COLLOQUIUM "DIFFERENTIAL GEOMETRY IN THE LARGE",

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A survey on differential geometries in the large.

In differential geometry in the large, we treat the following problem:

What are the relations between the local properties and the global properties of a space?

There are two main problems in differential geometry in the large, that is:

(1) Problem of prolongation:

We are given a small portion of the space. Prolong it in such a way that we get the whole space.

(2) Problem of metrization:

We are given a topological space, find a Riemannian metric in such a way that the whole space is metrized without singularities.

In a certain extent, these two problems are converse to each other. We mention here an example of the problem which belongs to the problem of metrization:

What are topological spaces which we can metrize by a metric whose sectional curvature has, at all points and for any 2-direction, the same sign ? In this statement, by a sectional curvature at a point P and for a 2-direction π we mean the following:

In a Riemannian space, we take a point P and a 2-direction π at this point and we consider all the geodesics passing through P and being tangent to π. Such geodesics constitute a 2-dimensional surface V_2 passing through P and being tangent to π. The Gaussian curvature of V_2 at P is called the sectional curvature of the space at the point P and for the 2-direction π.

If we denote by $K_{\nu\mu\lambda\kappa}$ the covariant components of the curvature tensor of the space, the sectional curvature k is given by

$$k = -K_{\nu\mu\lambda\kappa} u^\nu v^\mu u^\lambda v^\kappa$$

where u^κ and v^κ are two unit vectors orthogonal to each other defining the 2-direction.

In the above mentioned example, if the sectional curvature is constant, we get the famous Clifford-Klein's space problem. Clifford-Klein's space problems can be considered also in affinely, projectively or conformally flat spaces. To illustrate the problems in differential geometry in the large we wish to mention first of all:

Some results in classical differential geometry in the large.

It seems to us that almost all problems in more modern differential geometry in the large have their origins in these classical results. So we shall give here a rather extensive collection of the classical results.

W.Blaschke (1924), For any two points on an ovaloid, there exists always a shortest line which joins these two points and the shortest line is a geodesic. Here we mean by an ovaloid a closed convex regular analytic surface which has positive Gaussian curvature.

W.Blaschke (1912), H.Weyl (1917): If we can deform an ovaloid continuously and isometrically, then it is a motion of the ovaloid as a rigid body.

O.Bonnet (1855), When the Gauss curvature k at every point of an ovaloid satisfies $k \geq \frac{1}{a^2}$, the geodesic distance between two consecutive conjugate points is $\leq \pi a$. Here by conjugate points, we mean the following: Consider a geodesic on a point A on it. Suppose that a point P starts from A and goes in one direction of the geodesic. A point B on the geodesic is called a conjugate point on the geodesic if, for the point P which lies between geodesic arc AB , the geodesic AP gives the shortest distance between A and P for the sufficiently small variations of the geodesic arc (relative minimum), but for the point P which lies on B or on the prolongation of AB in this direction, the geodesic arc AP does not give the relative minimum.

C.Carathéodory (1912), The locus of the conjugate points of a fixed point on an ovaloid has at least 4 cusps.

T.Carleman (1921), We are given a closed space curve C of length L and we assume that, passing through the curve C , there exists a surface S of the minimum area. Then S is a minimal surface. We denote by O its area bounded by C . Then we have

$$L^2 - 4\pi O \geq 0$$

and the equality occurs if and only if the curve C is a circle. When the curve C reduces to a plane curve we get the answer to the classical isoperimetry problem.

Gauss-Bonnet (1848),

$$\frac{1}{2\pi}\int_{V_2} K do = \chi = 2(1-p)$$

where K is the Gaussian curvature of the surface V_2 ,   the volume element, χ Euler-Poincaré characteristic and p the genus of the surface.

D.Hilbert (1900), There does not exist an everywhere regular analytic surface without frontier which has negative constant Gauss curvature.
H.Liebmann (1899), Any regular closed surface with constant Gauss curvature is a sphere.(Consequently a sphere is rigid). Any ovaloid with constant mean curvature is a sphere.
H.Poincaré (1905), Every ovaloid has at least three closed geodesics on it.
J.Steiner- W.Gross (1836, 1917), Between the area O and the volume V of a ovaloid, there exists the relation

$$O^3 - 36\pi V^2 \geqq 0$$

and the equality occurs if and only if the ovaloid is a sphere.

We now come to the more modern differential geometries in the large and we will classify these geometries according to the topics treated in each branch as follows:

(1) Complete spaces.
(2) Behaviour of geodesics.
(3) Spaces of negative sectional curvature.
(4) Spaces of positive sectional curvature.
(5) Harmonic integrals.
(6) Curvature and Betti numbers.
(7) Gauss-Bonnet formula.
(8) Complex manifolds.
(9) Almost complex manifolds.
(10) Holonomy groups.
(11) Groups of transformations.
(12) Imbedding problems.
(13) Fibred spaces.

We will mention some typical results obtained in each of these branches of differential geometry in the large.

(1) Complete spaces.

A Riemannian space is said to be complete if it satisfies one of the four following postulates.

(i) Postulate of measuring. We can measure on each geodesic any length starting from a fixed point on the geodesic.

(ii) Postulate of infinity. We call a divergent line a one-to-one continuous image of a half-line in which, to any divergent series of points on a half-line, there corresponds a divergent series of points on the line in the space. Then the postulate of infinity can be stated as follows: Any divergent line of the space has infinitely long length .

(iii) Postulate of Cauchy. A Riemannian space is a metric space. In fact, we can define the distance (AB) between two points A and B as the infimum of the length of curves which join these two points. Thus we can define the fundamental sequence of Cauchy as follows: If a sequence of points $P_1, P_2, \ldots$ satisfies the condition

$$(P_m P_n) \to 0 \quad (m, n \to \infty)$$

we call this sequence fundamental sequence of Cauchy. Then the postulate of Cauchy can be stated as follows: All the fundamental sequences of Cauchy converge.

(iv) Postulate of compactness. Any bounded set of infinitely many points of the space admits at least a point of accumulation. We usually express this fact by saying that the space is locally compact.

E.Cartan adopted the fourth postulate and called a normal space a space which satisfies this postulate.

H.Hopf and W.Rinow proved the equivalence of these four postulates. Moreover they got

Theorem. We can join always two arbitrary points of a complete Riemannian space by an arc of minimal geodesic. Recently G.de Rham gave very simple proofs of the equivalence of these four postulates and of the above mentioned theorem of Hopf and Rinow.

(2) Behaviour of Geodesics.

A generalization of Bonnet's theorem to a complete Riemannian space was obtained by Synge (1935), Schoenberg (1932) and Myers (1935).

Theorem. A complete Riemannian space whose sectional curvature for any point and for any 2-direction is greater than a positive number k is a closed space with diameter less than $\pi/\sqrt{k}$. We can reduce the proof of this theorem to that of corresponding theorem in a two-dimensional space by the following lemma due to J.L.Synge.

Synge's lemma. The Gaussian curvature of a portion of a surface V_2 which passes through a geodesic of an n-dimensional Riemannian space V_n depends along the geodesic only on the ruban of planes tangent to V_2 along the geodesic. This curvature is less than or equal to the sectional curvature of V_n for the tangent plane of V_2. The equality occurs only when the ruban of tangent planes is obtained by the parallel displacement of a vector along the geodesic, the tangent plane being determined by the tangent to the geodesic and parallelly displaced direction. In the two-dimensional case, we know the following theorem:

<u>Theorem</u>. The arc of geodesic joining two points A and B such that any solution of Jacobi's equation

$$y'' + Ky = 0 \qquad \begin{pmatrix} K : \text{Gaussian curvature,} \\ y' = dy/ds \quad s : \text{arc length} \end{pmatrix}$$

has at most one zero-point in the interval AB is an arc of geodesic which realize the relative minimum of the distance, that is, which is the shortest among the curves joining A and B and lying in a sufficiently small neighbourhood of the geodesic.

On the other hand, we have the following comparison theorem of J.C.F. Sturm (1836):

<u>Theorem</u> Suppose that we have two differential equations

$$y'' + K(s)y = 0, \qquad (1)$$
$$z'' + L(s)z = 0 \qquad (2)$$

in which $K(s) \geq L(s)$, then the distance between two consecutive zero-points of (1) is shorter than that between two consecutive zero-points of (2). But, if $L(s) = k = \text{constant} > 0$, then we have, assuming $z = 0$ for $s = 0$

$$z = A \cdot \sin \sqrt{k} \cdot s$$

and the distance between two consecutive zero points is $\frac{\pi}{\sqrt{k}}$. Thus combining above two theorems, we get the classical theorem of Bonnet.

Now take a complete Riemannian space whose sectional curvature for any point and for any 2-direction is greater than a positive number k and fix a point O in the space. By a theorem of Hopf and Rinow, we can join any point P of the space to point O by a shortest geodesic. Take a direction at O which is not tangent to the geodesic and displace it parallelly along the geodesic. We consider then a 2-dimensional subspace which passes through the geodesic and which (is) tangent to the vector field parallel along the geodesic. Then by Synge's lemma, the sectional curvature of the space for a point on the geodesic and for a 2-direction tangent to the subspace is equal to the Gaussian curvature of the subspace along the geodesic. Since the sectional curvature of the space is supposed to be always greater than k, and consequently this geodesic which is also a geodesic of the subspace has a point conjugate to O in the interval $\pi/\sqrt{k}$ But $\overset{\frown}{OP}$ is the shortest geodesic, and consequently we have $\overset{\frown}{OP} < \pi/k$. This proves the generalization of Bonnet's theorem.

(3) Spaces of negative sectional curvature.
If the space has sectional curvatures always negative for any point and for any 2-direction, then, by calculating the second variation of the length of a geodesic, we can prove that any geodesic arc realizes the relative minimum of the length.

On the other hand, E.Cartan (1928) and S.B.Myers (1935) proved that in a simply connected space if there exists a point P such that it has no conjugate point on any geodesic passing through it, that is, any geodesic passing through P realizes a relative minimum, then the space is homeomorphic to the Euclidean space.

Thus we can state:

Theorem A simply connected space with negative sectional curvature is homeomorphic to the Euclideon space. We will quote here some of the interesting theorems on the spaces of negative sectional curvature obtained by Preissmann (1942).

Theorem Any geodesic which does not pass through a point P of a space of negative sectional curvature has a point which is the nearest to P . From this point, two branches of the geodesic move away from P monotonously to the infinity.

Theorem The sum of the angles of a geodesic triangle in a simply connected space of negative constant curvature is less than two right angles.

Theorem In a space of negative sectional curvature, there exists at least one closed geodesic of any type of homotopy.
As a consequence of the first theorem, we remark here

Theorem All the elements of the fundamental group of a space of negative sectional curvature are of infinite order.

(4) Spaces of positive sectional curvature.
We must mention first of all Bonnet's theorem stated at the beginning of (2).

Theorem A complete Riemannian space whose sectional curvature for any point and for any 2-direction is greater than a positive number k is a closed space with diameter less than $\pi/\sqrt{k}$. From which we have.

