\alpha^3+\lambda_5\alpha^5+\dots = \tfrac{1}{2}\left(\frac{\alpha}{e^\alpha-1}-\frac{-\alpha}{e^{-\alpha}-1}\right) = -\frac{\alpha}{2}$$

dus

$$\lambda_1 = -\tfrac{1}{2}\,,\quad \lambda_3=\lambda_5=\lambda_7=\dots=0$$

Voorts geldt $\frac{\alpha i}{2}\operatorname{cotg}h\frac{\alpha i}{2}=\frac{\alpha}{2}\operatorname{cotg}\frac{\alpha}{2}$, dus:

3.63m) $A^{\alpha}_{b}=\delta^{\alpha}_{a}\left\{A+\tfrac{1}{2}Et+E^{2}t^{2}\frac{1-\frac{\rho}{2}\operatorname{cotg}\rho/2}{\rho^{2}}\right\}^{a}_{b}$

3.63n) $A^{\alpha}_{B}=\delta^{\alpha b}_{AB}\left\{A-\tfrac{1}{2}Et+E^{2}t^{2}\frac{1-\frac{\rho}{2}\operatorname{cotg}\rho/2}{\rho^{2}}\right\}^{a}_{b}$

Verder vinden we uit (3.63a) wegens $c_{\gamma\beta}^{\cdot\cdot\alpha}=-2\overset{+}{\Gamma}{}^{\alpha}_{[\gamma\beta]}$:

o)

$c_{\gamma\beta}^{\cdot\cdot 1}$

$\beta \backslash \gamma$	1	2	3
1	0	0	0
2	0	0	s^{1}
3	0	$-s^{1}$	0

$c_{\gamma\beta}^{\cdot\cdot 2}$

$\beta \backslash \gamma$	1	2	3
1	0	$-c^{1}/s^{1}$	$-1/s^{1}$
2	c^{1}/s^{1}	0	0
3	$1/s^{1}$	0	0

$c_{\gamma\beta}^{\cdot\cdot 3}$

$\beta \backslash \gamma$	1	2	3
1	0	$1/s^{1}$	c^{1}/s^{1}
2	$-1/s^{1}$	0	0
3	$-c^{1}/s^{1}$	0	0

We vinden voor de niet-verdwijnende kentallen van $R_{dcb}^{\cdot\cdot\cdot a}$ in verband met (3.17) en (2.63 l):

$$R_{121}^{\cdot\cdot\cdot 2}=-R_{211}^{\cdot\cdot\cdot 2}=-R_{122}^{\cdot\cdot\cdot 1}=R_{212}^{\cdot\cdot\cdot 1}=-\tfrac{1}{4}$$

3.63p)
$$R_{131}^{\cdot\cdot\cdot 3}=-R_{311}^{\cdot\cdot\cdot 3}=-R_{133}^{\cdot\cdot\cdot 1}=R_{313}^{\cdot\cdot\cdot 1}=-\tfrac{1}{4}$$

$$R_{232}^{\cdot\cdot\cdot 3}=-R_{322}^{\cdot\cdot\cdot 3}=-R_{233}^{\cdot\cdot\cdot 2}=R_{323}^{\cdot\cdot\cdot 2}=-\tfrac{1}{4}$$

Evenzo vinden we, in verband met (3.63o) voor $R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha}$:

3.63q)

$R_{12\beta}^{\cdot\cdot\cdot\alpha}=-R_{21\beta}^{\cdot\cdot\cdot\alpha}$

$\alpha \backslash \beta$	1	2	3
1	0	$-\frac{1}{4}$	$\frac{1}{4}c^{1}$
2	$-\frac{1}{4}$	0	0
3	0	0	0

$R_{13\beta}^{\cdot\cdot\cdot\alpha}=-R_{31\beta}^{\cdot\cdot\cdot\alpha}$

$\alpha \backslash \beta$	1	2	3
1	0	$\frac{1}{4}c^{1}$	$\frac{1}{4}$
2	0	0	0
3	$-\frac{1}{4}$	0	0

$R_{23\beta}^{\cdot\cdot\cdot\alpha}=-R_{32\beta}^{\cdot\cdot\cdot\alpha}$

$\alpha \backslash \beta$	1	2	3
1	0	0	0
2	0	$\frac{1}{4}c^{1}$	$\frac{1}{4}$
3	0	$-\frac{1}{4}$	$-\frac{1}{4}c^{1}$

Na invoering van normaalcoordinaten $\eta^{\alpha}=\delta^{\alpha}_{a}e^{a}t$ neemt A^{α}_{B} de vorm aan (verg. (3.63g)):

3.63r) $A^{a}_{B}=\delta^{b}_{B}\left\{A^{a}_{b}-\eta^{\alpha}\delta^{c}_{\alpha}c_{cb}^{\cdot\cdot a}\frac{\sin\rho}{\rho}+\eta^{\alpha}\eta^{\beta}\delta^{cd}_{\alpha\beta}c_{cb}^{\cdot\cdot e}c_{de}^{\cdot\cdot a}\frac{1-\cos\rho}{\rho^{2}}\right\}$,

of in verband met (3.63e):

A^a_B

$B \backslash a$	1	2	3
1	$\cos\rho + \frac{\eta^1\eta^1}{\rho^2}(1-\cos\rho)$	$-\eta^3\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^2}{\rho^2}(1-\cos\rho)$	$\eta^2\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^3}{\rho^2}(1-\cos\rho)$
2	$\eta^3\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^2}{\rho^2}(1-\cos\rho)$	$\cos\rho + \frac{\eta^2\eta^1}{\rho^2}(1-\cos\rho)$	$-\eta^1\frac{\sin\rho}{\rho} + \frac{\eta^2\eta^3}{\rho^2}(1-\cos\rho)$
3	$-\eta^2\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^3}{\rho^2}(1-\cos\rho)$	$\eta^1\frac{\sin\rho}{\rho} + \frac{\eta^2\eta^3}{\rho^2}(1-\cos\rho)$	$\cos\rho + \frac{\eta^3\eta^3}{\rho^2}(1-\cos\rho)$

3.63s)

$$\rho^2 = \eta^1\eta^1 + \eta^2\eta^2 + \eta^3\eta^3.$$

De tabel van A^A_b wordt verkregen door vervanging van η^α door $-\eta^\alpha$ (vgl. (3.63h)). Dat geeft juist de gespiegelde van (3.63s).

Door vergelijking met (2.63h) vinden we η^α als functies van η^α (Hiermee is dan dus ook het stelsel (3.63d) opgelost voor krommen door η^α_0!):

$$\cos\eta^1 = \cos\rho + \frac{\eta^3\eta^3}{\rho^2}(1-\cos\rho) \quad ; \sin\eta^1 = \left\{\frac{\eta^1\eta^1+\eta^2\eta^2}{\rho^2}\left(\sin^2\rho + (1-\cos\rho)^2\frac{\eta^3\eta^3}{\rho^2}\right)\right\}^{1/2}$$

2.63t)
$$\cos\eta^2 = \left\{-\eta^2\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^3}{\rho^2}(1-\cos\rho)\right\}/\sin\eta^1 ; \sin\eta^2 = \left\{\eta^1\frac{\sin\rho}{\rho} + \frac{\eta^2\eta^3}{\rho^2}(1-\cos\rho)\right\}/\sin\eta^1$$

$$\cos\eta^3 = -\left\{\eta^2\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^3}{\rho^2}(1-\cos\rho)\right\}/\sin\eta^1 ; \sin\eta^3 = \left\{-\eta^1\frac{\sin\rho}{\rho} + \frac{\eta^2\eta^3}{\rho^2}(1-\cos\rho)\right\}/\sin\eta^1$$

De tabel van A^a_b (3.63i) wordt, in verband met (3.63e):

A^a_b

$b \backslash a$	1	2	3
$\bar{1}$	$\frac{\sin\rho}{\rho} + \frac{\eta^1\eta^1}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$	$-\eta^3\frac{1-\cos\rho}{\rho^2} + \frac{\eta^1\eta^2}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$	$\eta^2\frac{1-\cos\rho}{\rho^2} + \frac{\eta^1\eta^3}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$
$\bar{2}$	$\eta^3\frac{1-\cos\rho}{\rho^2} + \frac{\eta^1\eta^2}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$	$\frac{\sin\rho}{\rho} + \frac{\eta^2\eta^2}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$	$-\eta^1\frac{1-\cos\rho}{\rho^2} + \frac{\eta^2\eta^3}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$
$\bar{3}$	$-\eta^2\frac{1-\cos\rho}{\rho^2} + \frac{\eta^1\eta^3}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$	$\eta^1\frac{1-\cos\rho}{\rho^2} + \frac{\eta^2\eta^3}{\rho^3}\left(1-\frac{\sin\rho}{\rho}\right)$	$\frac{\sin\rho}{\rho} + \frac{\eta^3\eta^3}{\rho^2}\left(1-\frac{\sin\rho}{\rho}\right)$

3.63u)

Vervanging van η^α door $-\eta^\alpha$ (d.i. spiegeling t.o.v. de hoofdiagonaal) geeft weer A^A_b.

Voor A^{α}_{b} (3.63m) vinden we:

	A^{α}_{b}: $b \backslash \alpha$	$\bar{1}$	$\bar{2}$	$\bar{3}$
	1	$\frac{\rho}{2}\cot\frac{\rho}{2} + \frac{\eta^{\bar{1}}\eta^{1}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$-\frac{1}{2}\eta^{\bar{3}} + \frac{\eta^{\bar{1}}\eta^{\bar{2}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$\frac{1}{2}\eta^{\bar{2}} + \frac{\eta^{\bar{1}}\eta^{\bar{3}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$
3.63v)	2	$\frac{1}{2}\eta^{\bar{3}} + \frac{\eta^{\bar{1}}\eta^{\bar{2}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$\frac{\rho}{2}\cot\frac{\rho}{2} + \frac{\eta^{\bar{2}}\eta^{\bar{2}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$-\frac{1}{2}\eta^{\bar{1}} + \frac{\eta^{\bar{2}}\eta^{\bar{3}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$
	3	$-\frac{1}{2}\eta^{\bar{2}} + \frac{\eta^{\bar{1}}\eta^{\bar{3}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$\frac{1}{2}\eta^{\bar{1}} + \frac{\eta^{\bar{2}}\eta^{\bar{3}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$	$\frac{\rho}{2}\cot\frac{\rho}{2} + \frac{\eta^{\bar{3}}\eta^{\bar{3}}}{\rho^2}\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right)$

Vervanging van η^{α} door $-\eta^{\alpha}$ (d.i. spiegeling t.o.v. de hoofddiagonaal) geeft weer A^{α}_{b}.

§ 4. Invariants of a transformation group in the X_n of the ξ^κ.

In the X_m of the ξ^κ we have the following invariants:
invariant functions of the ξ^κ
invariant subspaces
invariant fields
invariant systems of partial differential equations.

1. Invariant functions [1)]

A function $\varphi(\xi^\kappa)$ is called invariant[2)] if

$$4.1) \qquad X_b \varphi = \Xi_b^\mu \partial_\mu \varphi = 0$$

The infinitesimal transformations X_b are independent but they may be connected. If $q \leq r$ is the b-rank of Ξ_b^μ the function φ is a solution of a system of q independent partial differential equations.
Because of

$$4.2) \qquad 2\,\Xi_{[c}^\nu \partial_{|\nu|} \Xi_{b]}^\mu \partial_\mu = (X_c X_b) = c_{cb}^{\ \ \ a} X_a = c_{cb}^{\ \ \ a} \Xi_a^\mu \partial_\mu$$

the system is complete (cf. II § 5), it has $n-q$ independent solutions $\overset{1}{\varphi},\dots \overset{n-q}{\varphi}$ and every solution is a function of these $n-q$.

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1) Lie-Engel I p.211 ff.
2) Eisenhart uses the term absolute invariant.