Theorem The fundamental group of a closed space of positive sectional curvature is finite. Before stating the following theorem, we define the orientability of the space. If a manifold of class C^n can be covered by a system of coordinate neighbourhoods in such a way that if any two coordinate neighbourhoods (ξ^{α}) and ($\xi^{\alpha'}$) overlap then in the overlapping domain the coordinate transformation

$$\xi^{x'} = f^{x'}(\xi^1, \ldots, \xi^n)$$

has always the same sign . Now Synge proved the following theorem.

<u>Theorem</u> An orientable closed Riemannian space of positive sectional curvature and of even dimension is simply connected. We now define the extremity of a space. We say that a sequence of points

$$P_1, P_2, P_3, \ldots\ldots \tag{1}$$

diverges to an extremity if, for any partial sequences

$$\begin{gathered} Q_1, Q_2, Q_3, \ldots \\ R_1, R_2, R_3, \ldots \end{gathered}$$

taken from (1) and for any compact domain D, it is always possible to find the curves joining

$$Q_1 \text{ to } R_1,\ Q_2 \text{ to } R_2,\ Q_3 \text{ to } R_3, \ldots\ldots$$

which have exeept a finite number of exceptions, no common points with D . Now we can state the following theorem of Preissmann (1942).

<u>Theorem</u> An open space of positive sectional curvature cannot have more than one extremity. We call a space with pole a space which contains a point P (pole) such that all the geodesic rays issueinf from P are minimum geodesic rays.

<u>Theorem</u> The sets of poles in a space with pole of positive sectional curvature ie bounded.

<u>(5) Harmonic Integrals.</u>

An n-dimensional manifold of class C^2 can be covered by a system of coordinte neighborhoods U_α. If, from any covering by the coordinate neighborhoods of the manifold, we can choose a finite number of coordinate neighborhoods $U_{i_1}, U_{i_2}, \ldots U_{i_N}$ whose union covers entirely the manifold, we say that the manifold is compact.

Now an exterior differential form of degree p

$$\frac{1}{p!}\varphi_{\lambda_1\lambda_2\cdots\lambda_p}\, d\xi^{\lambda_1}\wedge d\xi^{\lambda_2}\wedge\cdots\wedge d\xi^{\lambda_p}$$

is called harmonic form if the rotation and the divergence of the antisymmetric tensor $\varphi_{\lambda_1\lambda_2\cdots\lambda_3}$ vanish, that is, if

$$\begin{gathered} \nabla_{[\lambda}\varphi_{\lambda_1\lambda_2\cdots\lambda_p]} = 0, \\ g^{\mu\lambda}\nabla_\mu\varphi_{\lambda\lambda_2\cdots\lambda_p} = 0 \end{gathered}$$

where ∇ denotes the covariant derivative with respect to the Christoffel symbols.

Now we can state the famous theorem of W.V.D. Hodge.

Theorem. In an orientable compact Riemannian space of class C_r where r is a sufficiently large integer, the harmonic forms of degree p form a vector space whose dimension is equal to the pth Betti number of the space.

Take an exterior differential form

$$\varphi = \frac{1}{p!}\varphi_{\lambda_1\lambda_2\cdots\lambda_p}\, d\xi^{\lambda_1}\wedge d\xi^{\lambda_2}\wedge\cdots\wedge d\xi^{\lambda_p}.$$

We define the exterior differential $d\varphi$ of φ by

$$d\varphi = \frac{1}{p!}\nabla_{[\lambda}\varphi_{\lambda_1\lambda_2\cdots\lambda_p]}\, d\xi^{\lambda}\wedge d\xi^{\lambda_1}\wedge\cdots\wedge d\xi^{\lambda_p}$$

Now denote by $\varphi^{\lambda_1\lambda_2\cdots\lambda_p}$ the centravariant components of $\varphi_{\lambda_1\lambda_2\cdots\lambda_i}$ then we define the $*$-operation on φ by

$$*\varphi = i_{\lambda_1\lambda_2\cdots\lambda_i\lambda_{i+1}\cdots\lambda_n}\varphi^{\lambda_1\lambda_2\cdots\lambda_p}\, d\xi^{\lambda_{i+1}}\wedge\cdots\wedge d\xi^{\lambda_n}$$

and call $*\varphi$ the adjoint form of φ, where $i_{\lambda_1\cdots\lambda_n}$ is the unit n-vector of the Riemannian space.

The global inner product of two forms φ and ψ of degree p is defined as

$$(\varphi\,\psi) = \int_{V_n}\varphi\wedge(*\psi).$$

We define the exterior codifferential $\delta\varphi$ of φ by

$$\delta\varphi = (-1)^{np+n+1} * d * \varphi.$$

Now it is easily proved, by an integration by part,

$$(d\varphi,\psi) = (\varphi,\delta\psi)$$

If an exterior differential form φ is harmonic, we have $d\varphi = 0$, $\delta\varphi = 0$, and consequently

$$\Delta\varphi \equiv d\delta\varphi + \delta d\varphi = 0.$$

Conversely, if we have $\Delta\varphi \equiv d\delta\varphi + \delta d\varphi = 0$, then

$$0 = (\Delta\varphi,\varphi) = (d\delta\varphi,\varphi) + (\delta d\varphi,\varphi).$$

But, on the other hand, we have

$$(d\delta\varphi,\varphi) = (\delta\varphi,\delta\varphi),$$

$$(\delta d\varphi,\varphi) = (d\varphi, d\varphi),$$

and consequently

$$(\Delta\varphi, \varphi) = (d\varphi, d\varphi) + (\delta\varphi, \delta\varphi) = 0,$$

from which, since $(d\varphi, d\varphi) \geqq 0$, $(\delta\varphi, \delta\varphi) \geqq 0$,

$$(d\varphi, d\varphi) = 0, \quad (\delta\varphi, \delta\varphi) = 0$$

which shows that

$$d\varphi = 0, \quad \delta\varphi = 0.$$

Thus

Theorem. A necessary and sufficient condition that a form φ is harmonic, is $\Delta\varphi = 0$.

The expression $\Delta\varphi$ certainly contains the curvature tensor of the Riemannian space, but the curvature tensor of the space does not appear explicitly in the works of G. de Rham and W.V.D. Hodge.

We shall give here, as an example, the expression $\Delta\varphi$ for a covariant vector φ_λ.

First $d\varphi$ is given by

$$\nabla_\mu \varphi_\lambda - \nabla_\lambda \varphi_\mu$$

and consequently $\delta d\varphi$ is given by

$$g^{\nu\mu} \nabla_\nu (\nabla_\mu \varphi_\lambda - \nabla_\lambda \varphi_\mu)$$

Next $\delta\varphi$ is given by

$$g^{\nu\mu} \nabla_\nu \varphi_\mu$$

and consequently $d\delta\varphi$ is given by

$$\nabla_\lambda (g^{\nu\mu} \nabla_\nu \varphi_\mu).$$

Thus, for $d\delta\varphi + \delta d\varphi$, we have

$$g^{\nu\mu}\left[\nabla_\nu \nabla_\mu \varphi_\lambda - \nabla_\nu \nabla_\lambda \varphi_\mu + \nabla_\lambda \nabla_\nu \varphi_\mu\right]$$
$$= g^{\nu\mu}\left[\nabla_\nu \nabla_\mu \varphi_\lambda + K_{\nu\lambda\mu}^{\cdots\varkappa} \varphi_\varkappa\right]$$
$$= g^{\nu\mu} \nabla_\nu \nabla_\mu \varphi_\lambda - K_\lambda^{\cdot\varkappa} \varphi_\varkappa,$$

where $K_\lambda^{\cdot\varkappa}$ is the Ricci tensor of the space. Thus we can state

Theorem. A necessary and sufficient condition that a covariant vector φ_λ be harmonic, is that we have

$$g^{\nu\mu} \nabla_\nu \nabla_\mu \varphi_\lambda - K_\lambda^{\cdot\varkappa} \varphi_\varkappa = 0.$$

For a general antisymmetric tensor, we have

Theorem. A necessary and sufficient condition that a covariant antisymmetric tensor $\varphi_{\lambda_1, \lambda_2 \cdots \lambda_p}$ be harmonic is that

$$g^{\nu\mu} \nabla_\nu \nabla_\mu \varphi_{\lambda_1 \lambda_2 \cdots \lambda_p} - \sum_{s=1}^{p} K_{\lambda_s}^{\cdot\, \varkappa} \varphi_{\lambda_1 \cdots \lambda_{s-1} \varkappa \lambda_{s+1} \cdots \lambda_p}$$
$$+ \sum_{t>s}^{1 \cdots p} K_{\lambda_t \lambda_s}^{\cdot\cdot\, \sigma\rho} \varphi_{\lambda_1 \cdots \lambda_{t-1} \sigma \lambda_{t+1} \cdots \lambda_{s-1} \rho \lambda_{s+1} \cdots \lambda_p} = 0$$

The applications of the theory of harmonic integrals to complex spaces will be stated later.

A p-vector $w_{\lambda_p \cdots}$ is called harmonic if $\mathrm{Rot}\, w$ and $\mathrm{Div}\, w$ vanish:

$$\mathrm{Rot}\, w \; : \; (p+1)\, \nabla_{[\mu}\, w_{\lambda_p \cdots]} \qquad ; \quad \lambda_p \cdots \overset{def}{=} \lambda_p \cdots \lambda_1$$

$$\mathrm{Div}\, w \; : \; \nabla_\mu\, w^{\mu \kappa_{p-1} \cdots}$$

$v.w \overset{def}{=} v_{\lambda_p \cdots} w^{\lambda_p \cdots}$; $u.u \geq 0$ at every point if $g_{\lambda\kappa}$ is positive definite.

According to Stokes the volumeintegral over a compact space of every divergence $\nabla_\mu v^\mu$ is zero:

$$\int_{\tau_n} \mathrm{Div}\, v\, d\tau_n = \int_{\tau_n} \nabla_\mu v^\mu d\tau_n = \int_{\tau_{n-1}} n_\mu v^\mu d\tau_{n-1} = 0 \; ; \quad \tau_n = |\sqrt{g}|\, d\xi^1 \ldots d\xi^n$$

because $\tau_{n-1} = 0$. Hence for two multivectors $u_{\lambda_p \cdots}, v_{\lambda_{p+1} \cdots}$ we have

$$0 = \int_{\tau_n} \nabla_\mu u_{\lambda_p \cdots} v^{\mu\lambda_p \cdots} = \int_{\tau_n} \nabla_{[\mu} u_{\lambda_p \cdots]} v^{\mu\lambda_p \cdots} d\tau_n + \int_{\tau_n} u_{\lambda_p \cdots} \nabla_\mu v^{\mu\lambda_p \cdots} d\tau_n =$$

$$= \frac{1}{p+1} \int_{\tau_n} v.\mathrm{Rot}\, u + \int_{\tau_n} u.\mathrm{Div}\, v .$$

Now take a field $\varphi_{\lambda_p \cdots}$ for which

$$\Delta \varphi \overset{def}{=} \mathrm{Rot}\,\mathrm{Div}\,\varphi + \mathrm{Div}\,\mathrm{Rot}\,\varphi = 0 .$$

Then taking $\quad u = \mathrm{Div}\,\varphi \; , \; v = \varphi$:

$$\int v\, \mathrm{Rot}\,\mathrm{Div}\,\varphi = -(p+1) \int \mathrm{Div}\,\varphi . \mathrm{Div}\,\varphi\, d\tau \leq 0$$

and taking $\quad u = \varphi \; , \; v = \mathrm{Rot}\,\varphi$:

$$\int v\, \mathrm{Div}\,\mathrm{Rot}\,\varphi = -\frac{1}{p+1} \int \mathrm{Rot}\,\varphi . \mathrm{Rot}\,\varphi\, d\tau \leq 0$$

Hence in a compact space

$$\mathrm{Rot}\,\varphi = 0 \quad ; \quad \mathrm{Div}\,\varphi = 0 .$$

(6) Curvature and Betti numbers.

On applying Hodge's theorem, we can obtain some relations between the curvature and the Betti numbers of the space.

First of all, we shall quote here the famous theorem of Green:

Theorem: For an arbitrary continuous contravariant vector field φ^κ in a compact orientable riemannian space V_n, we have

$$\int_{V_n} \nabla_\kappa . \varphi^\kappa \, dv = 0,$$

where dv denotes the volume element of the space. As a corollary to this theorem, we have

Theorem: For an arbitrary scalar field φ in a compact orientable riemannian space V_n, we have

$$\int_{V_n} g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi \, dv = 0.$$

Now applying this theorem to φ^2, we obtain

$$2\int_{V_n} [\varphi g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi + g^{\lambda\kappa} (\nabla_\lambda \varphi)(\nabla_\kappa \varphi)] \, dv = 0.$$

Thus if $g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi \geqq 0$ everywhere in V_n, by the above theorem we must have $g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi = 0$, and consequently from the above formula, we get

$$\int_{V_n} g^{\lambda\kappa} (\nabla_\lambda \varphi)(\nabla_\kappa \varphi) \, dv = 0$$

But $g^{\lambda\kappa}(\nabla_\lambda \varphi)(\nabla_\kappa \varphi) \geqq 0$ for a positive definite riemannian metric, and consequently we must have $g^{\lambda\kappa}(\nabla_\lambda \varphi)(\nabla_\kappa \varphi) = 0$, from which $\nabla_\lambda \varphi = 0$, that is, $\varphi =$ const. Thus

Theorem: If a scalarfield φ in a compact orientable riemannian space V_n satisfies $g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi \geqq 0$, then φ is a constant, and $g^{\lambda\kappa} \nabla_\lambda \nabla_\kappa \varphi = 0$

Now suppose that φ_λ is a harmonic vector, then we have

$$g^{\nu\mu} \nabla_\nu \nabla_\mu \varphi_\lambda - K_\lambda^{\cdot\kappa} \varphi_\kappa = 0$$

We now calculate $g^{\nu\mu} \nabla_\nu \nabla_\mu (\varphi^\lambda \varphi_\lambda)$, then we have

$$\begin{aligned} g^{\nu\mu} \nabla_\nu \nabla_\mu (\varphi^\lambda \varphi_\lambda) &= 2\varphi^\lambda g^{\nu\mu} \nabla_\nu \nabla_\mu \varphi_\lambda + 2g^{\nu\mu} (\nabla_\nu \varphi^\lambda)(\nabla_\mu \varphi_\lambda) \\ &= 2[K_{\lambda\kappa} \varphi^\lambda \varphi^\kappa + g^{\nu\mu} (\nabla_\nu \varphi^\lambda)(\nabla_\mu \varphi_\lambda)]. \end{aligned}$$

Thus if the Ricci curvature $K_{\lambda\kappa} \varphi^\lambda \varphi^\kappa$ is positive definite, then we have $g^{\nu\mu} \nabla_\nu \nabla_\mu (\varphi^\lambda \varphi_\lambda) \geqq 0$ and consequently

$$K_{\lambda\kappa} \varphi^\lambda \varphi^\kappa = 0$$

which gives $\varphi^\lambda = 0$. Thus

Theorem: If the Ricci curvature of a compact orientable riemannian space is positive definite, the first Betti number of the space is zero. (S.B.Myers 1941, S.Bochner 1946).

In the same way we can prove

Theorem: If the quadratic form

$$K_{\lambda\kappa}\varphi^{\lambda\lambda_2\cdots\lambda_p}\varphi^{\kappa}_{\cdot\lambda_2\cdots\lambda_p}+\frac{p-1}{2}K_{\nu\mu\lambda\kappa}\varphi^{\nu\mu\lambda_3\cdots\lambda_p}\varphi^{\lambda\kappa}_{\cdot\cdot\lambda_3\cdots\lambda_p}$$

in a compact riemannian space is positive definite, then the Betti numbers B_p for $p=1,2,\dots n-1$ vanish.

As an application of this theorem, we can prove

Theorem: If the curvature tensor of a compact orientable riemannian space V_n satisfies

$$0<\frac{1}{2}k\leqq-\frac{K_{\nu\mu\lambda\kappa}\varphi^{\nu\mu}\varphi^{\lambda\kappa}}{\varphi^{\lambda\kappa}\varphi_{\lambda\kappa}}\leqq k \quad . \qquad (1)$$

for any anti-symmetric tensor $\varphi^{\lambda\kappa}$ where k is a positive constant, then the Betti numbers B_p for $p=1,2,\dots n-1$ vanish.

It is not known if this theorem is true or not when we assume tha. (1) is true only for any anti-symmetric tensor of the form $u^\lambda v^\kappa - u^\kappa v^\lambda$.

It may be interesting to compare this theorem with the following theorem which is obtained recently by H.E.Rauch.

Theorem: If in a complete riemannian space, the sectional curvature $K(P,\gamma)$ at a point P and for a 2-direction γ satisfies

$$hk\leqq K(P,\gamma)\leqq K-\varepsilon$$

for some constant , some $\varepsilon>0$ and for all P in the space and all γ, where $h\approx 3/4$ is the root of the equation $\sin\pi\sqrt{h}=\sqrt{h}/2$, then the simply connected covering space of the space is homeomorphic to the n-dimensional sphere S_n.

(7) Gauss-Bonnet formula.

The classical Gauss-Bonnet formula states that for a closed surface V_2 in a 3-dimensional euclidean space, we have

$$(1) \qquad \int_{V_2}K\,dO=\frac{1}{2}\cdot\pi\cdot\chi$$

where χ is the Euler number of V_2

H.Hopf (1925) generalized this theorem for a closed riemannian space V_n of even dimension which is a hypersurface in an n + 1 dimensional euclidean space. In this case the theorem takes the form

$$(2) \qquad \int_{V_n}K\,dO=\frac{1}{2}\omega_n\cdot\chi,$$

where K is the total curvature of V_n, ω_n is the area of an n-spher and χ is the Euler number of the space. The total curvature K is defined for a hypersurface as the product of the n principal curvatures and consequently it can be expressed as a polynomial in the componen of the curvature tensor $K_{\nu\mu\lambda\kappa}$ and in the components of the metric tensor $g_{\lambda\kappa}$.

W.Fenchel and C.B.Allendoerfer (1940) generalized the Gauss-Bonnet theorem for a closed riemannian manifold of even dimension which is a subspace of a euclidean space.

Theorem: In a closed riemannian manifold of even dimension n which is a subspace of a euclidean space E_{n+q} , we have

(2) $$\int_{V_n} K\, dO = \tfrac{1}{2}\omega_n \cdot \chi ,$$

where

(3) $$K = \frac{K_{\lambda_1\lambda_2\kappa_1\kappa_2} K_{\lambda_3\lambda_4\kappa_3\kappa_4} \cdots K_{\lambda_n\lambda_{n-1}\kappa_n\kappa_{n-1}} \varepsilon^{\lambda_1\lambda_2\cdots\lambda_n} \varepsilon^{\kappa_1\kappa_2\cdots\kappa_n}}{n!\, 2^{n/2} \,|g_{\lambda\kappa}|}$$

To prove this theorem, C.B.Allendoerfer utilized two notions, the Kronecker index and the notion of tubes.

The Kronecker index.

We suppose that at each point of an n-dimensional riemannian space V_n there are given n + 1 functions $V^i(\xi)$ of class C^1 satisfying $\sum V^i V^i = 1$. By means of this set of functions we can consider a continuous mapping of V_n upon the unit sphere S_n whose equation is $\sum V^i V^i = 1$. This mapping is of a definite degree d . If we put

(4) $$D = \begin{vmatrix} V^1 & V^2 \cdots & V^n \\ \frac{\partial V^1}{\partial \xi^1} & \frac{\partial V^2}{\partial \xi^1} \cdots & \frac{\partial V^n}{\partial \xi^1} \\ \cdots & \cdots & \\ \frac{\partial V^1}{\partial \xi^n} & \frac{\partial V^2}{\partial \xi^n} \cdots & \frac{\partial V^n}{\partial \xi^n} \end{vmatrix}$$

the number d is given by

(5) $$\int_{V_n} D\, d\xi^1 \cdots d\xi^n = \omega_n \cdot d .$$

This number d is called the Kronecker index.

If V_n is a hypersurface in an (n+1)-dimensional euclidean space and the V^i are components of the normal vector, we have

(6) $$\int_{V_n} K\, dO = \omega_n \cdot d .$$

But if n is even, we have $\chi = 2d$, and consequently

$$\int_{V_n} K\, dO = \tfrac{1}{2}\omega_n \cdot \chi .$$

which is the Gauss-Bonnet formula for a hypersurface.

The tubes.

Let $\eta^i = \eta^i(\xi^\kappa)$ be the parametric equations of a subspace V_n in an $(n+q)$-dimensional euclidean space, and $B_\alpha^{\cdot i}$ q unit normal vectors orthogonal to each other of the subspace such that $\left|\frac{\partial \eta^i}{\partial \xi^\lambda}, B_\alpha^{\cdot i}\right| > 0$, where $i, j, \dots = 1, 2, \dots, n+q$, $\kappa, \lambda = 1, 2, \dots, n$ and $\alpha, \beta = n+1, \dots n+q$.

The tube of unit radius of the subspace is defined by

$$\zeta^i = \eta^i(\xi) + B_\alpha^{\cdot i}(\xi)\, t^\alpha(v^A)$$

$$\sum t^\alpha t^\alpha = 1$$

where the $v^A (A = 1, 2, \dots, q-1)$ are parameters on a $(q-1)$-sphere S_{q-1}.

Denoting by $g_{\lambda\kappa}$ and $H_{\lambda\kappa}^{\cdot\cdot\alpha}$ the first and the second fundamental tensors and by t the terminant $\left|\frac{\partial t^\alpha}{\partial v^A} \frac{\partial t^\alpha}{\partial v^B}\right|$, we have

$$(7) \quad \int_{V_n, S_{q-1}} \frac{|1 - \sum t^\alpha . H_{\lambda\kappa}^{\cdot\cdot\alpha}|}{|g_{\lambda\kappa}|} \sqrt{t} \sqrt{g}\, dv^1 \dots dv^{q-1} d\xi^1 \dots d\xi^n = \frac{1}{2}\omega_{n+q-1} \cdot \bar{\chi},$$

where $\bar{\chi}$ is the Euler number of the tube, provided that n+q-1 is even.