Zijn $\underset{y}{v}^{\kappa}$; $y=1,\dots q$ lineair onafhankelijke vectoren dan zijn de oplossingen van het stelsel

4.2a) $$\underset{y}{v}^{\mu}\partial_{\mu}\varphi(\xi^{\kappa})=0$$

tevens oplossingen van de $\binom{q}{2}$ lineaire vergelijkingen

4.2b) $$\underset{[z}{v}^{\nu}\partial_{\nu}\underset{y]}{v}^{\mu}\partial_{\mu}\varphi=(\underset{[z}{v}^{\nu}\partial_{\nu}\underset{y]}{v}^{\mu})\partial_{\mu}\varphi=0$$

Zijn deze vergelijkingen elk een gevolg van (4.2a) dan heet het stelsel (4.2a) volledig. Het kan worden bewezen dat het stelsel dan $n-q$ onafhankelijke oplossingen heeft en dat iedere oplossing een functie van deze $n-q$ is. Is het stelsel niet compleet dan voegt men de onafhankelijke vergelijkingen uit (4.2b) aan (4.2a) toe en verkrijgt zo een nieuw stelsel dat weer op dezelfde manier behandeld wordt. Tenslotte krijgt men altijd na een eindig aantal stappen of een volledig stelsel of een stelsel van n vergelijkingen dat alleen de triviale oplossing φ = const bezit. Literatuur, zie b.v. Phaffs Problem Hfdst. III. In het geval dat (4.2a) een volledig systeem is bestaan er vergelijkingen van de vorm

$$\underset{[z}{v}^{\nu}\partial_{\nu}\underset{y]}{v}^{\mu}=\beta_{yz}^{x}(\xi^{\kappa})\underset{x}{v}^{\mu}$$

Het stelsel van $n-q$ oplossingen vormt, constant gesteld, de vergelijking van een stelsel ∞^{n-q} X_q's en (4.2a) brengt tot uitdrukking dat de E_q die door de vectoren $\underset{y}{v}^{\kappa}$ wordt opgespannen met de plaatselijke E_q samenvalt. Het E_q-veld is dus X_q-vormend en de noodzakelijke en voldoende voorwaarde hiervoor is dat het stelsel (4.2a) volledig is.

Equalizing these $n-q$ functions to constants we get the equations of a system of ∞^{n-q} X_q's each of which is invariant for all infinitesimal transformations of the group. In order to prove that each X_q is also invariant for all finite transformations

4.3) $$'\xi^{\kappa} = f^{\kappa}(\xi^{\nu}, \eta^{\alpha})$$

of the group we consider the transformations of the one-parametrical subgroup generated by $e^{\ell}X_{\ell}$. For these transformations we have according to (2.17)

4.4) $$\frac{d'\xi^{\kappa}}{dt} = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\, e^{\ell} dt \qquad ; '\xi^{\kappa} = \xi^{\kappa} \quad \text{for} \quad t = 0$$

hence, in consequence of (4.1)

4.5) $$\frac{d}{dt}\,\varphi('\xi^{\kappa}) = \Xi^{\kappa}_{\ell}('\xi^{\lambda})\,\frac{\partial\varphi('\xi^{\kappa})}{\partial'\xi^{\kappa}}\,e^{\ell} = 0.$$

The invariant X_q's can also be derived easily from the finite equations (4.3) of the group. According to (2.17) the rank of $\partial_{\beta}{'\xi^{\kappa}}$ is q and A^{ℓ}_{β} has the rank r. Hence from the theorem of elimination[1] we know, that from (4.3) just $n-q$ independent equations between ξ^{κ} and $'\xi^{\kappa}$ can be derived from which $n-q$ of the $'\xi^{\kappa}$ (suitably chosen) can be solved

4.6) $$'\xi^{\alpha} = \psi^{\alpha}('\xi^{\varphi}, \xi^{\kappa}) \quad ; \alpha = 1, \dots n-q \; ; \varphi = n-q+1, \dots n$$

Now we know that there exist $n-q$ invariant functions $\overset{\alpha}{\varphi}$; $\alpha = 1, \dots n-q$ and thus $n-q$ equations

4.7) $$\overset{\alpha}{\varphi}(\xi^{\kappa}) = \overset{\alpha}{\varphi}('\xi^{\kappa})$$

and this set of equations must be equivalent to (4.6). Hence if in (4.6) we give the $'\xi^{\kappa}$ constant values $c^{1}, \dots c^{n}$ it follows that

4.8) $$\psi^{\alpha}(c^{\varphi}, \xi^{\kappa}) = \text{constant}$$

and in this way we have found for each choice of the c^{φ} a set of independent invariant functions.

1) P.P. p.42 {R.S. theorem VII}.

2. Invariant subspaces[1]

A set of independent functions $\overset{\alpha}{\varphi}(\xi^\kappa)$ is called relative invariant[2] if the equations

$$4.9) \qquad X_i \overset{\alpha}{\varphi} = \Xi^\mu_l \partial_\mu \overset{\alpha}{\varphi} = 0$$

is a consequence of

$$4.10) \qquad \overset{\alpha}{\varphi} = 0$$

This means geometrically that the X_q represented by (4.10) is invariant for all transformations of the group. To the one-parametrical subgroup generated by $\epsilon^l X_l$ belongs a system of ∞^{m-1} curves of X_m, the streamlines of the vector field $\epsilon^l \Xi^\kappa_l$ and (4.9) expresses that the X_q is built up on streamlines and that every streamline having one point in common with the X_q lies wholly in X_q.

The group interchanges the points of the X_q and we may ask for the group of the transformations of these points. Let

$$4.11) \qquad \xi^\kappa = \phi^\kappa(y^h) \; ; \quad h = 1, \dots q,$$

$$B^\kappa_i \overset{\text{def}}{=} \partial_i \xi^\kappa = \text{connecting quantity of rank } q \, .$$

be the parametrical equations of the X_q and let $Y_l = Y^j_l \partial_j$ be the infinitesimal point transformations in X_q corresponding to $X_l = \Xi^\mu_l \partial_\mu$. Then we have on the X_q for a function $f(\xi^\kappa) = f(\phi^\kappa(y^h))$ and for the subgroup generated by $\epsilon^l X_l$

$$4.12) \qquad \frac{df}{dt} = \epsilon^l \Xi^\mu_l \partial_\mu f$$

but at the other hand

$$4.13) \qquad \frac{df}{dt} = \epsilon^l Y^j_l B^\mu_j \partial_\mu f$$

hence

$$4.14) \qquad Y^j_l B^\kappa_j = \Xi^\kappa_l$$

1) Lie-Engel I p. 222 ff.
2) Eisenhart, Continuous groups, p.63.

In general such an equation can-not exist because the κ -domain of Ξ^{κ}_{ℓ} does not coincide with the κ -domain of B^{κ}_{j}, But in our case the X_q is invariant and its tangent E_q is the support of both domains. That implies that in this case the y^{j}_{ℓ} can be solved from (4.14). The easiest way to do this is to take $\xi^1, \ldots, \xi^q$ (suitably chosen) as parameters on the X_q. Then $h = 1, \ldots q$ and consequently

4.15) $$B^{\kappa}_{h} = \delta^{\kappa}_{h} \;;\; y^{h}_{\ell} = \Xi^{h}_{\ell} \;;\; \Xi^{q+1}_{\ell} = 0, \ldots \Xi^{n}_{\ell} = 0.$$

That means that we get the y^{h}_{ℓ} by dropping in $\Xi^{\mu}_{\ell}\partial_{\mu}$ all differentiations with respect to $\xi^{q+1}, \ldots, \xi^{n}$. The group of point transformations in X_q found in this way, and called the group induced in , has the r infinitesimal transformations y_{ℓ} with

4.16) $$(y_c y_{\ell}) = C_{c\ell}^{\cdot\cdot a}\, y_a$$

but the y_{ℓ} need not be independent. Take for instance the case where the X_{ℓ} are connected $\psi^{\ell}(\xi^{\kappa})\, X_{\ell} = 0$. Then it may occur that the $\psi^{\ell}(\xi^{\kappa})$ are constant on X_q. Now suppose the induced group to be r' -parametrical $r' < r$.

Then there must be $r - r'$ relations of the form

4.17) $$\underset{p}{c}^{\ell}\, y^{h}_{\ell} = 0 \quad ; \; p = 1, \ldots, r - r'$$

with constants $\underset{p}{c}^{\ell}$. From this it results that there exist $r - r'$ relations

4.18) $$\underset{p}{c}^{\ell}\, X_{\ell} = 0$$

valid in all points of the X_q. But this implies that there exist $r - r'$ independent infinitesimal transformation of the original group leaving <u>every point</u> of X_q invariant. Of course these transformations form an $(r - r')$-parametrical subgroup. Hence:

<u>The group induced in an invariant X_q is r'-parametrical if and only if there exists an $(r - r')$-parametrical subgroup, leaving each point of X_q individually invariant.</u>

Any point $\underset{0}{\xi}^{\kappa}$ is transformed into the points

4.19) $$\xi^{\kappa} = f^{\kappa}(\underset{0}{\xi}^{\nu}, \eta^{\alpha})$$

depending on r parameters η^{α}. If q is the rank of $\partial_{\beta}\xi^{\kappa}$ it results from the theorem of elimination[1] that there exist just $n-q$ independent equations between the ξ^{κ}. These equations represent an X_q through $\overset{\circ}{\xi}{}^{\kappa}$, called the smallest invariant subspace of $\overset{\circ}{\xi}{}^{\kappa}$. Obviously this X_q is smallest invariant subspace of everyone of its points. In all these points the ranks of $\partial_{\beta}\xi^{\kappa}$ and Ξ^{μ}_{b} are equal to q. Hence, in order to find all invariant subspaces of a group we have only to construct all points where Ξ^{κ}_{b} has a rank q for all values $q \leq r$. For each value we find either an invariant X_q or a subspace consisting of invariant X_q's[2].

A group is called transitive if the rank of Ξ^{κ}_{b} is n in all points of an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$. If the group is transitive, to every set of two points P and Q in a sufficiently small $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ there exists at least one transformation of the group transforming P into Q. For a transitive group $r \geq n$. The group is called simply transitive if $r=n$.

3. Invariant contravariant vectorfields.

A contravariant vectorfield can be looked upon as an infinitesimal transformation $\xi^{\kappa} \rightarrow \xi^{\kappa} + \zeta^{\kappa}\,dt$. Hence invariance of infinitesimal transformations is essentially the same problem. Now we may ask for

a. invariance to within an arbitrary factor;

b. invariance to within a constant factor;

c. complete invariance.

If ζ^{κ} is subjected to the infinitesimal transformation $X = e^{b}\Xi^{\mu}_{b}\partial_{\mu}$ of the group, we know already that the increase of the field is the negative Lie differential of ζ^{κ} with respect to $e^{b}\Xi^{\kappa}_{b}$

$$4.20) \qquad -dt\,\underset{L}{D}\,\zeta^{\kappa} = -e^{b}\Xi^{\mu}_{b}\partial_{\mu}\zeta^{\kappa}\,dt + e^{b}\zeta^{\mu}\partial_{\mu}\Xi^{\mu}_{b}\,dt$$

If ζ^{κ} is considered as an infinitesimal transformation its symbol is $Z \overset{\text{def}}{=} \zeta^{\mu}\partial_{\mu}$ and (4.20) can be written in the form

$$4.21) \qquad dZ = (ZX)\,dt$$

Instead of one vectorfield ζ^{κ} we take now a set of p vectorfields $\zeta^{\kappa}_{y};\ y=1\ldots p$ and consider the cases where the $(Z_y X)$ depend linearly on the Z_x.

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1) P.P. p.42 { R.S. theorem VII }

2) Lie-Engel 1888.1 I p.228, 237.

Case a (invariant complete systems)[1].

$$4.22) \qquad (Z_y X) = \alpha_y^x(\xi^\kappa) Z_x$$

This equation expresses that the E_p-field spanned by the p vectors Z_y is invariant. In the special case where the equations $Z_y^\mu \partial_\mu \varphi = 0$ form a complete system the E_p's form a set of $\infty^{n-p} X_p$'s . In this case there exist equations of the form

$$4.23) \qquad (Z_y Z_z) = \beta_{yz}^x(\xi^\kappa) Z_x$$

and the equations of the X_p's are obtained by equalizing the solutions of $Z_y \varphi = 0$. to constants. Now according to (4.22)

$$4.24) \qquad Z_y X \varphi = \alpha_y^x(\xi^\kappa) Z_x \varphi + X Z_y \varphi$$

and this proves that $\psi = X\varphi$ is a solution of $Z_y \psi = 0$ if $\psi = \varphi$ is a solution. Every solution of the complete system is a function of $n-p$ independent solutions, hence X transforms the set of all solutions in itself. This proves once more that the set of all $\infty^{n-p} X_p$'s (but not every X_p individually!) is invariant for X .

If a group has an invariant complete system (that is an invariant set of $\infty^{n-p} X_p$'s) it is called imprimitive and in the other case primitive. If a group has an invariant X_p it can not be transitive, but an imprimitive group may be transitive.

Now we will prove that a complete system which is invariant for every infinitesimal transformation $e^b X_b$ of a group, is also invariant for all finite transformations of the group. If $\overset{\alpha}{\varphi}(\xi^\kappa)$; $\alpha = 1,\dots n-p$ are $n-p$ independent solutions of $Z_y \varphi = 0$, X transforms the set of all solutions in itself and accordingly there exists an equation of the form

$$4.25) \qquad X \overset{\alpha}{\varphi} = \overset{\alpha}{\psi}(\overset{b}{\varphi})$$

Hence from (4.5) we get

$$4.26) \qquad \frac{d}{dt}\overset{\alpha}{\varphi}('\xi^\kappa) = e^b \Xi_b^\kappa('\xi^\lambda) \frac{\partial \overset{\alpha}{\varphi}('\xi^\rho)}{\partial '\xi^\kappa} = \overset{\alpha}{\psi}(\overset{b}{\varphi}('\xi^\kappa))$$

$\overset{\alpha}{\varphi}('\xi^\kappa) = \overset{\alpha}{\varphi}(\xi^\kappa)$ for $t=0$

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1) Lie-Engel 1888.1 I p. 220 ff.

and from this equation the $\overset{\alpha}{\varphi}('\xi^\kappa)$ can be solved as functions of the $\overset{m}{\varphi}(\xi^\kappa)$ and a parameter t. This expression does not contain the ξ^κ explicitly but only in the form $\overset{\alpha}{\varphi}(\xi^\kappa)$ and this proves the proposition.

There exists a kind of practical rule that "everything" invariant for the infinitesimal transformations of a group is also invariant for all finite transformation of the group germ. In an "intuitive" (= not rigorous) way this is quite clear, every finite transformation being obtainable by applying an infinitesimal transformation an "infinite number of times". The rigorous proof always goes the same way as we have seen now in two examples. But the practical rule has only some heuristic value and the rigorous proof must be given in every special case.

Case b [1)]

4.27) $$(\chi_y X) = K_y^x \chi_x \quad ; \quad K_y^x = \text{constants} \quad ; x, y = 1, \dots p$$

For $p=1$ not only the stream lines of χ^κ are invariant but the whole field χ^κ can only get a constant factor. If on each streamline a parameter s is fixed by the equation $d\xi^\kappa/ds = \chi^\kappa$, the parameter on any streamline is fixed to within an additive constant ...

For general values of p every χ_y represents an infinitesimal transformation generating a one-parametrical group. But in general the χ_y together do not generate a group. The transformations $s^y\chi_y$ with constant coefficients s^y generate each a one-parametrical group. These transformations together are said to form a linear ser (Schaar) of infinitesimal transformations. Lie[2)] has proved that this linear set always generates ∞^p transformations. In the special case that the χ_y though not building a group are transformations of one and the same group this is obvious because in the space of this group every one-parametrical group is represented by a geodesic through $\overset{0}{\eta}{}^\alpha$ and these geodesics form an X_p in an $\mathfrak{M}(\eta^\alpha)$.

If the infinitesimal transformations $s^y\chi_y$ are subjected to the infinitesimal transformation X we have from (4.21)

4.28) $$d\, s^y\chi_y = s^y(\chi_y X)dt = s^y K_y^x$$

Here the s^y were consideredas constants and the χ_y^κ were transformed.

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1) Lie-Engel 1888.1 I p.246 ff.
2) Lie-Engel 1888.1 I p.62.

But we can interprete the result in another way by looking upon the χ_y^k as a kind of fixed measuring vectors and considering the s^y as variable components of the variable vector $s^y \chi_y^k$. Then we get

4.29) $$ds^x = K_y^x s^y dt$$

Now we consider the one-parametrical group generated by the infinitesimal transformation $X = e^\ell \Xi_\ell^\mu \partial_\mu$ for some fixed values of the e^ℓ. Then the transformation of the ξ^k is given by (2.31) and any field v^k having the components $v^k(0, \xi^\nu)$. for $t = 0$ is transformed into a field $v^k(t, \xi^\nu)$ depending on t. Let $s^y \chi_y^k$ be such a field for a definite value of t . If t changes into $t + dt$ and if the χ_y^k are considered as measuring vectors, ds^x is given by (4.29) and this leads to the differential equation

4.30) $$\frac{ds^x(t)}{dt} = K_y^x s^y(t)$$

with the solution

4.31) $$s^x(t) = (e^{Kt})_y^x s^y(0)$$

from which it is evident that the linear set of infinitesimal transformations is invariant for all finite transformations of the group germ.

A finite transformation of the one-parametrical group generated by $s^y(0) \chi_y$ has the form

4.32) $$'\xi^k = e^{u\, s^y(0) \chi_y} \xi^k$$

with some parameter u . Now if a finite transformation of the group generated by X is applied, the transformation (4.32) is tranformed into

4.33) $$'\xi^k = e^{u\, s^y(t) \chi_y} \xi^k$$

and this is a finite transformation of the group generated by $s^y(t) \chi_y$. Hence the one-parametrical groups generated by $s^y \chi_y$ for all different values of the ς^y are interchanged by the transformations of the given group. That proves that (4.27) is the necessary and sufficient condition for the invariance of the linear set $\varsigma^y Z_y$ for all transformations of the group and also for the invariance of the set of all one-parametrical groups generated by $\varsigma^y Z_y$ for different values of the ς^y.

If the Z_y generate a group (4.27) is the necessary and sufficient condition for this group to be an invariant subgroup.

As a corollary we get that a linear set generates a group if and only if it is invariant for all of its own infinitesimal transformations.

Case c. (commutative transformations).

4.34)
$$(Z_1 Z_2) = 0$$

The infinitesimal transformation Z_1 and Z_2 are commutative

4.35)
$$Z_1 Z_2 f = Z_2 Z_1 f$$

Now we consider two finite transformations of the one-parametrical groups generated by Z_1 and by Z_2 and apply these one after the other

$$'\xi^K = \xi^K + t Z_1 \xi^K + \tfrac{1}{2} t^2 Z_1^2 \xi^K + \dots$$

$$''\xi^K = {}'\xi^K + u Z_2 {}'\xi^K + \tfrac{1}{2!} u^2 Z_2^2 {}'\xi^K + \dots$$

$$= \xi^K + t Z_1 \xi^K + \tfrac{1}{2} t^2 Z_1^2 \xi^K + \dots$$

$$+ u Z_2 \xi^K + ut Z_2 Z_1 \xi^K + \tfrac{1}{2} u t^2 Z_2 Z_1^2 \xi + \dots$$

$$+ \tfrac{1}{2} u^2 Z_2^2 \xi^K + \tfrac{1}{2} u^2 t Z_2^2 Z_1 \xi^K + \tfrac{1}{4} u^2 t^2 Z_2^2 Z_1^2 \xi^K + \dots$$

$$+ \dots\dots$$

Then we see that also the finite transformations are commutative:

The finite transformations generated by Z_1 are commutative with the finite transformations generated by Z_2 if and only if $(Z_1 Z_2) = 0$

Rotatiegroep in R_3.

Invariante functies.

De operatoren X_1, X_2, X_3 (2.63j) zijn onafhankelijk, niet onverbonden (pag. 13); er geldt nl. de betrekking

(45a) $$\xi^1 X_1 + \xi^2 X_2 + \xi^3 X_3 = 0$$

Invariante functies worden dus gevonden door oplossing van het stelsel

(4.35b) $$\begin{aligned} X_1\varphi &\equiv \xi^2 \partial_3 \varphi - \xi^3 \partial_2 \varphi = 0 \\ X_2\varphi &\equiv \xi^3 \partial_1 \varphi - \xi^1 \partial_3 \varphi = 0 \end{aligned}$$

Alle oplossingne zijn functies van één niet-triviale oplossing (pag. 32a). Meetkundig is reeds duidelijk dat

(4.35c) $$\varphi = \xi^1\xi^1 + \xi^2\xi^2 + \xi^3\xi^3$$

een oplossing is - hetgeen bevestigd wordt bij substitutie in (4.35b).

De invariante onderruimten zijn:

(4.35d) $$\xi^1\xi^1 + \xi^2\xi^2 + \xi^3\xi^3 = \text{const}$$

Relatief invariante onderruimten (pag. 34) zijn er niet, behalve de invariante bollen (4.35d), immers in elk punt van R_3 vullen de stroomlijnen een 2-richting en die raakt precies de bol door dat punt.

De op de eenheidsbol geinduceerde groep wordt als volgt bepaald: we nemen coördinaten y^h $(h = 1, 2)$:

(4.35e) $$\begin{aligned} \xi^1 &= \sin y^2 \cos y^1 \\ \xi^2 &= \cos y^2 \\ \xi^3 &= \sin y^2 \sin y^1 \end{aligned}$$

en vinden voor $B_i^k = \partial_i \xi^k$:

(4.35f)

B_i^k ($i \backslash k$)	1	2	3
1	$-\sin y^1 \sin y^2$	0	$\cos y^1 \sin y^2$
2	$\cos y^1 \cos y^2$	$-\sin y^1$	$\sin y^1 \cos y^2$

terwijl Ξ^{κ}_{ℓ} wordt:

(4.35g)

Ξ^{κ}_{ℓ}: $\ell \backslash \kappa$	1	2	3
1	0	$-\sin y^1 \sin y^2$	$\cos y^2$
2	$\sin y^1 \sin y^2$	0	$-\cos y^1 \sin y^2$
3	$-\cos y^2$	$\cos y^1 \sin y^2$	0

zodat y^h_a tenslotte de gedaante krijgt:

(4.35h)

y^h_a: $a \backslash h$	1	2
1	$\cos y^1 \cot y^2$	$\sin y^1$
2	-1	0
3	$-\sin y^1 \cot y^2$	$\cos y^1$

Dus:

(4.35i)
$$Y_1 = \cos y^1 \cot y^2\, \partial_1 + \sin y^1 \partial_2$$
$$Y_2 = -\partial_1$$
$$Y_3 = -\sin y^1 \cot y^2 \partial_1 + \cos y^1 \partial_2$$

De operatoren Y_ℓ zijn, evenals de X_ℓ, wel onafhankelijk, niet onverbonden:

(4.35j)
$$\cos y^1\, Y_1 + \cot y^2\, Y_2 - \sin y^1\, Y_3 = 0$$

Dit was te verwachten daar de rotaties in R_3 in (1,1) verwantschap staan met de bewegingen op de bol. Er is dan ook geen niet-triviale ondergroep die de bol punt voor punt invariant laat.(vgl. p. 35 onderaan).