H.Weyl (1939) proved that

$$(8) \quad I = \int_{S_{q-1}} \frac{|1 - \sum t^\alpha H_{\lambda\kappa}^{\cdot\cdot\alpha}|}{|g_{\lambda\kappa}|} \sqrt{t}\, dv^1 \dots dv^{q-1} = \omega_{q-1} \frac{(n-1)(n-3)\dots 3 . 1}{q(q+2)\dots(n+q-2)} K$$

where K is given by (3).

Combining (7) and (8), one gets

$$\int_{V_n} K \sqrt{g}\, d\xi^1 \dots d\xi^n = \frac{\omega_n}{2} \frac{\bar{\chi}}{2}.$$

But we know from topology that $\bar{\chi} = 2\chi$. Thus we have (2) for q = even. The proof for q odd is given by imbedding the subspace in an $(n+q+1)$- dimensional euclidian space.

An intrinsic proof, that is a proof in which we do not utilize the fact that a riemannian space is imbedded in a higher dimensional euclidean space was given by Allendoerfer-Weil (1943) and S.S.Chern (...

A.A.Abramov's theorem.

Let V_n be a riemannian space with positive definite fundamental tensor $g_{\lambda\kappa}$. We consider the anti-symmetric tensor field

$$w_{\lambda_1\lambda_2\dots\lambda_p} = f_{\lambda_1\lambda_2\dots\lambda_p}(g_{\lambda\kappa}, \partial_\mu g_{\lambda\kappa}, \dots \partial_{\mu_s \dots \mu_1} g_{\lambda\kappa})$$

where the $f_{\lambda_1 \dots \lambda_p}$ are analytic functions of $g_{\lambda\kappa}; \partial_\mu g_{\lambda\kappa}, \dots$. Let V_p be a compact orientable sufficiently many times differentiable subspace of V_n. If the integral

$$\int_{C_p} w_{\lambda_1\lambda_2\dots\lambda_p}\, d\xi^{\lambda_1} d\xi^{\lambda_2} \dots d\xi^{\lambda_p}$$

is independent the choice of $g_{\lambda\kappa}$, $w_{\lambda_1\lambda_2\cdots\lambda_p}$ is a topological invariant. It is not proved that all topological invariants can be written in this way. If

$$\int_{C_p}' w_{\lambda_1\lambda_2\cdots\lambda_p}\, d\xi^{\lambda_1}\, d\xi^{\lambda_2} \cdots d\xi^{\lambda_p} = \int_{C_p} {}'w_{\lambda_1\lambda_2\cdots\lambda_p}\, d\xi^{\lambda_1}\, d\xi^{\lambda_2} \cdots d\xi^{\lambda_p}$$

we say that the fields w and $'w$ are equivalent. We put

$$\Pi_{\lambda_1\lambda_2\cdots\lambda_{4p}} = K_{[\lambda_1\lambda_2|\kappa|}^{\cdots\lambda}\, K_{\lambda_3\lambda_4|\lambda|}^{\cdots\mu} \cdots K_{\lambda_{4p-1}\lambda_{4p}]\nu}^{\cdots\kappa}$$

$$\tilde{\Pi}_{\lambda_1\lambda_2\cdots\lambda_n} = K_{[\lambda_1\lambda_2|\kappa_1\kappa_2}\, K_{\lambda_3\lambda_4|\kappa_3\kappa_4|} \cdots K_{\lambda_{n-1}\lambda_n]\kappa_{n-1}\kappa_n}\, \varepsilon^{\kappa_1\cdots\kappa_n}$$

Theorem: Tensor fields of the type (for n even)

$$\sum C\, \Pi_{[\lambda_1\cdots} \cdots \Pi_{\cdots\lambda_p]}$$

and

$$\sum C\, \tilde{\Pi}_{\lambda_1\lambda_2\cdots\lambda_n}$$

are only topological invariant up to an equivalence.

(8) Complex manifolds (kählerian spaces).

We consider a 2n-dimensional real manifold covered by a set of coordinate neighbourhoods. We denote the coordinates of a point P in a certain coordinate neighbourhood containing P , by η^κ, ξ^κ; $\kappa = 1, \ldots, n$. If we put

$$z^\kappa \overset{\text{def}}{=} \eta^\kappa + i\xi^\kappa$$

we can regard z^κ as complex coordinates of the point P in the coordinate neighbourhood $(\eta^\kappa, \xi^\kappa)$. If we can choose a set of coordinate neighbourhoods in such a way that, when the point P lies in the overlapping domain of two coordinate neighbourhoods (κ') and (κ) we have always

$$z^{\kappa'} = f^{\kappa'}(z^\kappa) \tag{1}$$

where $f^{\kappa'}(z)$ are complex analytic functions of z^κ and

$$z^{\kappa'} \overset{\text{def}}{=} \eta^{\kappa'} + i\xi^{\kappa'}$$

we say that the manifold has a complex analytic structure or simply a complex structure. Putting

$$z^{\bar\kappa} \overset{\text{def}}{=} \eta^\kappa - i\xi^\kappa$$

we have from (1)

$$z^{\bar\kappa'} = f^{\bar\kappa'}(z^{\bar\kappa}) \tag{2}$$

where $f^{\bar{\kappa}'}(\bar{x})$ denote the complex conjugate functions of $f^{\kappa'}$. The x^{κ} and $x^{\bar{\kappa}}$ can be considered as 2n independent coordinates and the equations (1) and (2) can be written as

$$(3) \quad x^{h'} = f^{h'}(x^h) \; ; \qquad \begin{array}{l} h = 1, \dots, n, \bar{1}, \dots, \bar{n} \\ h' = 1', \dots, n', \bar{1}', \dots \bar{n}' \end{array}$$

with functions $f^{h'}$ that have the special forms (1) and (2). Each coordinate system (h') obtained in this way is called a <u>complex coordinate system</u> in contradistinction to (general) coordinate systems (h') that arise if the functions $f^{h'}$ in (3) are analytic in x^h but do not satisfy (1) and (2).

Because of this special form of coordinate transformations, the tensors and affine connections in a space with complex structure show many interesting properties which usual tensors and connections do not have. Geometrically the introduction of the preferred coordinate systems means that in the 2n-dimensional manifold are fixed

1. the set of ∞^n n-dimensional manifolds $x^{\bar{\kappa}} = 0$
2. dito $x^{\kappa} = 0$
3. the principal n-dimensional manifold $x^{\bar{\kappa}} = \overline{x^{\kappa}}$.

A complex manifold is always of class C^{ω}, that is, it is a real analytic manifold of even dimension.

Since the Jacobian of the coordinate transformation (3) is real and positive, a complex manifold is always orientable.

In a complex manifold, we can define a special tensor $F_i^{\cdot h}$ which has, in any complex coordinate system, the components

$$(4) \quad F_i^{\cdot h} = \begin{pmatrix} +i\delta_\lambda^\kappa & 0 \\ 0 & -i\delta_\lambda^\kappa \end{pmatrix} \text{ or } F_\lambda^{\cdot\kappa} = i \; ; \; F_{\bar{\lambda}}^{\cdot\kappa} = 0 \; ; \; F_\lambda^{\cdot\bar{\kappa}} = 0 \; ; \; F_{\bar{\lambda}}^{\cdot\bar{\kappa}} = -i$$

This tensor plays a very important rôle in the theory of complex manifolds.

If a complex manifold is endowed with a hermetian metric

$$(5) \qquad ds^2 = 2 g_{\bar{\lambda}\kappa}\, d\xi^{\bar{\lambda}}\, d\xi^{\kappa} \quad \left(g_{\bar{\lambda}\kappa} = g_{\kappa\bar{\lambda}} \overset{\text{def}}{=} \overline{g_{\bar{\kappa}\lambda}}\right)$$

which is positive definite, it is called a hermetian manifold.

If we denote by $\left\{ {h \atop j\; i} \right\}$ the Christoffel symbols formed with

$$g_{ih} = \begin{pmatrix} 0 & g_{\bar{\lambda}\kappa} \\ g_{\lambda\bar{\kappa}} & 0 \end{pmatrix} \text{ or } g_{\lambda\kappa} = 0 \; ; \; g_{\bar{\lambda}\bar{\kappa}} = 0$$

and by ∇_j the covariant differentiation with respect to $\left\{ {h \atop j\; i} \right\}$, we have of course

$$\nabla_j g_{ih} = 0, \text{ but in general not } \nabla_j F_i^{\cdot h} = 0$$

If we have

(6) $$\nabla_j g_{ih} = 0, \quad \nabla_j F_i^{\ h} = 0$$

the hermetian manifold is called a kählerin manifold. The condition (6) is equivalent to

(7) $$\partial_{\bar{\mu}} g_{\bar{\lambda}\kappa} = \partial_{\bar{\lambda}} g_{\bar{\mu}\kappa}\ ; \ \partial_{\bar{\lambda}} = \frac{\partial}{\partial z^{\bar{\lambda}}} .$$

We quote here some results in differential geometry in the large in kählerian manifolds.

Bochner (1946):

Theorem: In a compact kählerian manifold with $R_{\bar{\lambda}\kappa} v^{\bar{\lambda}} v^{\kappa} > 0$, there does not exist a covariant vector field whose components are complex analytic functions of complex coordinates except a zero vector.

Theorem: In a complex kählerian manifold with $R_{\bar{\lambda}\kappa} v^{\bar{\lambda}} v^{\kappa} < 0$, there does not exist a contravariant vector field whose components are complex analytic functions of complex coordinates except a zero vector.

Hodge (1941), Eckmann and Guggenheimer (1949) :

Theorem: In a compact kählerian manifold an anti-symmetric tensor field of the form

$$w_{i_1 i_2 \ldots i_p} = (w_{\lambda_1\lambda_2\ldots\lambda_p}, 0, 0, \ldots, 0, w_{\bar{\lambda}_1\bar{\lambda}_2\ldots\bar{\lambda}_p})$$

is harmonic if and only if the components $w_{\lambda_1\lambda_2\ldots\lambda_p}$ are complex analytic functions of the z^{κ} and the components $w_{\bar{\lambda}_1\bar{\lambda}_2\ldots\bar{\lambda}_p}$ are complex analytic functions of the $z^{\bar{\kappa}}$.

Hodge (1941):

In a compact kählerian manifold of complex dimension n, we have for the Betti numbers B_p :

$$B_0 \leqq B_2 \leqq B_4 \leqq \cdots \leqq B_{2[\frac{n}{2}]},$$

$$B_1 \leqq B_3 \leqq B_5 \leqq \cdots \leqq B_{2[\frac{n}{2}]+1},$$

and the number $B_{p+2}-B_p\ (p+2 \leqq n)$ is equal to the number of linearly independent self-adjoint effective harmonic tensors (with constant coeffients) of order $p+2$.

In this statement an effective harmonic tensor is a harmonic tensor $w_{i_1 i_2 \ldots i_p}$ satisfying

$$F^{i_1 i_2} w_{i_1 i_2 \ldots i_p} = 0,$$

where

$$F^{ih} = F_l^{\ h} g^{li}.$$

Using this theorem of Hodge, we can estimate the Betti numbers of a kählerian manifold of positive constant curvature.