Invariante contravariante vectorvelden.

a) Invariantie op een factor na. Het veld

(4.35k)
$$\eta^{\kappa} = \varphi(\xi^{\nu})\, \xi^{\kappa}$$

waarbij φ géén oplossing van (4.35b), is invariant op een niet-constante factor na:

(4.35.l) $$(\mathcal{X} X_a) = -(X_a \varphi)\,\xi^\nu \partial_\nu = -(X_a \log \varphi)\,\mathcal{X}$$

De invariante onderruimten worden gevonden uit

(4.35m) $$\xi^1 \partial_1 \psi + \xi^2 \partial_2 \psi + \xi^3 \partial_3 \psi = 0$$

ψ moet dus een homogene functie van graad nul zijn, dus is ψ een functie van de twee oplossingen ξ^3/ξ^2 en ξ^3/ξ^1:

(4.35n) $$\psi = \chi(\xi^3/\xi^1, \xi^3/\xi^2)$$

De rechten door de oorsprong zijn dus invariant bij $\mathcal{X}$.

b) <u>Invariantie op een constante factor na</u>. De velden $\underset{1}{e}^k$, $\underset{2}{e}^k$, $\underset{3}{e}^k$ met de operatoren $\mathcal{X}_x$ $(x = 1,2,3)$:

(4.35o) $$\mathcal{X}_1 = \partial/\partial\xi^1 \;,\; \mathcal{X}_2 = \partial/\partial\xi^2 \;,\; \mathcal{X}_3 = \partial/\partial\xi^3$$

(toevallig een groep) zijn ijvariant op een constante factor na:

(4.35p) $$(\mathcal{X}_y X_\ell) = K_{y\ell}^{\cdot\cdot x}\mathcal{X}_x$$

waar

(4.35q) $$K_{12}^{\cdot\cdot 3} = K_{23}^{\cdot\cdot 1} = K_{31}^{\cdot\cdot 2} = +1$$
$$K_{21}^{\cdot\cdot 3} = K_{13}^{\cdot\cdot 2} = K_{32}^{\cdot\cdot 1} = -1$$

Door de transformatie $e^t X_\ell$ gaat dus $s^x \mathcal{X}_x$ over in $(s^x + ds^x)\mathcal{X}_x$ met

(4.35r) $$ds^x = s^y K_{yc}^{\cdot\cdot x} e^c\, dt$$

dus

(4.35s) $$ds^1 = (s^2 e^3 - s^3 e^2)\, dt \quad (\text{cycl.})$$

De eindige transformaties zijn, als we K_c schrijven voor de matrix van $K_{yc}^{\cdot\cdot x}$:

(4.35t) $$'s^x = (e^{tK_c e^c})^x_y\, s^y$$

Daar nu $K_{y\ell}^{\cdot\cdot x} = -\delta^{cx}_{ya} c_{c\ell}^{\cdot\cdot a}$ (2.63.1), geldt:

(4.35u) $$e^c K_{yc}^{\cdot\cdot x} = \delta^{bx}_{ya} e^c c_{cb}^{\cdot\cdot a} = \delta^{bx}_{ya} E_\ell^{\cdot a}$$

zodat we vinden door vergelijking met (3.63h):

(4.35v) $$\left(e^{te^cK_c}\right)^x_y = \left(A + te^cK_c \frac{\sin\rho}{\rho} + t^2 (e^cK_c)^2 \frac{1-\cos\rho}{\rho^2}\right)^x_y$$
$$\rho^2 = (e^1e^1 + e^2e^2 + e^3e^3)t^2$$

Op de ξ^x (infinitesimale translaties) worden dus dezelfde transformaties uitegvoerd als op de η^c van een element der rotatiegroep bij toepassing van de transformatie van de geadjungeerde groep behorende bij te^a (pag. 28 bovenaan).

Daar bij toepassing van het element van de geadjungeerde groep behorende bij $\eta^\alpha = \delta^\alpha_a te^a$ de geodetische lijn door $\eta^\alpha = 0$ en $\eta^\alpha = \delta^\alpha_a e^a$ op zijn plaats blijft vinden we hier als corollarium:

Bij de rotatie $e^a X_a$ is de rechte langs de vector

(4.35w) $$v^\varkappa = \delta^\varkappa_a e^a$$

de rotatie-as.

Door als speciaal geval te nemen $e^1=0$, $e^2=0$, $e^3=1$, en $\xi^1=1$, $\xi^2=0$, $\xi^3=0$ en te letten op (4.35v) en (3.63s) vindt men gemakkelijk:

(4.35x) $$'\xi^1 = A^1_1 = A^1_1 = \cos\rho = \cos t$$
$$'\xi^2 = A^2_1 = A^1_2 = \frac{\sin\rho}{\rho}\eta^3 = \sin t$$
$$'\xi^3 = A^3_1 = A^1_3 = 0$$

dus de rotatiehoek is t. Samenvattend vinden we dus:

De coördinaten $\eta^\varkappa$ van een eindige rotatie in R_3 zijn numeriek gelijk aan de kentallen van de vector $v^\varkappa$ in R_3 waarbij de rechte langs $v^\varkappa$ de rotatie-as is, en de lengte van $v^\varkappa$ de rotatie-hoek.

c) <u>Volledige</u> <u>invariantie</u>: Men neme nu voor $\varphi(\xi^\nu)$ in (4.35k) een oplossing van (4.35b), b.v. $\varphi(\xi^\nu)=1$. Dan is dus:

(4.35y) $$X = \xi^1\partial_1 + \xi^2\partial_2 + \xi^3\partial_3$$

de operator; hij stelt voor een infinitesimale gelijkvormigheidstransformatie.

§ 5. Invariants of a group in group space[1].

In § 3 we have seen that the differential comitants of group space are the ordinary comitants of $c_{\gamma\beta}^{\cdot\cdot\alpha}$ and that accordingly all these comitants have constant components with respect to the anholonomic systems (a) en (A). The infinitesimal transformation of the linear adjoint group corresponding to the transformation $s^{\ell}X_{\ell}$ (s^{ℓ} = constant) of the group is

$$5.1) \qquad y \overset{\text{def}}{=} s^{\ell} y_{\ell}^{a} \partial_a \; ; \qquad y_{\ell}^{a} \overset{\text{def}}{=} e^{c} c_{cb}^{\cdot\cdot a}$$

e^{α} represents here the radius vector in the local E_n of η^{α} . Hence (5.1) represents a homogeneous linear vector transformation in this E_n. An example of a finite transformation of the linear adjoint group is the transformation of the $\underset{c}{e}^{\alpha}$ into the $\underset{b}{e}^{\alpha}$ in a local E_n of a point of group space. Now the invariance of a quantity for a linear homogeneous vector transformation is the same as the invariance of its components for the corresponding coordinate transformation. In § 3 we have already seen that the components $c_{cb}^{\cdot\cdot a}$ are invariant for a coordinate transformation belonging to the linear adjoint group. Hence it is already sure that the quantity $c_{c\ell}^{\cdot\cdot a}$ is invariant for vector transformations belonging to this group. This can be checked as follows. Let $S_{\ell}^{\cdot a}$ stand for $-s^{\ell} c_{c\ell}^{\cdot\cdot a}$. Then

$$5.2) \qquad \begin{aligned} v^{a} &\rightarrow (A_{\ell}^{a} - S_{\ell}^{\cdot a} dt) v^{\ell} \\ w_{\ell} &\rightarrow (A_{\ell}^{a} + S_{\ell}^{\cdot a} dt) w_{a} \end{aligned}$$

is the transformation of contra- and covariant vector by (5.1). Hence the transformation of $c_{c\ell}^{\cdot\cdot a}$ is given by (cf. (1.30b))

$$\begin{aligned} dc_{c\ell}^{\cdot\cdot a} &= (S_{c}^{\cdot d} c_{d\ell}^{\cdot\cdot a} + S_{\ell}^{\cdot d} c_{cd}^{\cdot\cdot a} - S_{d}^{\cdot a} c_{c\ell}^{\cdot\cdot d})\, dt = \\ 5.3) \qquad &= -s^{e}(c_{ec}^{\cdot\cdot d} c_{db}^{\cdot\cdot a} + c_{e\ell}^{\cdot\cdot d} c_{cd}^{\cdot\cdot a} - c_{ed}^{\cdot\cdot a} c_{c\ell}^{\cdot\cdot d})\, dt = \\ &= 0. \end{aligned}$$

Now $c_{cb}^{\cdot\cdot a}$ being a comitant of the linear adjoint group and the adjoint group itself being fixed by $c_{cb}^{\cdot\cdot a}$, it follows that the comitants of $c_{cb}^{\cdot\cdot a}$ and the comitants of the adjoint group are the same. Hence

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1) Lie Engel 1888, 1, I p.270 ff.

<u>The differential comitants of group space are the comitants of the linear adjoint group.</u>

There is a set of comitants of $c_{cb}^{\cdot\cdot a}$ that are very important for the theory:

a) $g_b = c_{cb}^{\cdot\cdot c}$

5.4 b) $g_{ba} = c_{cb}^{\cdot\cdot d}\, c_{da}^{\cdot\cdot c}$

c) $g_{cba} = c_{dc}^{\cdot\cdot e}\, c_{eb}^{\cdot\cdot f}\, c_{fa}^{\cdot\cdot d}$

They are all invariant for cyclical permutations of the indices. Hence g_{ba} is a tensor.

If a comitant of $c_{cb}^{\cdot\cdot a}$ is transvected by one or more factors e^a and if the expression is equated to zero we get an equation that is invariant for the linear adjoint group as was proved by Cartan. We prove this here for the transvection $P^{ab}_{\cdot\cdot cd}\, e^c e^d$:

$$\begin{aligned}\frac{d}{dt}\left(P^{ab}_{\cdot\cdot cd}\, e^c e^d\right) &= - S_e^{\cdot a} P^{eb}_{\cdot\cdot cd}\, e^c e^d - S_e^{\cdot b} P^{ae}_{\cdot\cdot cd}\, e^c e^d + \\ &\quad + S_c^{\cdot e} P^{ab}_{\cdot\cdot ed}\, e^c e^d + S_d^{\cdot e} P^{ab}_{\cdot\cdot ce}\, e^c e^d - \\ &\quad - P^{ab}_{\cdot\cdot cd}\, S_e^{\cdot c} e^e e^d - P^{ab}_{\cdot\cdot cd}\, e^c S_e^{\cdot d} e^e = \\ &= -\left(S_e^{\cdot a} A_f^b + S_f^{\cdot b} A_e^a\right) P^{ef}_{\cdot\cdot cd}\, e^c e^d. \end{aligned} \tag{5.5}$$

This property can be used to form invariant equations from the set (5,4) by transvecting with factors e^a. If only one factor occurs and if the equations are consistent, they represent a flat subspace of E_n through the origin and are invariant for the linear adjoint group.

To every point e^a in the local E_n of η^α corresponds the point of group space with the normal coordinates $\overset{\circ}{\eta}{}^\alpha = A_a^\alpha e^a$. Because the e^a and the η^α transform in the same way, to an E_p in the E_n through the origin corresponds an X_p in group space consisting of ∞^{p-1} geodesics, each representing a one-parametrical subgroup. In normal coordinates this X_p is given by $n-p$ homogeneous linear equations. Let the normal coordinates be chosen in such a way that these equations are

5.6) $\eta^{\mathfrak{x}} = 0 \quad ; \quad \mathfrak{x}, \mathfrak{y}, \mathfrak{z} = \overline{p+1}, \ldots \overline{n}$

and that the $\eta^{\mathfrak{m}}$; $\mathfrak{m}, \mathfrak{n}, \mathfrak{p} = \overline{1}, \ldots \overline{p}$ can be used as coordinates in the X_p.

An X_p is geodesic in R_n (cf. IV....) if and only if every vector of X_p remaines in X_p if it is displaced pseudoparallel along X_p. Let v^{α} be the vector and let $d\eta^{\alpha}$ be the displacement. Then $v^{\psi}=0$ and $d\eta^{\psi}=0$. For the pseudoparallel displacement we have:

5.7) $$dv^{\alpha} = -\Gamma_{cb}^{\alpha} v^{b} d\eta^{c} = -\Gamma_{\mathfrak{p}\mathfrak{n}}^{\alpha} v^{\mathfrak{n}} d\eta^{\mathfrak{p}} \quad ; \mathfrak{p},\mathfrak{n} = \bar{1},\dots\bar{p}.$$

and dv^{ψ} must be zero. Hence

5.8) $$\Gamma_{\mathfrak{p}\mathfrak{n}}^{\psi} = 0$$

in every point of X_p and this condition is necessary and sufficient. That implies that according to the expansion (3.57)

5.9) $$\underset{\circ}{S}_{\mathfrak{m}(\mathfrak{p}}{}^{\vartheta}\,\underset{\circ}{S}_{\mathfrak{n}),\vartheta}{}^{\psi} = 0$$

is a <u>necessary</u> condition. But this equation expresses that $\underset{\circ}{S}_{\mathfrak{m}\mathfrak{p}}{}^{\vartheta}\underset{\circ}{S}_{\mathfrak{n}\vartheta}{}^{\psi}$ is alternating in $\mathfrak{p}\mathfrak{n}$ and because the same expression is also alternating in $\mathfrak{m}\mathfrak{p}$, it is alternating in $\mathfrak{m}\mathfrak{p}\mathfrak{n}$. Hence (cf. (1.30b)) (5.9) is equivalent to

5.10) $$\underset{\circ}{S}_{\mathfrak{m}\mathfrak{p}}{}^{\vartheta}\,\underset{\circ}{S}_{\mathfrak{n}\vartheta}{}^{\psi} = 0$$

Now we will prove that (5.10) is not only necessary but also <u>sufficient</u>. Every term of the expansion (3.57) (cf. (3.55)) contains one factor C_c and an odd number of factors P, for instance $P^2 C_c P^3$. If this term is written out we get for its contribution to $\Gamma_{\mathfrak{p}\mathfrak{n}}^{\psi}$:

$$\eta^{\mathfrak{a}_1}\underset{\circ}{S}_{\mathfrak{a}_1\dot{\alpha}_1(\mathfrak{n}}{}^{\alpha_1}\,\eta^{\mathfrak{a}_2}\underset{\circ}{S}_{|\mathfrak{a}_2\dot{\alpha}_1|}{}^{\alpha_2}\,C_{\dot{\mathfrak{p}})\alpha_2}{}^{\alpha_3}\,\eta^{\mathfrak{a}_3}\underset{\circ}{S}_{\mathfrak{a}_3\dot{\alpha}_3}{}^{\alpha_4}\,\eta^{\mathfrak{a}_4}\underset{\circ}{S}_{\mathfrak{a}_4\dot{\alpha}_4}{}^{\alpha_5}\times$$

5.11) $$\times\,\eta^{\mathfrak{a}_5}\underset{\circ}{S}_{\mathfrak{a}_5\dot{\alpha}_5}{}^{\psi} \quad ;$$

$$\alpha_1,\dots\alpha_5 = \bar{1},\dots\bar{n}$$

$$\mathfrak{n},\mathfrak{p},\mathfrak{a}_1,\dots\mathfrak{a}_5 = \bar{1},\dots\bar{p}$$

$$\psi = \overline{p+1},\dots\bar{n}.$$

Now it follows from (5.10) that $\mathfrak{p}_2$ can be written instead of α_2. Then by the same argument $\mathfrak{p}_4$ can be written instead of α_4. But then it results from (5.8) that the whole expression (5.11) is zero because the last factor is. The reasoning is independent of the odd number of factors P and of the place of C_c in the product. Hence (5.9) has as a consequence that not only the first term in the expansion of $\Gamma_{pn}^{\prime\,u}$ vanishes but also all other terms.

The E_p in η^α corresponding to the X_p (5.6) is spanned by vectors e^α_{n}. Hence the one-parametrical subgroups (geodesics) constituting the X_p are generated by the linear set of infinitesimal transformations $X_n \overset{def}{=} e^\alpha_n \partial_\alpha$. Writing out the expression $(X_n(X_m X_p)$ we get

5.12)
$$(X_n(X_m X_p)) = c_{mp}^{\;\;\cdot o}\, c_{no}^{\;\;\cdot \alpha} X_\alpha$$
$$m, n, p = \bar{1}, \ldots \bar{p}$$

Hence (5.10) is the necessary and sufficient condition that $(X_n(X_m X_p))$ depends only on $X_{\bar 1}, \ldots X_{\bar p}$ [1].

It may happen that already $(X_n X_m)$ depends only on $X_{\bar 1}, \ldots X_{\bar p}$. Necessary and sufficient condition is that $c_{pn}^{\;\;\cdot u} = 0$. That means that the X_p represents a subgroup with the structural constants $c_{pn}^{\;\;\cdot m}$. Of course the X_p remains geodesic but in this special case it has another remarkable geometric property. If the vector $v^\alpha, v^u = 0$ is displaced (+) or (-) - parallel along the X_p we have

5.13)
$$dv^\alpha = -\Gamma_{c b}^{\alpha} d\eta^c v^b \pm \tfrac{1}{2} c_{cb}^{\;\;\cdot\alpha} d\eta^c v^b =$$
$$= -\Gamma_{pn}^{\alpha} d\eta^p v^n \pm \tfrac{1}{2} c_{pn}^{\;\;\cdot\alpha} d\eta^p v^n,$$

hence

5.14)
$$dv^u = \pm \tfrac{1}{2} c_{pn}^{\;\;\cdot u} d\eta^p v^n$$

Now we use the expansion (3.58) (cf. (3.55))

5.15)
$$c_{cb}^{\;\;\cdot\alpha} = c_{cb}^{\;\;\cdot\alpha} + \ldots$$

and remark that all terms contain one factor C_c and an <u>even</u> number of factors P, for instance PC_cP.

If this term is written out its contribution to is

5.16)

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1) Cartan 1927 .... p. 72 ff.

Because of $\overset{\circ}{S}_{pu}{}^{\mathfrak{l}} = 0$, instead of α_1 we can write p_1 and by the same argument p_2 instead of α_2 and p_3 instead of α_3. Then we see that the last factor is zero and therefore the whole expression. The reasoning is independent of the (even) number of factors P and of the place of the factor C_c. Hence <u>the X_p represents a subgroup if and only if it is not only geodesic, that is (0)-parallel in itself but also $(+)$-parallel and $(-)$-parallel</u>[1].

If the vectors $\underset{u}{e}^{\alpha}$ are subjected to the infinitesimal transformation of the linear adjoint group corresponding to $X_b \stackrel{*}{=} \underset{b}{e}^{\beta}\partial_\beta$ we get

$$\text{5.17)}\qquad d\underset{u}{e}^{a} = \underset{b}{e}^{b}\underset{u}{e}^{c} c_{cb}^{\cdot\cdot a}\, dt \stackrel{*}{=} \overset{\circ}{S}_{bu}{}^{a}\, dt \qquad b = \bar{1},\dots\bar{n};\ u = \bar{1},\dots\bar{p}.$$

Hence $\overset{\circ}{S}_{pu}{}^{\mathfrak{l}} = 0$ is the necessary and sufficient condition that the E_p spanned by the $\underset{u}{e}^{a}$ is invariant for <u>those</u> transformations of the linear adjoint group that correspond to the infinitesimal transformations of the linear set belonging to the E_p itself.

It may happen that not only $(X_u\, X_m)$ but also $(X_b\, X_m)$ depends only on $X_{\bar{1}},\dots, X_{\bar{p}}$. That means that the E_p is invariant for all infinitesimal transformations of the linear adjoint group. But in § 4 (case b) we have seen that this means also that $X_{\bar{1}},\dots X_{\bar{p}}$ generate an invariant subgroup. In this case the X_p of the subgroup has another remarkable property. In X_n we form a $(+)$-constant field v^{α}; $v^{\mathfrak{l}} = 0$ Then the components v^{a} are constants and $\overset{+}{\delta} v^{a} = 0$. Now we have

$$\text{5.18)}\qquad \overset{-}{\delta} v^{\alpha} = \overset{+}{\delta} v^{\alpha} - (\overset{+}{\Gamma}{}_{cb}^{\alpha} - \overset{-}{\Gamma}{}_{cb}^{\alpha})\, v^{b} d\eta^{c} = \; = c_{cb}^{\cdot\cdot\alpha}\, v^{b} d\eta^{c}$$

hence

$$\text{5.19)}\qquad \begin{aligned} \overset{-}{\delta} v^{m} &= c_{cu}^{\cdot\cdot m}\, v^{u} d\eta^{c} \\ \overset{-}{\delta} v^{\mathfrak{l}} &= c_{cu}^{\cdot\cdot \mathfrak{l}}\, v^{u} d\eta^{c} = 0 \end{aligned}$$

and accordingly the p-direction of the X_p is not only $(-)$-constant for displacements in the X_p but for all displacements. In the same way it can be proved that it is $(+)$-constant.

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1) Cartan - Schouten 1926 ••• Verslag Kon.Ak.v.Wet. 35 (1926) 387-399, p. 397.

A $(-)$-displacement is also a dragging along a $(+)$-constant field (cf. exercise on p. 4 bottom), and holomity is invariant for the process of dragging along, hence the E_p-field dragged along generates $\infty^{r-p}\ X_p$'s

That proves that an X_p in X_r through $\overset{0}{\eta}{}^\alpha$ represents an invariant subgroup if and only if it is geodesic and $(+)$-parallel and $(-)$-parallel in itself and if every X_p that is $(+)$-parallel to it is also $(-)$-parallel.

This property is clear from the point of view of group theory: The X_p's parallel to the one through $\overset{0}{\eta}{}^x$ are the co-sets (Nebenklassen)[1] of the invariant subgroup corresponding to the X_p through $\overset{0}{\eta}{}^\alpha$. This is clear because e.g. a $(+)$-parallel displacement is a dragging along a $(-)$ constant field (p. 4 bottom) and that is an element of the second parameter group (p. 4 middle).

Every set of equations linear homogeneous in e^a formed by means of comitants of $c_{\dot{c}\dot{b}}{}^a$ represents a flat manifold in E_r invariant for all infinitesimal transformations of the linear adjoint group. Hence this manifold and the corresponding manifold in X_r represent an invariant subgroup. The following invariant subgroups are important

a. The group of g_b

5.20) $$g_b e^b = 0$$

If $g_b \neq 0$ this group is $(r-1)$-parametric

b. The centre

5.21) $$e^c c_{\dot{c}\dot{b}}{}^a = 0$$

This is the group of all exceptional transformations (cf. § 4.), The centre is abelian and vanishes if the linear adjoint group is r-parametrical.

c. The first derived group represented by the support of the a-domain of $c_{\dot{c}\dot{b}}{}^a$. If q is the a-rank of $c_{\dot{c}\dot{b}}{}^a$ its equation is

5.22) $$c_{\dot{c}_1\dot{b}_1}{}^{[a_1} \dots c_{\dot{c}_q\dot{b}_q}{}^{a_q} e^{a]} = 0$$

Its infinitesimal transformations are those transformations of the group that can be written as Lie brackets. It has the property that it is generative by the commutators of the infinitesimal elements of the group: be T and U two such elements and $e^b X_b$ and $f^b X_b$ their operators, then to $TUT^{-1}U^{-1}$ there corresponds the operator $e^c f^b (X_c X_b) = e^c f^b c_{\dot{c}\dot{b}}{}^a X_a$

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1) B.L.v.d.Waerden, Moderne Algebra I 1930, p. 32 ff.