Theorem: In a compact kählerian manifold of positive constant curvature, we have

$$B_{2l} = 1 \ , \ B_{2l+1} = 0 \quad (0 \leqq 2l, 2l+1 \leqq n).$$

8. Complex manifolds.[1]

We consider a 2n-dimensional real manifold covered by a set of coordinate neighbourhoods $U(\eta^\kappa, \zeta^\kappa)$, where $\kappa, \lambda, \mu, \ldots = 1, 2, \ldots, n$.

If we put

(8.1) $$\xi^\kappa = \eta^\kappa + i\zeta^\kappa,$$

then we can regard (ξ^κ) as complex coordinates of a point in the coordinate neighbourhoods $U(\eta^\kappa, \zeta^\kappa)$.

If we can choose a set of coordinate neighbourhoods in such a way that, when a point P lies in the overlapping domain of two coordinate neighbourhoods $U(\eta^\kappa, \zeta^\kappa)$ and $U'(\eta^{\kappa'}, \zeta^{\kappa'})$ then we have always

(8.2) $$\xi^{\kappa'} = f^{\kappa'}(\xi^\kappa),$$

where $f^{\kappa'}(\xi^\kappa)$ are complex analytic functions of complex variables ξ^κ and

(8.3) $$\xi^{\kappa'} = \eta^{\kappa'} + i\zeta^{\kappa'},$$

we say that the manifold admits a <u>complex analytic structure</u> or simply <u>complex structure</u> and we call such a manifold an n-dimensional <u>complex manifold</u>. Such a manifold is of class C^ω.

Let X_n be an n-dimensional complex manifold, then we define its conjugate manifold $\bar{X}_n$ as a manifold satisfying the following conditions[2].

The $\bar{X}_n$ is a complex manifold related to the manifold X_n by a homeomorphism, mapping each point $P \in X_n$ into a point $\bar{P} \in \bar{X}_n$, such that for each coordinate neighbourhood $U(\xi^\kappa)$ of P, there exists a coordinate neighbourhood $\bar{U}(\xi^{\bar{\kappa}})$ of $\bar{P}$, which is the image of $U(\xi^\kappa)$ under the conjugation and which satisfies

(8.4) $$\xi^{\bar{\kappa}}(\bar{P}) = \overline{\xi^\kappa(P)}$$

In the product complex manifold $\mathfrak{X}_{2n} = \mathfrak{X}_n \times \bar{\mathfrak{X}}_n$, the diagonal set of points $(P, \bar{P})$ form a real 2n-dimensional real analytic submanifold and this submanifold is homeomorphic to X_n.

This submanifold has a neighbourhood that can be covered by a special class of coordinate systems $(\xi^\kappa, \xi^{\bar{\kappa}})$ in $\mathfrak{X}_n \times \bar{\mathfrak{X}}_n$ as follows: for each coordinate neighbourhood $U(\xi^\kappa)$ of X_n, let $\bar{U}(\xi^{\bar{\kappa}})$ its conjugate coordinate neighbourhood defined by (8.4). Then the sets $U \times \bar{U}$ cover a neighbourhood of the points $(P, \bar{P})$ and for any one such set, made up of points $(P, \bar{Q})$ where $P, Q \in U$ let

$$\xi^\kappa(P, \bar{Q}) = \xi^\kappa(P), \quad \xi^{\bar{\kappa}}(P, \bar{Q}) = \xi^{\bar{\kappa}}(\bar{Q}) = \overline{\xi^\kappa(Q)},$$

then the points $(P, \bar{P})$ of the diagonal submanifold are characterized by the equation

(8.5) $$\xi^{\bar{\kappa}} = \overline{\xi^\kappa}.$$

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1) cf. J.A. Schouten: Ricci Calculus (1954), 388-420.

2) E. Calabi: Isometric imbedding of complex manifolds. Ann. of Math., 58 (1953), 1-23.

We call this submanifold the principal X_n of X_{2n}.

The transformation of coordinates in $\mathfrak{X}_{2n}=\mathfrak{X}_n\times\overline{\mathfrak{X}}_n$ has the form

(8.6) $$\xi^{\kappa'}=f^{\kappa'}(\xi^{\kappa}),\quad \xi^{\bar\kappa'}=f^{\bar\kappa'}(\xi^{\bar\kappa})$$

where $f^{\bar\kappa'}$ are complex conjugate functions of $f^{\kappa'}$. The transformation (8.6) reduces to

(8.7) $$\xi^{\kappa'}=f^{\kappa'}(\xi^{\kappa}),\quad \overline{\xi^{\kappa'}}=f^{\bar\kappa'}(\overline{\xi^{\kappa}})$$

on the principal X_n.

The Jacobian of (8.6) is

$$\Delta=\left|\frac{\partial f^{\kappa'}}{\partial\xi^{\kappa}}\right|\cdot\left|\frac{\partial f^{\bar\kappa'}}{\partial\xi^{\bar\kappa}}\right|$$

which is real and positive on the principal X_n. Thus the principal X_n and consequently the complex X_n is always orientable.

Now, when we have in X_n a function $f(\eta^{\kappa},\zeta^{\kappa})$ analytic in η^{κ} and ζ^{κ} we can consider it as a function of ξ^{κ} and $\overline{\xi^{\kappa}}$ which can be expanded in power series of ξ^{κ} and $\overline{\xi^{\kappa}}$. Then substituting $\overline{\xi^{\kappa}}$ by $\xi^{\bar\kappa}$ we can regard this as a complex analytic function of ξ^{κ} and $\xi^{\bar\kappa}$.

Conversely, if we have a complex analytic function $f(\xi^{\kappa},\xi^{\bar\kappa})$ of ξ^{κ} and $\xi^{\bar\kappa}$, this gives a function $f(\xi^{\kappa},\overline{\xi^{\kappa}})$ in the principal X_n. We say that $f(\xi^{\kappa},\overline{\xi^{\kappa}})$ are semi-analytic in the complex X_n.

In the following we assume that the components of all geometric objects in $\mathfrak{X}_{2n}=\mathfrak{X}_n\times\overline{\mathfrak{X}}_n$ are analytic.

If we have a function $f(\xi^{\kappa},\overline{\xi^{\kappa}})$ in X_n, we consider this as a function $f(\xi^{\kappa},\xi^{\bar\kappa})$ in $\mathfrak{X}_{2n}=\mathfrak{X}_n\times\overline{\mathfrak{X}}_n$, then we can consider

$$\partial_{\mu}f=\frac{\partial f}{\partial\xi^{\mu}},\quad \partial_{\bar\mu}f=\frac{\partial f}{\partial\xi^{\bar\mu}}.$$

After this partial differentiation, we put $\xi^{\bar\mu}=\overline{\xi^{\mu}}$. We call this the partial derivatives of $f(\xi^{\kappa},\overline{\xi^{\kappa}})$ in $\mathfrak{X}_n$ with respect to ξ^{κ} and $\xi^{\bar\kappa}$ respectively.

Now we define, for instance, a contravariant vector in $\mathfrak{X}_{2n}$ as a geometric object which is represented by its 2n components $(v^{\kappa},v^{\bar\kappa})$ with respect to each coordinate system $(\xi^{\kappa},\xi^{\bar\kappa})$ whose transformation law under (8.6) is

(8.8) $$v^{\kappa'}=A_{\kappa}^{\kappa'}v^{\kappa},\quad v^{\bar\kappa'}=A_{\bar\kappa}^{\bar\kappa'}v^{\bar\kappa},$$

where

$$A_{\kappa}^{\kappa'}=\partial_{\kappa}\xi^{\kappa'},\quad A_{\bar\kappa}^{\bar\kappa'}=\partial_{\bar\kappa}\xi^{\bar\kappa'}$$

and consequently $A_{\kappa}^{\kappa'}$ are functions of ξ^{κ} only and $A_{\bar\kappa}^{\bar\kappa'}$ functions of $\xi^{\bar\kappa}$ only.

If a contravariant vector of $\mathfrak{X}_{2n}$ is defined at a point of the principal X_n, that is, if $v^{\kappa}=v^{\kappa}(\xi,\bar\xi)$, $v^{\bar\kappa}=v^{\bar\kappa}(\xi,\bar\xi)$, we call it a contravariant vector of X_n.

Equation (8.8) shows that, if $(v^{\kappa},v^{\bar\kappa})$ are components of a contravariant vector in $\mathfrak{X}_{2n}$, then

(8.9) $$(v^\kappa, 0),\ (0, v^{\bar\kappa}),\ (\overline{v^{\bar\kappa}}, \overline{v^\kappa})$$
are also components of contravariant vectors in X_{2n}.

Through each point $(\xi^\kappa, \xi^{\bar\kappa})$ of X_{2n}, there pass a X_n and a $\overline{X}_n$. We denote the tangent planes to X_n and to $\overline{X}_n$ at $(\xi^\kappa, \xi^{\bar\kappa})$ by $\mathcal{E}_n$ and $\overline{\mathcal{E}}_n$ respectively. Then $(v^\kappa, 0)$ is the projection of $(v^\kappa, v^{\bar\kappa})$ on $\mathcal{E}_n$, and $(0, v^{\bar\kappa})$ is the projection of $(v^\kappa, v^{\bar\kappa})$ on $\overline{\mathcal{E}}_n$.

Between a neighbourhood of X_n and that of $\overline{X}_n$, we have a mapping defined by (8.4). This mapping induces a mapping between $\mathcal{E}_n$ and $\overline{\mathcal{E}}_n$. By this mapping $(v^\kappa, 0)$ corresponds to $(0, \overline{v^\kappa})$ and $(0, v^{\bar\kappa})$ corresponds to $(\overline{v^{\bar\kappa}}, 0)$. This fact gives a geometrical interpretation of the correspondence between $(v^\kappa, v^{\bar\kappa})$ and $(\overline{v^{\bar\kappa}}, \overline{v^\kappa})$. That is, we decompose first $(v^\kappa, v^{\bar\kappa})$ into $(v^\kappa, 0)$ and $(0, v^{\bar\kappa})$, then we consider the images of these vectors under the correspondence between $\mathcal{E}_n$ and $\overline{\mathcal{E}}_n$. Thus we obtain $(0, \overline{v^\kappa})$ and $(\overline{v^{\bar\kappa}}, 0)$ whose sum is equal to $(\overline{v^{\bar\kappa}}, \overline{v^\kappa})$. We call $(\overline{v^{\bar\kappa}}, \overline{v^\kappa})$ the conjugate of the vector $(v^\kappa, v^{\bar\kappa})$.

If

(8.10) $$(v^\kappa, v^{\bar\kappa}) = (\overline{v^{\bar\kappa}}, \overline{v^\kappa}),$$

we call this vector a self-conjugate vector. Such a vector is tangent to the principal X_n. If we calculate the components of this vector in the real coordinate system $(\eta^\kappa, \zeta^\kappa)$ in X_n, we get

$$\tfrac{1}{2}(v^\kappa + v^{\bar\kappa}),\quad \tfrac{1}{2i}(v^\kappa - v^{\bar\kappa}).$$

These are real for a self-conjugate vector. So a self-conjugate vector is sometimes called a real vector.