This group is contained in the group of g_b . In fact

5.23) $$c_{cb}^{\cdot\cdot a} g_a = c_{cb}^{\cdot\cdot a} c_{da}^{\cdot\cdot d} = - c_{bd}^{\cdot\cdot a} c_{ca}^{\cdot\cdot d} - c_{dc}^{\cdot\cdot a} c_{ba}^{\cdot\cdot d} = 0$$

It may happen that a group and its first derived group are identical. From the definition it follows that the first derived group of a subgroup is a subgroup of the first derived group of the group itself. Let $X_1, \dots X_m$ be the transformations of an invariant subgroup. Then $(X_a X_p)$ $(p = 1, \dots m)$ belongs to this subgroup and $(X_p X_q)$; $p, q = 1, \dots m$ belongs to its first derived group. Hence, in the identity

5.24) $$((X_p X_q) X_a) + ((X_q X_a) X_p) + ((X_a X_p) X_q) = 0$$

the last two terms belong to the first derived group of the subgroup and this proves that the first derived group of an invariant subgroup is an invariant subgroup of the group. The following propositions are obvious.

<u>The first derived group of a subgroup is a subgroup of the first derived group of the group</u>.

<u>The first derived group of an invariant subgroup is an invariant subgroup</u>.

The first derived group of the first derived group is called the <u>second derived</u> group, etc . All derived groups are invariant subgroups of the group under consideration.

d. The <u>group of</u> g_{ba} .

5.25) $$g_{ba} e^b = 0$$

is $(n-q)$- parametrical if q is the rank of g_{ba} and vanishes if g_{ba} has rank n .

e. The <u>group of</u> $c_{cb}^{\cdot\cdot d} g_{da}$.

5.26) $$c_{cb}^{\cdot\cdot d} g_{da} e^a = 0$$

Obviously this group contains the group of g_{ba} . c_{cba} is always a <u>trivector</u>. In fact

5.27) $$\begin{aligned} c_{cb}^{\cdot\cdot d} g_{da} &= c_{cb}^{\cdot\cdot d} c_{ed}^{\cdot\cdot f} c_{fa}^{\cdot\cdot e} = \\ &= - c_{be}^{\cdot\cdot d} c_{cd}^{\cdot\cdot f} c_{fa}^{\cdot\cdot e} - c_{ec}^{\cdot\cdot d} c_{bd}^{\cdot\cdot f} c_{fa}^{\cdot\cdot e} = \\ &= -2\, g_{[bc]a} = -2\, g_{[abc]} \end{aligned}$$

because g_{cba} is invariant for cyclical permutations:

5.28) $$g_{abc} = g_{[abc]} + g_{(abc)}.$$

§ 6. Properties of integrable groups.

If we form the derived groups of a given group it may happen that the provess stops because the derived group of the th derived group is identical with this latter group. If this is not the case the last derived group is zero and the last but one is abelian. In this special case the group is said to be integrable. Every abelian group is integrable and so is every two-parametrical group. In fact, if $(X_1 X_2) = \alpha X_1 + \beta X_2$, we have $(X_1(\alpha X_1+\beta X_2)) = \beta(\alpha X_1+\beta X_2)$; $(X_2(\alpha X_1+\beta X_2)) = -\alpha(\alpha X_1+\beta X_2)$.

We prove the proposition:

An r-parametric group G_r is integrable if and only if it is possible to find a sequence of groups $G_1, \dots G_{r-1}, G_r$ such that G_p is p-parametrical and an invariant subgroup of G_{p+1}, $p = 1, \dots r-1$.

Let G_r be integrable and its q-th derived group $G^{(q)}$ be s-parametrical. Then every transformation of $G^{(q-1)}$ not belonging to $G^{(q)}$ determines with $G^{(q)}$ an $(s+1)$-parametrical group, and this group though not necessarily an invariant subgroup of G_r is always an invariant subgroup of $G^{(q-1)}$.

Proof. Be $X_1, \dots X_s$ the operators of $G^{(q)}$ and $X_1, \dots X_t$ those of $G^{(q-1)}$ $(s<t)$ then, because $G^{(q)}$ is the derived group of $G^{(q-1)}$: $c_{cb}{}^{a} = 0$ for $c, b = 1, \dots t$, $a = s+1, \dots t$

Hence:

1. the $(s+1)$-parametrical set generated by $X_1, \dots X_{s+1}$ is a group.
2. this group is an invariant subgroup of $G^{(q-1)}$.

A transformation of $G^{(q-1)}$ not belonging to the $(s+1)$-parametrical group determines with this latter group an $(s+2)$ parametrical invariant subgroup of $G^{(q-1)}$ containing the $(s+1)$-parametrical group as an invariant subgroup. Proceeding in this way for all values of q we get at last a sequence of groups $G_1, \dots G_r$ such that always G_p is an invariant subgroup of G_{p+t} and that G_p is an invariant subgroup of G_{p+t} if no derived groups occur between G_p and G_{p+t}.

Conversely let there be a sequence $G_1, \dots G_r$, such that G_p is always an invariant subgroup of G_{p+1}. If $X_1, \dots X_p$ are the operators G_p and $X_1, \dots X_{p+1}$ those of G_{p+1}, then, because G_p is a group $c_{cb}{}^{p+1} = 0$ for $c, b = 1, \dots p$ and because G_p is invariant in G_{p+1}: $c_{p+1\, b}{}^{p+1} = 0$ for $b = 1, \dots p$, i.e. $c_{cb}{}^{p+1} = 0$ for $c, b = 1, \dots p+1$ (either c or b must be $< p+1$!) which espresses that the derived group of G_{p+1} is contained in G_p, hence the first derived subgroup of G_r is a subgroup of G_{r-1}, the second is a subgroup of G_{r-2} and so on. Hence it can never occur that one of the derivatives is its own derivative and the group must be integrable.

If the sequence $\mathcal{G}_1, \dots \mathcal{G}_r$ is known the system (a) can be chosen in such a way that X_1 belongs to $\mathcal{G}_1$ and X_2 to $\mathcal{G}_2$ etc. Then we have for the integrable group

$$(X_1 X_2) = c_{12}^{\cdot\cdot 1} X_1$$

6.1)

$$(X_1 X_3) = c_{13}^{\cdot\cdot 1} X_1 + c_{13}^{\cdot\cdot 2} X_2 \, ; (X_2 X_3) = c_{23}^{\cdot\cdot 1} X_1 + c_{23}^{\cdot\cdot 2} X_2$$

....

and accordingly

6.2)

$$(X_p X_{p+t}) = c_{p\;p+t}^{\cdot\cdot\;q} X_q \quad ; \quad t > 0 \; ; \; p = 1, \dots n-1,$$

$$q = 1, \dots p+t-1 < n$$

It is remarkable that the centre, the group of g_{ba} and the group of $c_{cb}^{\cdot\cdot d} g_{da}$ of any group are always integrable[1]. The proof of these propositions is rather long and makes use of the theory of elementary divisors applied to the matrix $e^c c_{cb}^{\cdot\cdot a}$. So we can not bring this proof here though we need in the following the integrability of the group of g_{ba}.

§ 7. Simple and semisimple groups.

A group is called simple if it does not contain any invariant subgroup. It is called semi-simple if it does not contain any integrable invariant subgroup. Hence a simple group is also semi-simple. As the group of g_{ba} is an integrable invariant subgroup it follows that for a semi-simple group g_{ba} has rank n. But this condition is not only necessary but also sufficient. We prove that the rank of g_{ba} is always $< n$ if the group contains an integrable invariant subgroup. If such a subgroup exists, either it is abilian or one of its derived groups in abelian. This latter group is an invariant subgroup of the group in question. Let $X_1, \dots X_n$ be chosen in such a way that $X_p \,; p = 1, \dots m$ belong to this invariant subgroup. Then we have

7.1)

$$c_{sq}^{\cdot\cdot a} = 0 \; ; \quad c_{sb}^{\cdot\cdot x} = 0 \; ; \quad p, q, s = 1, \dots m \; ; \; x = m+1, \dots n.$$

and consequently

7.2)

$$g_{sb} = c_{cs}^{\cdot\cdot d} c_{db}^{\cdot\cdot c} = c_{cs}^{\cdot\cdot p} c_{pb}^{\cdot\cdot c} = c_{qs}^{\cdot\cdot p} c_{pb}^{\cdot\cdot q} = 0$$

and this proves that the rank of g_{ba} is $< n$. Hence the groupspaces of semi.simple groups have a Riemannian geometry.

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1) cf. Eidenhart 33.1, p. 173.

Rotatiegroep in R_3 . De kentallen $\overset{+}{\Gamma}{}^{\alpha}_{cb}$ en Γ^{α}_{cb} zijn tot nu toe niet berekend omdat de zeer bewerkelijke berekening hiervan enigszins verkort kan worden met behulp van $g_{b\alpha}$. Uit (5.4) en (2.63.1) volgt gemakkelijk voor g_{ba} dat

$$(7.2a) \qquad g_{11} = g_{22} = g_{33} = -2 \; ; \; g_{12} = g_{23} = g_{31} = 0$$

De rotatiegroep van R_3 is dus half-enkelvoudig. Daar de overbrenging niet verandert bij vermenigvuldiging van de fundamentaaltensor met een constante factor, nemen we liever in het volgende als uitgangspunt

$$(7.2b) \qquad g_{11} = g_{22} = g_{33} = 1 \; ; \; g_{12} = g_{23} = g_{31} = 0$$

Dit doet geen geweld aan aan (5.4), en geeft tevens aan ρ een eenvoudige meetkundige betekenis. In verband met (2.63e) volgt hieruit voor $g_{\beta\alpha}$:

$$(7.2c) \qquad g_{11} = g_{22} = g_{33} = 1 \; ; \; g_{12} = g_{31} = 0 \; ; \; g_{23} = -\cos\eta^1$$

Uit (3.63u) volgt nu voor $g_{b\alpha}$:

$$(7.2d) \qquad \begin{aligned} g_{\bar 1 \bar 1} &= 2\,\frac{1-\cos\rho}{\rho^2} + \frac{\eta^{\bar 1}\eta^{\bar 1}}{\rho^2}\left(1 - 2\,\frac{1-\cos\rho}{\rho^2}\right) \qquad (\text{cycl.}) \\ g_{\bar 1 \bar 2} &= \frac{\eta^{\bar 1}\eta^{\bar 2}}{\rho^2}\left(1 - 2\,\frac{1-\cos\rho}{\rho^2}\right) \qquad (\text{cycl.}) \end{aligned}$$

en voor $g^{\alpha b}$ uit (3.63v):

$$(7.2e) \qquad \begin{aligned} g^{\bar 1 \bar 1} &= \tfrac{1}{2}\,\frac{\rho^2}{1-\cos\rho} + \frac{\eta^{\bar 1}\eta^{\bar 1}}{\rho^2}\left(1 - \tfrac{1}{2}\,\frac{\rho^2}{1-\cos\rho}\right) \\ g^{\bar 1 \bar 2} &= \frac{\eta^{\bar 1}\eta^{\bar 2}}{\rho^2}\left(1 - \tfrac{1}{2}\,\frac{\rho^2}{1-\cos\rho}\right) \end{aligned}$$

In verband met de komende berekeningen het volgende:

1. We zullen, waar dat doelmatig is, η_α in plaats van η^α schrijven.
2. We zullen eveneens soms $\delta_{b\alpha}$ of $\delta^{\alpha b}$ zetten voor δ^α_b . Het verplaatsen van de indices hier is natuurlijk juist een overschuiving met $g_{b\alpha}$ in $\eta^\alpha = 0$. Deze hebben de waarden $\delta_{b\alpha}$. Alle optredende grootheden - inclusief η^α - transformeren zich bij overgang tot een ander stel normaalcoordinaten in $\eta^\alpha = 0$ als grootheden in $\eta^\alpha = 0$.
3. We voeren in de symmetrische matrix $P_b^{\cdot\alpha}$ (ook te schrijven als $P^{b\alpha}$ en $P_{b\alpha}$):

$$P_{b}^{\cdot a} = \eta_b \eta^a - \delta_b^a \rho^2$$

(7.2f)

$P_{b}^{\cdot a}$ $\;{}^{a}\diagdown{}_{b}$	$\bar 1$	$\bar 2$	$\bar 3$
$\bar 1$	$\eta^{\bar 1}\eta^{\bar 1}-\rho^2$	$\eta^{\bar 1}\eta^{\bar 2}$	$\eta^{\bar 1}\eta^{\bar 3}$
$\bar 2$	$\eta^{\bar 1}\eta^{\bar 2}$	$\eta^2\eta^2-\rho^2$	$\eta^2\eta^{\bar 3}$
$\bar 3$	$\eta^{\bar 1}\eta^{\bar 3}$	$\eta^{\bar 2}\eta^{3}$	$\eta^{\bar 3}\eta^{\bar 3}-\rho^2$

Deze heeft (vgl. b.v. (3.63e)) de volgende eigenschappen:

(a) $$\eta^b P_{b}^{\cdot a} = 0$$

(b) $$P_{b}^{\cdot a} P_{a}^{\cdot b} = -\rho^2 P_{c}^{\cdot a}$$

4) We schrijven ter afkorting:

(7.2g)
$$\alpha \stackrel{def}{=} \frac{1}{\rho^2}\left(1 - 2\,\frac{1-\cos\rho}{\rho^2}\right); \qquad \alpha_\rho \stackrel{def}{=} \frac{\partial \alpha}{\partial \rho}$$
$$\beta \stackrel{def}{=} \frac{1}{\rho^2}\left(1 - \tfrac{1}{2}\,\frac{\rho^2}{1-\cos\rho}\right)$$

5) We kunnen (7.2d,e) nu afkortend schrijven:

(7.2h)
$$g_{ba} = \delta_{ba} + \alpha P_{ba}$$
$$g^{ab} = \delta^{ab} + \beta P^{ab}$$

6) We merken op dat

(7.2i) $$\partial_c \rho = \rho^{-1}\eta_c$$

(7.2j) $$\partial_c P_{ba} = \eta_b \delta_{ca} + \eta_a \delta_{bc} - 2\eta_c \delta_{ba}$$

We berekenen nu Γ_{cb}^{a} uit:

(7.2k) $$\Gamma_{cb}^{a} = \tfrac{1}{2} g^{a\vartheta}\left(\partial_c g_{b\vartheta} + \partial_b g_{c\vartheta} - \partial_\vartheta g_{cb}\right)$$

$$\Gamma_{cb}^{a} = \tfrac{1}{2}\left(\delta^{a\vartheta} + \beta P^{a\vartheta}\right)\left\{\partial_c \alpha P_{b\vartheta} + \partial_b \alpha P_{c\vartheta} - \partial_\vartheta \alpha P_{cb}\right) =$$

(7.2.1)
$$= \tfrac{1}{2}\left(\delta^{a\vartheta} + \beta P^{a\vartheta}\right)\left\{\rho^{-1}\alpha_\rho\left(P_{b\vartheta}\eta_c + P_{c\vartheta}\eta_b - P_{cb}\eta_\vartheta\right) + \right.$$
$$\left. + \alpha\left(4\eta_\vartheta \delta_{cb} - 2\eta_c \delta_{b\vartheta} - 2\eta_b \delta_{c\vartheta}\right)\right\} =$$

$$= \tfrac{1}{2}\rho^{-1}\alpha_\rho\,(P_{\dot b}^{\ \alpha}\eta_c + P_{\dot c}^{\ \alpha}\eta_b - P_{cb}\,\eta^\alpha) + \tfrac{\alpha}{2}\,(4\eta^\alpha\delta_{cb} - 2\eta_c\delta_b^\alpha - 2\eta_b\delta_c^\alpha) +$$

(7.2.1)

$$+ \tfrac{1}{2}\alpha_\rho\beta\rho^{-1}(-\rho^2\eta_c P_{\dot b}^{\ \alpha} - \rho^2\eta_b P_{\dot c}^{\ \alpha}) + \tfrac{1}{2}\alpha\beta\,(-2\eta_c P_{\dot b}^{\ \alpha} - 2\eta_b P_{\dot c}^{\ \alpha}).$$

Nu vinden we gemakkelijk met enig rekenwerk:

$$\Gamma_{\bar 1\bar 1}^{\bar 1} = \tfrac{1}{2}\rho^{-1}\alpha_\rho P_{\bar 1}^{\cdot\bar 1}\eta^{\bar 1} - \beta\rho\alpha_\rho\eta^{\bar 1}P_{\bar 1}^{\cdot\bar 1} - 2\alpha\beta\eta^{\bar 1}P_{\bar 1}^{\cdot\bar 1} =$$

$$= \eta^{\bar 1}(2\alpha\beta\rho^2 - \tfrac{1}{2}\alpha_\rho\rho + \alpha_\rho\beta\rho^3) + (\eta^{\bar 1})^3(-2\alpha\beta + \tfrac{1}{2}\alpha_\rho\rho^{-1} - \alpha_\rho\beta\rho) =$$

$$= -\eta^{\bar 1}\rho^{-2}\left(1 + \frac{\sin\rho}{\rho} - \rho\cot g\,\rho/2\right) + (\eta^{\bar 1})^3\rho^{-4}\left(1 + \frac{\sin\rho}{\rho} - \rho\cot g\,\rho/2\right).$$

$$\Gamma_{\bar 1\bar 2}^{\bar 1} = \Gamma_{\bar 2\bar 1}^{\bar 1} = \eta^{\bar 2}(-\alpha + \alpha\beta\rho^2 - \tfrac{1}{2}\alpha_\rho\rho + \tfrac{1}{2}\alpha_\rho\beta\rho^2) + \eta^{\bar 1}\eta^{\bar 1}\eta^{\bar 2}(-2\alpha\beta + \tfrac{1}{2}\alpha_\rho\rho^{-1} - \alpha_\rho\beta\rho) =$$

(7.2m)

$$= -\eta^{\bar 2}\rho^{-2}(1 - \rho/2\,\cot g\,\rho/2) + \eta^{\bar 1}\eta^{\bar 1}\eta^{\bar 2}\rho^{-4}\left(1 + \frac{\sin\rho}{\rho} - \rho\cot g\,\rho/2\right)$$

$$\Gamma_{\bar 1\bar 1}^{\bar 2} = \eta^{\bar 2}(2\alpha + \tfrac{1}{2}\rho\alpha_\rho) + \eta^{\bar 1}\eta^{\bar 1}\eta^{\bar 2}(-2\alpha\beta + \tfrac{1}{2}\alpha_\rho\rho^{-1} - \alpha_\rho\beta\rho) =$$

$$= \eta^{\bar 2}\rho^{-2}\left(1 - \frac{\sin\rho}{\rho}\right) + \eta^{\bar 1}\eta^{\bar 1}\eta^{\bar 2}\rho^{-4}\left(1 + \frac{\sin\rho}{\rho} - \rho\cot g\,\rho/2\right).$$

$$\Gamma_{\bar 1\bar 2}^{\bar 3} = \eta^{\bar 1}\eta^{\bar 2}\eta^{\bar 3}(-2\alpha\beta + \tfrac{1}{2}\alpha_\rho\rho^{-1} - \alpha_\rho\beta\rho) =$$

$$= \eta^{\bar 1}\eta^{\bar 2}\eta^{\bar 3}\rho^{-4}\left(1 + \frac{\sin\rho}{\rho} - \rho\cot g\,\rho/2\right).$$

De andere Γ's worden uit deze gevonden door cyclische verwisseling der indices.

De berekening van de overige grootheden levert nu niet veel moeilijkheden meer. $c_{cb}^{\ \alpha}$ kan het best worden gevonden door uit te gaan van het feit dat $c_{cb\alpha}$ een trivector is. Nu is, als $i_{\gamma\beta\alpha}$ de eenheids-trivector is (vgl. (2.63.1) en (7.2b))

(7.2n) $$c_{cba} = -i_{cba} \quad ; \quad c_{cb\alpha} = -i_{cb\alpha} \quad ; \quad c_{123} = -1.$$

dus:

(7.2o) $$c_{\bar 1\bar 2\bar 3} = \Delta^{-1}c_{123} = -\Delta^{-1} \quad ; \quad \Delta = \det(A_b^{\alpha})$$

Daar $\Delta^{-1} > 0$ ($\Delta^{-1} = 1$ voor $\eta^\alpha = 0$, vgl. (3.63u)) vinden we:

(7.2p)

$$c_{\bar 1\bar 2\bar 3} = -\sqrt{g^{(\alpha)}/g^{(0)}} = -\left\{2^3\left(\frac{1-\cos\rho}{\rho^2}\right)^3 + \frac{\eta^{\bar 1}\eta^{\bar 1} + \eta^{\bar 2}\eta^{\bar 2} + \eta^{\bar 3}\eta^{\bar 3}}{\rho^2}\left(1 - 2\frac{1-\cos}{\rho^2}\right)2^2\left(\frac{1-\cos\rho}{\rho^2}\right.\right.$$

$$= -2\left(\frac{1-\cos}{\rho^2}\right).$$

dus:

(7.2q)

$$c_{\bar{1}\bar{2}}^{\cdot\cdot\;\bar{3}} = -2g^{\bar{3}\bar{3}}\left(\frac{1-\cos\rho}{\rho^2}\right) = -1 - \frac{\eta^{\bar{3}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right) \qquad (\text{cycl.})$$

$$c_{\bar{1}\bar{2}}^{\cdot\cdot\;\bar{2}} = -2g^{\bar{2}\bar{3}}\left(\frac{1-\cos\rho}{\rho^2}\right) = -\frac{\eta^{\bar{2}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right) \qquad (\text{cycl.})$$

We vinden dus voor $\overset{+}{\Gamma}{}_{cb}^{a} = \Gamma_{cb}^{a} - \frac{1}{2}c_{cb}^{\cdot\cdot a}$

$$\overset{+}{\Gamma}{}_{\bar{1}\bar{1}}^{\bar{1}} = -\eta^{\bar{1}}\rho^{-2}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right) + (\eta^{\bar{1}})^3\rho^{-4}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right)$$

$$\overset{+}{\Gamma}{}_{\bar{1}\bar{1}}^{\bar{2}} = \eta^{\bar{2}}\rho^{-2}\left(1-\frac{\sin\rho}{\rho}\right) + \eta^{\bar{1}}\eta^{\bar{1}}\eta^{\bar{2}}\rho^{-4}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right)$$

(7.2r)

$$\overset{+}{\Gamma}{}_{\bar{1}\bar{2}}^{\bar{1}} = -\eta^{\bar{2}}\rho^{-2}\left(1-\frac{\rho}{2}\,\mathrm{cotg}\,\frac{\rho}{2}\right) + \frac{1}{2}\frac{\eta^{\bar{1}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right) + \eta^{\bar{1}}\eta^{\bar{1}}\eta^{\bar{2}}\rho^{-4}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right)$$

$$\overset{+}{\Gamma}{}_{\bar{1}\bar{2}}^{\bar{2}} = -\eta^{\bar{1}}\rho^{-2}\left(1-\frac{\rho}{2}\,\mathrm{cotg}\,\frac{\rho}{2}\right) + \frac{1}{2}\frac{\eta^{\bar{2}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right) + \eta^{\bar{1}}\eta^{\bar{2}}\eta^{\bar{2}}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right)$$

$$\Gamma_{\bar{1}\bar{2}}^{\bar{3}} = \frac{1}{2} + \frac{1}{2}\frac{\eta^{\bar{3}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right) + \eta^{\bar{1}}\eta^{\bar{2}}\eta^{\bar{3}}\left(1+\frac{\sin\rho}{\rho}-\rho\,\mathrm{cotg}\,\frac{\rho}{2}\right)$$