More generally, if a tensor $T_{\gamma\beta}^{\cdot\cdot\alpha}$ $(\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n; \bar 1, \bar 2, \ldots, \bar n)$ satisfies the condition

(8.11) $$\overline{T_{\gamma\beta}^{\cdot\cdot\alpha}} = T_{\bar\gamma\bar\beta}^{\cdot\cdot\bar\alpha}$$

then we call it self-conjugate or real where $\bar\alpha = \bar\kappa$ when $\alpha = \kappa$ and $\bar\alpha = \kappa$ when $\alpha = \bar\kappa$.

Now, as the equation (8.8) shows that when $(v^\kappa, v^{\bar\kappa})$ are components of a contravariant vector, $(iv^\kappa, -iv^{\bar\kappa})$ are also components of a contravariant vector and moreover that when $(v^\kappa, v^{\bar\kappa})$ is self-conjugate, $(iv^\kappa, -iv^{\bar\kappa})$ is also self-conjugate.

This shows that the tensor $F_\beta^{\cdot\alpha}$ with

(8.12) $$F_\beta^{\cdot\alpha} = \begin{pmatrix} i\delta_\lambda^\kappa & 0 \\ 0 & -i\delta_{\bar\lambda}^{\bar\kappa} \end{pmatrix}$$

or

$$F_\lambda^{\cdot\kappa} = i\delta_\lambda^\kappa,\quad F_{\bar\lambda}^{\cdot\kappa} = 0,\quad F_\lambda^{\cdot\bar\kappa} = 0,\quad F_{\bar\lambda}^{\cdot\bar\kappa} = -i\delta_{\bar\lambda}^{\bar\kappa},$$

are components of a self-conjugate mixed tensor of valence two.

If we put

(8.13) $$B_\beta^\alpha = \tfrac{1}{2}(A_\beta^\alpha - iF_\beta^{\cdot\alpha}),\quad C_\beta^\alpha = \tfrac{1}{2}(A_\beta^\alpha + iF_\beta^{\cdot\alpha}),$$

then the projections $(v^\kappa, v^{\bar\kappa}) \to (v^\kappa, 0)$ and $(v^\kappa, v^{\bar\kappa}) \to (0, v^{\bar\kappa})$ are respectively represented by

$$v^\alpha \to B^\alpha_\beta v^\beta, \quad v^\alpha \to C^\alpha_\beta v^\beta. \tag{8.14}$$

Now an affine connexion in X_{2n} [1] is defined by $(2n)^3$ functions $\Gamma^\alpha_{\gamma\beta}(\xi)$ and the covariant differential of a contravariant vector is given by

$$\delta v^\alpha = dv^\alpha + \Gamma^\alpha_{\gamma\beta}\, d\xi^\gamma v^\beta \tag{8.15}$$

or

$$\begin{cases} \delta v^\kappa = dv^\kappa + (\Gamma^\kappa_{\mu\lambda} v^\lambda + \Gamma^\kappa_{\mu\bar\lambda} v^{\bar\lambda}) d\xi^\mu + (\Gamma^\kappa_{\bar\mu\lambda} v^\lambda + \Gamma^\kappa_{\bar\mu\bar\lambda} v^{\bar\lambda}) d\xi^{\bar\mu}, \\ \delta v^{\bar\kappa} = dv^\kappa + (\Gamma^{\bar\kappa}_{\mu\lambda} v^\lambda + \Gamma^{\bar\kappa}_{\mu\bar\lambda} v^{\bar\lambda}) d\xi^\mu + (\Gamma^{\bar\kappa}_{\bar\mu\lambda} v^\lambda + \Gamma^{\bar\kappa}_{\bar\mu\bar\lambda} v^{\bar\lambda}) d\xi^{\bar\mu}. \end{cases} \tag{8.16}$$

If we put the conditions $\overline{d\xi^\mu} = d\xi^{\bar\mu}$, formula (8.16) gives the covariant differential of v^α in the principal X_n.

The transformation law of the components of the linear connexion is given by

$$\Gamma^{\alpha'}_{\gamma'\beta'} = A^{\alpha'}_\alpha (A^\gamma_{\gamma'} A^\beta_{\beta'} \Gamma^\alpha_{\gamma\beta} + \partial_{\gamma'} A^\alpha_{\beta'}) \tag{8.17}$$

But because of the special form (8.6) of the coordinate transformation, (8.17) shows that

$$\begin{cases} \Gamma^{\kappa'}_{\mu'\lambda'} = A^{\kappa'}_\kappa (A^\mu_{\mu'} A^\lambda_{\lambda'} \Gamma^\kappa_{\mu\lambda} + \partial_{\mu'} A^{\kappa'}_{\lambda'}), \\ \Gamma^{\bar\kappa'}_{\bar\mu'\bar\lambda'} = A^{\bar\kappa'}_{\bar\kappa} (A^{\bar\mu}_{\bar\mu'} A^{\bar\lambda}_{\bar\lambda'} \Gamma^{\bar\kappa}_{\bar\mu\bar\lambda} + \partial_{\bar\mu'} A^{\bar\kappa'}_{\bar\lambda'}) \end{cases} \tag{8.18}$$

and that $\Gamma^\kappa_{\mu\bar\lambda}, \Gamma^{\bar\kappa}_{\mu\lambda}, \Gamma^{\bar\kappa}_{\mu\bar\lambda}, \Gamma^\kappa_{\bar\mu\lambda}, \Gamma^\kappa_{\bar\mu\bar\lambda}, \Gamma^{\bar\kappa}_{\bar\mu\lambda}$ are all transformed like components of tensors.

In the following, we assume that

$$\text{I)} \quad \Gamma^\kappa_{\mu\bar\lambda} = 0, \; \Gamma^{\bar\kappa}_{\mu\lambda} = 0; \; \Gamma^\kappa_{\bar\mu\bar\lambda} = 0, \; \Gamma^{\bar\kappa}_{\bar\mu\lambda} = 0. \tag{8.19}$$

These equations have the following geometrical meaning:

$\Gamma^\kappa_{\mu\bar\lambda} = 0 \rightleftarrows \bar{\mathcal{E}}_n$ is parallel along X_n.

$\Gamma^{\bar\kappa}_{\mu\lambda} = 0 \rightleftarrows$ The invariant X_n's build by $\mathcal{E}_n$ are totally geodesic.

$\Gamma^\kappa_{\bar\mu\bar\lambda} = 0 \rightleftarrows$ The invariant $\bar X_n$'s build by $\bar{\mathcal{E}}_n$ are totally geodesic.

$\Gamma^{\bar\kappa}_{\bar\mu\lambda} = 0 \rightleftarrows \mathcal{E}_n$ is parallel along $\bar X_n$.

We remark here that the conditions (8.19) are equivalent to

$$\nabla_\gamma F_\beta{}^\alpha = 0. \tag{8.20}$$

This is geometrically evident because the $\mathcal{E}_n(\bar{\mathcal{E}}_n)$ is spanned by the n eigenvectors corresponding to eigenvalues $+i(-i)$ of $F_\beta{}^\alpha$.

Because of $\nabla_\gamma A^\alpha_\beta = 0$, (8.20) is equivalent to

$$\nabla_\gamma B^\alpha_\beta = 0, \quad \nabla_\gamma C^\alpha_\beta = 0 \tag{8.21}$$

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1) J.A. Schouten and D. van Dantzig: Über unitäre Geometrie. Math. Ann. 103 (1930), 319-346.

... which is tangent to the principal X_n (a self-con-
jugate vector) is displaced parallelly along the principal X_n, the necessary and sufficient condition that the result be also tangent to the principal X_n, in other words, that the principal X_n be totally geodesic is that II) holds.

In $X_{2n}=X_n\times\overline{X}_n$, a figure in one X_n and the corresponding figure in another X_n which has the same components as the first one, are said to be **equipollent**.

We also assume often

(8.23) III) $$\Gamma_{\bar{\mu}\lambda}^{\kappa}=0,\quad \Gamma_{\mu\bar{\lambda}}^{\bar{\kappa}}=0.$$

If we transport parallelly a contravariant vector $v^\alpha=(v^\kappa, 0)$ in $\mathcal{E}_n$ along $\overline{\mathcal{E}}_n$, we get

$$\delta v^\kappa = dv^\kappa + d\xi^{\bar{\mu}}\Gamma_{\bar{\mu}\lambda}^{\kappa}v^\lambda = 0$$

or

$$dv^\kappa = -d\xi^{\bar{\mu}}\Gamma_{\bar{\mu}\lambda}^{\kappa}v^\lambda.$$

Hence (8.23) means that two equipollent figures are parallel.

In a complex manifold, the fact that the components $v^\kappa(w_\lambda)$ of a contravariant (covariant) vector field are complex analytic functions of the ξ^κ has an invariant meaning. We call such a vector field <u>analytic</u>.

If we adopt a linear connexion which satisfies I) and III) the fact that v^κ and w_λ are analytic fields is represented by the following equations:

(8.24) $$\nabla_{\bar{\mu}}v^\kappa = \partial_{\bar{\mu}}v^\kappa = 0$$

(8.25) $$\nabla_{\bar{\mu}}w_\lambda = \partial_{\bar{\mu}}w_\lambda = 0.$$

If a complex manifold is endowed with a Hermitian metric

(8.26) $$ds^2 = 2g_{\bar{\lambda}\kappa}(\xi,\bar{\xi})d\xi^{\bar{\lambda}}d\xi^\kappa$$

where

(8.27) $$g_{\bar{\lambda}\kappa} = g_{\kappa\bar{\lambda}} = \overline{g_{\bar{\kappa}\lambda}}$$

is positive definite, we call the manifold a <u>Hermitian manifold</u>.

If we write (8.26) in the form

(8.28) $$ds^2 = g_{\beta\alpha}(\xi^\kappa,\xi^{\bar{\kappa}})d\xi^\beta d\xi^\alpha$$

then this gives a metric in $X_{2n}=X_n\times\overline{X}_n$ and $g_{\beta\alpha}$ has the form

(8.29) $$g_{\beta\alpha} = \begin{pmatrix} 0 & g_{\bar{\lambda}\kappa} \\ g_{\lambda\bar{\kappa}} & 0 \end{pmatrix} \quad \text{or} \quad g_{\lambda\kappa}=0,\ g_{\bar{\lambda}\bar{\kappa}}=0.$$

If we put

(8.30) $$F_{\beta\alpha} = F_{\beta}^{\cdot\,\varepsilon} g_{\varepsilon\alpha}$$

the tensor $F_{\beta\alpha}$ has the components

(8.31) $$F_{\beta\alpha} = \begin{pmatrix} 0 & -ig_{\bar{\lambda}\kappa} \\ +ig_{\lambda\bar{\kappa}} & 0 \end{pmatrix} \quad \text{or} \quad F_{\lambda\kappa} = 0\,; \ F_{\bar{\lambda}\kappa} = -ig_{\bar{\lambda}\kappa}\,; \ F_{\lambda\bar{\kappa}} = +ig_{\lambda\bar{\kappa}}\,; \ F_{\bar{\lambda}\bar{\kappa}} = 0\,,$$

and is a bivector.

From $F_{\alpha}^{\cdot\,\varepsilon} g_{\varepsilon\beta} = -F_{\beta}^{\cdot\,\varepsilon} g_{\varepsilon\alpha}$ we get by transvection with $F_{\gamma}^{\cdot\,\alpha}$

(8.32) $$g_{\gamma\beta} = F_{\gamma}^{\cdot\,\varepsilon} F_{\beta}^{\cdot\,\delta} g_{\varepsilon\delta}$$

because of

(8.33) $$F_{\gamma}^{\cdot\,\alpha} F_{\alpha}^{\cdot\,\varepsilon} = -A_{\gamma}^{\varepsilon}\,.$$

Now we consider the Riemannian connexion $\left\{{\alpha \atop \gamma\beta}\right\}$ of X_{2n}. We have

(8.34) $$\begin{cases} \left\{{\kappa \atop \mu\lambda}\right\} = g^{\bar{\rho}\kappa}\partial_{[\mu} g_{\lambda]\bar{\rho}}\,, \quad \left\{{\kappa \atop \bar{\mu}\lambda}\right\} = g^{\bar{\rho}\kappa}\partial_{[\bar{\mu}} g_{\bar{\rho}]\lambda}\,, & \text{conj.} \\ \left\{{\kappa \atop \mu\bar{\lambda}}\right\} = g^{\bar{\rho}\kappa}\partial_{[\bar{\lambda}} g_{\bar{\rho}]\mu}\,, \quad \left\{{\kappa \atop \bar{\mu}\bar{\lambda}}\right\} = 0 & \text{conj.} \end{cases}$$

This connexion satisfies, of course, $\overset{\circ}{\nabla}_{\gamma} g_{\beta\alpha} = 0$ but not necessarily $\overset{\circ}{\nabla}_{\gamma} F_{\beta}^{\cdot\,\alpha} = 0$, where $\overset{\circ}{\nabla}_{\gamma}$ denotes covariant differentiation with respect to Christoffel symbols.

There are many connexions which satisfy

(8.35) $$\nabla_{\gamma} g_{\beta\alpha} = 0\,, \quad \nabla_{\gamma} F_{\beta}^{\cdot\,\alpha} = 0\,, \quad \overline{\Gamma_{\gamma\beta}^{\alpha}} = \Gamma_{\bar{\gamma}\bar{\beta}}^{\bar{\alpha}}\,,$$

and whose torsion tensor

(8.36) $$\Gamma_{[\gamma\beta]}^{\alpha} = S_{\gamma\beta}^{\cdot\cdot\,\alpha}$$

does not vanish. From $\nabla_{\gamma} g_{\beta\alpha} = 0$ we get [1)]

$$\text{(8.37)}\quad \begin{aligned} &a) \quad \Gamma_{\mu\lambda}^{\kappa} = g^{\bar{\rho}\kappa}\partial_{[\mu} g_{\lambda]\bar{\rho}} + S_{\mu\lambda}^{\cdot\cdot\,\kappa} - S_{\mu\cdot\lambda}^{\cdot\,\kappa} - S_{\lambda\cdot\mu}^{\cdot\,\kappa} && \text{conj.} \\ &b) \quad \Gamma_{\bar{\mu}\lambda}^{\kappa} = g^{\bar{\rho}\kappa}\partial_{[\bar{\mu}} g_{\bar{\rho}]\lambda} + S_{\bar{\mu}\lambda}^{\cdot\cdot\,\kappa} - S_{\bar{\mu}\cdot\lambda}^{\cdot\,\kappa} - S_{\lambda\cdot\bar{\mu}}^{\cdot\,\kappa} && \text{conj.} \\ &c) \quad \Gamma_{\mu\bar{\lambda}}^{\kappa} = g^{\bar{\rho}\kappa}\partial_{[\bar{\lambda}} g_{\bar{\rho}]\mu} + S_{\mu\bar{\lambda}}^{\cdot\cdot\,\kappa} - S_{\mu\cdot\bar{\lambda}}^{\cdot\,\kappa} - S_{\bar{\lambda}\cdot\mu}^{\cdot\,\kappa} && \text{conj.} \\ &d) \quad \Gamma_{\bar{\mu}\bar{\lambda}}^{\kappa} = \qquad\qquad\qquad S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\,\kappa} - S_{\bar{\mu}\cdot\bar{\lambda}}^{\cdot\,\kappa} - S_{\bar{\lambda}\cdot\bar{\mu}}^{\cdot\,\kappa} && \text{conj.} \end{aligned}$$

Here the terms on the right hand side not containing S give the Riemannian connexion $\left\{{\alpha \atop \gamma\beta}\right\}$ with the fundamental tensor $g_{\beta\alpha}$.

When $S_{\gamma\beta}^{\cdot\cdot\,\alpha}$ is self-conjugate, the linear connexions defined by (8.37) satisfy $\nabla_{\gamma} g_{\beta\alpha} = 0$ and $\overline{\Gamma_{\gamma\beta}^{\alpha}} = \Gamma_{\bar{\gamma}\bar{\beta}}^{\bar{\alpha}}$ but not always $\nabla_{\gamma} F_{\beta}^{\cdot\,\alpha} = 0$. Among these connexions, the followings which also satisfy $\nabla_{\gamma} F_{\beta}^{\cdot\,\alpha} = 0$ are known.

1) J.A. Schouten: Ricci Calculus (1954), P.396.

1°. If we assume that the parallel displacement of a vector in $\mathcal{E}_n$ or or $\bar{\mathcal{E}}_n$ with respect to $\Gamma^{\alpha}_{\gamma\beta}$ is the same as that with respect to $\left\{{\alpha \atop \gamma\beta}\right\}$ we find

(8.38) $$\Gamma^{\kappa}_{\mu\lambda} = \left\{{\kappa \atop \mu\lambda}\right\}, \quad \Gamma^{\kappa}_{\bar{\mu}\lambda} = \left\{{\kappa \atop \bar{\mu}\lambda}\right\} \qquad \text{conj.}$$

all the other components of $\Gamma^{\alpha}_{\gamma\beta}$ being zero. [The connexion of Lichnerowicz][1].

2° If we add the condition $\nabla_{\gamma} F_{\beta}^{\cdot\alpha} = 0$ and $\Gamma^{\kappa}_{\bar{\mu}\lambda} = 0$ to (8.37), we get

(8.39) $$\Gamma^{\kappa}_{\mu\lambda} = g^{\bar{\rho}\kappa}\partial_{\mu} g_{\lambda\bar{\rho}}$$

all the other components of $\Gamma^{\alpha}_{\gamma\beta}$ being zero. [The connexion of Schouten-Van Dantzig[2]].

3° If we assume $S_{\gamma\cdot\beta}^{\cdot\alpha} = S_{\beta\cdot\gamma}^{\cdot\alpha}$ (geodesics with respect to $\Gamma^{\alpha}_{\gamma\beta}$ coincide with geodesics with respect to the Riemannian connexion), then we get [3]

(8.40) $$\Gamma^{\kappa}_{\mu\lambda} = g^{\bar{\rho}\kappa}\partial_{\lambda} g_{\mu\bar{\rho}}, \quad \Gamma^{\kappa}_{\bar{\mu}\lambda} = 2g^{\bar{\rho}\kappa}\partial_{[\bar{\mu}} g_{\bar{\rho}]\lambda}.$$

When the Riemannian connexion satisfies $\nabla_{\gamma} F_{\beta}^{\cdot\alpha} = 0$, the metric is called a Kählerian metric. For a Kählerian metric, we have $S_{\beta\gamma}^{\cdot\cdot\alpha} = 0$ and

$$\Gamma^{\kappa}_{\mu\bar{\lambda}} = g^{\bar{\rho}\kappa}\partial_{[\bar{\lambda}} g_{\bar{\rho}]\mu} = 0$$

from which

$$\partial_{[\bar{\lambda}} g_{\bar{\rho}]\mu} = 0 \quad \text{or} \quad \partial_{[\lambda} g_{\rho]\bar{\mu}} = 0.$$

This shows that there exists locally a function $\Phi(\xi,\bar{\xi})$ such that

(8.41) $$g_{\bar{\mu}\lambda} = \partial_{\bar{\mu}}\partial_{\lambda}\Phi.$$

We quote here some of the results in the differential geometry in the large.

Theorem (Bochner)[4]: In a compact Kählerian manifold for which the Hermitian tensor $R_{\bar{\mu}\lambda}$ is positive (negative) definite, that is, $R_{\bar{\mu}\lambda} v^{\bar{\mu}} v^{\lambda} > 0$ ($R_{\bar{\mu}\lambda} v^{\bar{\mu}} v^{\lambda} < 0$) , there does not exist a covariant (contravariant) analytic vector field except a zero vector.

Theorem[5]: In a compact Kähler manifold, a vector $(w_{\lambda}, 0)$ is harmonic if and only if w_{λ} is analytic in ξ^{κ} and $(0, w_{\bar{\lambda}})$ is harmonic if and only if $w_{\bar{\lambda}}$ is analytic in $\xi^{\bar{\kappa}}$.

Therefore, since $(w_{\lambda}, w_{\bar{\lambda}})$ is harmonic if and only if $(w_{\lambda}, 0)$ and $(0, w_{\bar{\lambda}})$ are analytic, $(w_{\lambda}, w_{\bar{\lambda}})$ is harmonic if and only if w_{λ} are analytic in ξ^{κ} and $w_{\bar{\lambda}}$ are analytic in $\xi^{\bar{\kappa}}$.

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1) A. Lichnerowicz: Un théorème sur les espaces homogènes complexes. Archiv der Math., 5(1954), 207-215.

2) J.A. Schouten and D. van Dantzig: loc.cit.

3) K. Yano: On three remarkable affine connexions in almost Hermitian spaces. Indag.Math. 12(1954).

4) S. Bochner: Vector fields and Ricci curvature. Bull.A.M.S., 52(1946), 776-797.

5) W.V.D. Hodge: The theory of harmonic integrals and its application (1940).

Theorem (Eckmann and Guggenheimer)[1]: In a compact Kähler manifold, a p-vector $w_{\alpha_1 \cdots \alpha_p}$, whose non zero components are only $w_{\lambda_1 \cdots \lambda_p}$ and $w_{\bar{\lambda}_1 \cdots \bar{\lambda}_p}$, is harmonic if and only if the components $w_{\lambda_1 \cdots \lambda_p}$ are analytic in ξ^κ and $w_{\bar{\lambda}_1 \cdots \bar{\lambda}_p}$ are analytic in $\xi^{\bar{\kappa}}$.

Theorem (Hodge)[2]: In a compact Kähler manifold of real dimension 2n, we have

$$B_0 \leqq B_2 \leqq B_4 \leqq \cdots \leqq B_{2[\frac{n}{2}]},$$
$$B_1 \leqq B_3 \leqq B_5 \leqq \cdots \leqq B_{2[\frac{n}{2}]+1},$$

and $B_{p+2} - B_p$ $(p+2 \leqq n)$ is equal to the number of linearly independent (with constant coefficients) self-conjugate effective harmonic tensors of valence $p+2$.