De kromtegrootheid vinden we als volgt: c^{cba} is een trivector met

(7.2s)
$$c^{\bar{1}\bar{2}\bar{3}} = \Delta c^{123} = -\Delta = -\frac{1}{2}\frac{\rho^2}{1-\cos\rho}$$

dus (vgl. (3.17))

(7.2t)
$$R_{\vartheta c}^{\cdot\cdot\, ba} = -\frac{1}{4}\,c_{\vartheta c n}\,c^{nba} = -\frac{1}{4}\left(A_{\vartheta}^{b}A_{c}^{a} - A_{c}^{b}A_{\vartheta}^{a}\right)$$

waaruit volgt:

(7.2u)
$$R_{\vartheta c b}^{\cdot\cdot\cdot\, a} = -\frac{1}{4}\left(A_{c}^{a}g_{\vartheta b} - A_{\vartheta}^{a}g_{cb}\right)$$

dus

$$R_{\bar{1}\bar{1}b}^{\cdot\cdot\cdot\,\bar{3}} = 0$$

$$R_{\bar{1}\bar{2}\bar{1}}^{\cdot\cdot\cdot\,\bar{1}} = -R_{\bar{2}\bar{1}\bar{1}}^{\cdot\cdot\cdot\,\bar{1}} = \frac{1}{2}g_{\bar{1}\bar{2}} = \frac{1}{2}\frac{\eta^{\bar{1}}\eta^{\bar{2}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right)$$

$$R_{\bar{1}\bar{2}\bar{2}}^{\cdot\cdot\cdot\,\bar{1}} = -R_{\bar{2}\bar{1}\bar{2}}^{\cdot\cdot\cdot\,\bar{1}} = \frac{1}{2}g_{\bar{2}\bar{2}} = \frac{1-\cos\rho}{\rho^2} + \frac{1}{2}\frac{\eta^{\bar{2}}\eta^{\bar{2}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right)$$

$$R_{\bar{1}\bar{2}\bar{3}}^{\cdot\cdot\cdot\,\bar{1}} = -R_{\bar{2}\bar{1}\bar{3}}^{\cdot\cdot\cdot\,\bar{1}} = \frac{1}{2}g_{\bar{2}\bar{3}} = \frac{1}{2}\frac{\eta^{\bar{2}}\eta^{\bar{3}}}{\rho^2}\left(1-2\,\frac{1-\cos\rho}{\rho^2}\right)$$

De andere kentallen worden hieruit gevonden door cyclische verwisseling der indices.

Uit (7.2t) blijkt dat de groepruimte een <u>ruimte van constante kromming</u> is.

Wat betreft de comitanten van $c_{cb}^{\;\;a}$ kan het volgende worden opgemerkt: deze behoren tot het stelsel van de comitanten van de trivector c^{cba} en van g_{ba} en g^{ab} . Daar al deze grootheden invariant zijn bij rotaties, d.w.z. een richting kan in iedere andere richting worden overgevoerd, hebben de comitanten geen richtingsvoorkeur. Alle grootheden g_b , g_{ba} , g_{cba} ,... etc. zijn dus nul of hebben de maximale rang in alle indices. Er zijn dus geen bevoorrechte richtingen , dus ook geen invariante onderruimten door , dus geen invariante ondergroepen.

Dit laatste wordt ook als volgt duidelijk: De rotatiegroep van R_3 laat ook een rotatiegroep om $\eta^\alpha=0$ toe omdat dit een ruimte van constante kromming is. Dit blijkt doordat de $g_{b\alpha}$ in (7.2d) bij orthogonale transformaties niet veranderen. De groepruimte representeert dus ook zijn eigen rotatiegroep om $\eta^\alpha=0$. Uit de (1,1) correspondentie tussen de rotaties in de groepruimte en in R_3 , en uit het feit dat in R_3 alleen bollen om de oorsprong dus géén onderruimten dóór de oorsprong - invariant zijn (p. 40a) volgt dat er in de groepruimte geen invariante ondergroepen zijn. De rotatiegroep van R_3 is dus enkelvoudig.

In de algemene theorie, en ook bij dit voorbeeld, hebben we ons beperkt tot de kiem van de groep, Op normaalcoordinaten blijken nu echter alle voorkomende reeksen analytisch voortzetbare functies te zijn, en dit is een reden, te kijken hoe het gaat met grote rotaties. Een rotatie over 2π radialen komt overeen met de identieke transformatie; is hiervan iets terug te vinden in de groepruimte? Daartoe de volgende beschouwing. Men merke op:

1. ρ is de booglengte langs de geodetische lijnen door $\eta^\alpha=0$.
2. een rotatie over een hoek ρ correspondeert met een vector van lengte ρ in de groepruimte (p.40d).
3. Het volume van een bol met straal R in de groepruimte is:

$$(7.2w) \qquad \tau_3(R)=\int\limits_{\rho^2<R^2}\sqrt{g}\,d\eta^1 d\eta^2 d\eta^3=\int_0^R 4\pi\rho^2\cdot 2\left(\frac{1-\cos\rho}{\rho^2}\right)d\rho=8\pi\int_0^R(1-\cos\rho)\,d\rho$$

4. De oppervlakte van een bol met straal R is:

$$(7.2x) \qquad \tau_2(R)=\frac{d\tau_3(R)}{dR}=8\pi(1-\cos\rho)$$

De oppervlakte van een bol met straal 2π is dus nul, d.w.z. alle rotaties over een hoek 2π hebben, onafhankelijk van de rotatie-as, hetzelfde beeldpunt. Identificeren we dus het punt $\eta^\alpha=0$, dus $\rho=0$, met $\rho=2\pi$, zo dat tevens de corresponderende maatvectoren $\underset{a}{e}^\alpha$ en ook $\underset{A}{e}^\alpha$ van de localruimten geidentificeerd zijn, dan hebben we daarmee een volkomen (1,1) correspondentie tussen de rotaties in R_3 en de punten van de aldus geconstrueerde ruimte verkregen.

Het gelijktijdig tot dekking brengen van $\underset{a}{e}^{\alpha}$ en $\underset{A}{e}^{\alpha}$ is mogelijk doordat zowel voor $\rho = 0$ als $\rho = 2\pi$ geldt $A^{a}_{\lambda} = \delta^{a}_{A}$ (vgl.(3.63s)).

Men vergelijke dit proces met het afbeelden van de rotaties in op een rechte lijn, en daarna het "oprollen" van die lijn om een cirkel met omtrek .

Enige andere groepen waarbij de berekening van de verschillende grootheden niet te ingewikkeld wordt, zijn de volgende:

a) De bewegingen in R_2 :

$$'x = \cos \eta^1 - y \sin \eta^1 + \eta^2$$

$$'y = \sin \eta^1 + y \cos \eta^1 + \eta^3$$

b) De projectieve transformaties op de rechte lijn:

$$'x = \frac{x + \eta^1}{\eta^2 x + \eta^3}.$$

B. Meetkunde der eindige continue transformatiegroepen.

§ 1 N. Groepen. Onder een groep wordt een verzameling van elementen $A, B, C, \dots$ verstaan waarvoor een "vermenigvuldiging" is gedefinieerd zodanig dat

1. het product van twee elementen steeds weer tot de verzameling behoort;
2. er een element ,de eenheid, bestaat zodat

1 N. 1) $$\mathfrak{I} A = A \mathfrak{I} = A$$

voor iedere A;

3. er tot elk element A een invers element $\overset{-1}{A}$ bestaat;

1 N. 2) $$A \overset{-1}{A} = \overset{-1}{A} A = \mathfrak{I}$$

4. de associatieve wet geldt

1 N. 3) $$(AB)C = A(BC)$$

Zijn de elementen transformaties dan is $\mathfrak{I}$ de identieke transformatie en (4) is vanzelf vervuld.

Enige definities:

a) Abelsche groep: $AB = BA$;

b) Ondergroep $\mathfrak{U}$ van groep $\mathfrak{G}$ is een groep $\mathfrak{U}$ waarvan alle elementen elementen van $\mathfrak{G}$ zijn, en waar dezelfde "product" definitie geldt;

c) Homoloog heten twee elementen A en A' indien er een element B bestaat zodat $A' = B A \overset{-1}{B}$;

d) Twee ondergroepen heten homoloog wanneer zij door $B \dots \overset{-1}{B}$ en $\overset{-1}{B} \dots B$ in elkaar overgevoerd worden. Symbool $\mathfrak{U} \rightarrow \mathfrak{U}' = B \mathfrak{U} \overset{-1}{B}$;

e) Een ondergroep $\mathfrak{U}$ is invariant in $\mathfrak{G}$ indien $\mathfrak{U} = A \mathfrak{U} \overset{-1}{A}$ voor iedere A ; (een andere naam is normaaldeler)

f) $\mathfrak{G}$ en $\mathfrak{G}'$ heten isomorph wanneer er een correspondentie tussen $A, B, C, \dots$ en $A', B', C', \dots$ bestaat, zodat als A' bij A en B' bij B behoort, steeds $A'B'$ bij AB behoort. De isomorphie heet holoëdrisch als de correspondentie één-éénduidig is en anders meroëdrisch.
(Dikwijls worden in deze betekenissen ook de termen isomorphie resp. homomorphie gebruikt.)

Enige eigenschappen:

1) Ondergroep van ondergroep van $\mathfrak{G}$ is ondergroep van $\mathfrak{G}$;
2) Doorsnede van twee ondergroepen van $\mathfrak{G}$ is ondergroep van $\mathfrak{G}$;
3) Doorsnede van twee in $\mathfrak{G}$ invariante ondergroepen is in $\mathfrak{G}$ invariante ondergroep;
4) Invariante ondergroep van invariante ondergroep van $\mathfrak{G}$ is in het algemeen geen invariante ondergroep van $\mathfrak{G}$.

Voorbeelden:

eindig

1. Permutaties van 2 dingen (2 elementen)
2. Permutaties van 3 dingen (6 elementen), holoëdrisch isomorph met de groep der 6 transformaties van 1 variabele

$$'x = x \; ; \; 'x = \frac{1}{1+x} \; ; \; 'x = \frac{x-1}{x} \; ; \; 'x = \frac{1}{x} \; ; \; 'x = 1-x \; ; \; 'x = \frac{x}{x-1}$$

eindig continu

3. Draaiïngen om een punt in een vlak. ∞^1 elementen afhankelijk van 1 parameter;
4. Bewegingen in een vlak, 3 parameters;
5. Draaiïngen om een punt in de ruimte. 3 parameters;
6. Bewegingen in de ruimte, 6 parameters;
7. Projectieve transformaties

in lijn	3 parameters
in vlak	8 parameters
in ruimte	15 parameters

In deze eindige continue groepen laat zich iedere transformatie vanuit de identieke transformatie bereiken door continue verandering der parameters. In de gemengde continue groepen kan dit niet, voorbeeld:

gemengd continu

8. Draaiïngen om een punt in een vlak en spiegelingen aan een lijn door dat punt, 2 definitievergelijkingen elk met 1 parameter:

$$'x = x\cos\alpha + y\sin\alpha$$

$$'y = \mp x\sin\alpha \pm y\cos\alpha$$

De transformaties van een oneindige transformatiegroep laten zich niet met behulp van een eindig aantal parameter vastleggen. Voorb.:

oneindig

9. $$\left.\begin{aligned} 'x &= f(x,y) \\ 'y &= \varphi(x,y) \end{aligned}\right] ; \quad \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial\varphi/\partial x & \partial\varphi/\partial y \end{vmatrix} \neq 0$$