In this statement an effective harmonic tensor means a harmonic tensor $w_{\alpha_1 \cdots \alpha_p}$ satisfying

(8.42) $$F^{\beta\alpha} w_{\beta\alpha\alpha_3 \cdots \alpha_p} = 0 \quad \text{or} \quad g^{\bar{\mu}\lambda} w_{\bar{\mu}\lambda\alpha_3 \cdots \alpha_p} = 0.$$

Now if two vectors u^α and v^β satisfy the conditions

$$v^\kappa = iu^\kappa, \quad v^{\bar{\kappa}} = -iu^{\bar{\kappa}} \quad \text{or} \quad v^\alpha = F_\beta^{\ \alpha} u^\beta,$$

the $\mathcal{E}_2$ determined by u^α and v^α is called an invariant $\mathcal{E}_2$ or a holomorphic $\mathcal{E}_2$ and the sectional curvature belonging to such an $\mathcal{E}_2$ a holomorphic sectional curvature.

For a holomorphic $\mathcal{E}_2$, we have

$$k = -\frac{K_{\delta\gamma\beta\alpha} u^\delta v^\gamma u^\beta v^\alpha}{(g_{\delta\alpha} g_{\gamma\beta} - g_{\gamma\alpha} g_{\delta\beta}) u^\delta v^\gamma u^\beta v^\alpha} = \frac{2K_{\bar{\nu}\mu\bar{\lambda}\kappa} u^{\bar{\nu}} u^\mu u^{\bar{\lambda}} u^\kappa}{(g_{\bar{\nu}\mu} g_{\bar{\lambda}\kappa} + g_{\bar{\nu}\kappa} g_{\bar{\lambda}\mu}) u^{\bar{\nu}} u^\mu u^{\bar{\lambda}} u^\kappa}.$$

Thus, if we assume that, at all points of the manifold, the holomorphic sectional curvature are all the same, then we must have

(8.43) $$K_{\bar{\nu}\mu\bar{\lambda}\kappa} = \frac{k}{2}(g_{\bar{\nu}\mu} g_{\bar{\lambda}\kappa} + g_{\bar{\nu}\kappa} g_{\bar{\lambda}\mu}).$$

It can easily be proved that k is a constant. Such a space is called a Kählerian manifold of constant holomorphic sectional curvature.[3]

The equation (8.43) can also be written as [4]

(8.44) $$K_{\delta\gamma\beta\alpha} = \frac{k}{4}[(g_{\delta\alpha} g_{\gamma\beta} - g_{\gamma\alpha} g_{\delta\beta}) - (F_{\delta\alpha} F_{\gamma\beta} - F_{\gamma\alpha} F_{\delta\beta}) - 2F_{\delta\gamma} F_{\beta\alpha}].$$

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1) B. Eckmann and H. Guggenheimer: Formes différentielles et métrique hermitienne sans torsion, I,II. C.R. 229(1949), 464-466; 489-491.

2) W.V.D. Hodge: loc. cit.

3) J.A. Schouten and D.van Dantzig: Über unitäre Geometrien konstanter Krümmung. Proc. Kon.Akad.v.Wet. 34(1931), 1293-1304.

N.S. Hawley: Constant holomorphic curvature: Canadian J.of Math., 5(1953),53-56.

4) K. Yano and I. Mogi: Sur les variétés pseudokählériennes à courbure holomorphique constante. C.R., 237(1953), 962-964.

Theorem (Bochner)[1]: In a Kählerian manifold of constant holomorphic curvature, for a general sectional curvature K, we have

$$0 < \tfrac{1}{4}k \leqq K \leqq k \qquad \text{if} \quad k > 0,$$

and

$$k \leqq K \leqq \tfrac{1}{4}k < 0 \qquad \text{if} \quad k < 0,$$

where the upperlimit in the first case (lower limit in the second case) is attained when the section is holomorphic and the lower limit in the first case (upper limit in the second case) is attained when the inner product of two vectors defining the section is real valued.

Theorem: In a compact Kähler manifold of positive constant holomorphic curvature, we have

$$B_{2l} = 1, \quad B_{2l+1} = 0 \quad (0 \leqq 2l,\ 2l+1 \leqq n).$$

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1) S. ~~Bochner~~ner: Curvature in Hermitian Manifolds. Bull. A.M.S., 53(1947). ~~179~~-195.

9. Almost complex spaces.

In a complex space X_n covered by a set of neighbourhoods with complex coordinates $\xi^\kappa = \eta^\kappa + i\zeta^\kappa$ $(\xi^{\bar\kappa} = \eta^\kappa - i\zeta^\kappa)$; $\kappa, \lambda, \mu, \ldots = 1, 2, \ldots, n$; there exists a tensor field $F_\beta^{\cdot\alpha}$; $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n; \bar 1, \bar 2, \ldots, \bar n$, whose components are given by

$$(9.1) \qquad F_\lambda^{\cdot\kappa} = +i\delta_\lambda^\kappa, \quad F_{\bar\lambda}^{\cdot\kappa} = 0, \quad F_\lambda^{\cdot\bar\kappa} = 0, \quad F_{\bar\lambda}^{\cdot\bar\kappa} = -i\delta_{\bar\lambda}^{\bar\kappa}$$

and which satisfies

$$(9.2) \qquad F_\gamma^{\cdot\beta} F_\beta^{\cdot\alpha} = -A_\gamma^\alpha .$$

In such a space, the differential equations

$$(9.3) \quad (a) \quad \tfrac{1}{2}(A_\beta^\alpha - iF_\beta^{\cdot\alpha})d\xi^\beta = 0 \qquad (b) \quad \tfrac{1}{2}(A_\beta^\alpha + iF_\beta^{\cdot\alpha})d\xi^\beta = 0$$

are completely integrable. In fact, (a) admits the solutions $\xi^\kappa =$ constants and (b) admits the solutions $\xi^{\bar\kappa}$ = constants.

When, in a 2n-dimensional real space X_{2n} of class C^r $(r \geqq 2)$, there is given a mixed tensor field $F_i^{\cdot h}$; $h, i, j, \ldots = 1, 2, \ldots, 2n$ satisfying

$$(9.4) \qquad F_i^{\cdot l} F_l^{\cdot h} = -A_i^h ,$$

we say that the space X_{2n} admits an almost complex structure and call such a space an almost complex space.[1]

If there exists a complex coordinate system with respect to which the tensor $F_i^{\cdot h}$ has the components (9.1), then in a domain in which two such coordinate systems ξ^α and $\xi^{\alpha'}$ are valid, we have

$$(9.5) \qquad \frac{\partial \xi^\alpha}{\partial \xi^{\alpha'}} F_{\beta'}^{\cdot\alpha'} = \frac{\partial \xi^\beta}{\partial \xi^{\beta'}} F_\beta^{\cdot\alpha} ,$$

from which it follows that ξ^κ are functions of $\xi^{\kappa'}$ only and $\xi^{\bar\kappa}$ are functions of $\xi^{\bar\kappa'}$ only. Thus the space is a complex space. In this case, we say that the almost complex structure is induced by a complex structure.

In order that an almost complex structure $F_i^{\cdot h}$ be induced by a complex structure, it is necessary that the space be of class C^ω, that the $F_i^{\cdot h}$ be also of class C^ω and that the equations

$$(9.6) \quad (a) \qquad B_i^h d\xi^i = 0 , \qquad (b) \qquad C_i^h d\xi^i = 0$$

be completely integrable, where

$$(9.7) \qquad B_i^h \stackrel{\text{def}}{=} \tfrac{1}{2}(A_i^h - iF_i^{\cdot h}) , \qquad C_i^h \stackrel{\text{def}}{=} \tfrac{1}{2}(A_i^h + iF_i^{\cdot h})$$

and consequently

$$(9.8) \qquad A_i^h = B_i^h + C_i^h , \qquad F_i^{\cdot h} = i(B_i^h - C_i^h).$$

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1) C. Ehresmann: Sur les variétés presque complexes. Proc.Int.Congr. Math. (1952), 412-419.

Conversely, if in a 2n-dimensional space with an almost complex structure $F_i^{\cdot h}$ of class C^ω, the equations (9.6.a) and (9.6.b) are completely integrable, we get, denoting the solutions of (9.6.a) and (9.6.b) by $\xi^{\kappa'} = \xi^{\kappa'}(\xi^i)$ = const. and $\xi^{\bar\kappa'} = \xi^{\bar\kappa'}(\xi^i)$ = const. respectively,

(9.9) $$\frac{\partial \xi^h}{\partial \xi^{\bar\kappa'}} = iF_i^{\cdot h}\frac{\partial \xi^i}{\partial \xi^{\bar\kappa'}}, \quad \frac{\partial \xi^h}{\partial \xi^{\kappa'}} = -iF_i^{\cdot h}\frac{\partial \xi^i}{\partial \xi^{\kappa'}},$$

which shows that $F_i^{\cdot h}$ has the components (9.1) in the coordinate system $(\xi^{\kappa'}, \xi^{\bar\kappa'})$. Thus we have [2)]

Theorem 9.1. In order that an almost complex structure $F_i^{\cdot h}$ of class C^ω be induced by a complex structure, it is necessary and sufficient that (9.6.a) and (9.6.b) be completely integrable.

Such a complex structure is sometimes said to be integrable.

Since the conditions of complete integrability of (9.6.a) and (9.6.b) are given by

(9.10) $$\begin{cases} C_j^\ell C_i^k \partial_{[\ell} B_{k]}^h = \frac{1}{8}(N_{ji}^{\cdot\cdot h} - iN_{ji}^{\cdot\cdot \ell}F_\ell^{\cdot h}) = 0, \\ B_j^\ell B_i^k \partial_{[\ell} C_{k]}^h = \frac{1}{8}(N_{ji}^{\cdot\cdot h} + iN_{ji}^{\cdot\cdot \ell}F_\ell^{\cdot h}) = 0, \end{cases}$$

where $N_{ji}^{\cdot\cdot h}$ denotes the tensor

(9.11) $$N_{ji}^{\cdot\cdot h} \overset{\text{def}}{=} 2F_{[j}^{\cdot \ell}(\partial_{|\ell|}F_{i]}^{\cdot h} - \partial_{i]}F_\ell^{\cdot h}),$$

we have

Theorem 9.2.[3)] In order that an almost complex structure $F_i^{\cdot h}$ of class C^ω be integrable, it is necessary and sufficient that $N_{ji}^{\cdot\cdot h}$ vanish identically.

A Concomitant of the form (9.11) was found for the most general case by A. Nijenhuis. [4)] We call $N_{ji}^{\cdot\cdot h}$ the Nijenhuis tensor of $F_i^{\cdot h}$. The Nijenhuis tensor satisfies the following identities:

(9.12) $$N_{(ji)}^{\cdot\cdot\cdot h} = 0, \quad N_{ji}^{\cdot\cdot i} = 0,$$

(9.13) $$N_{j\ell}^{\cdot\cdot h}F_i^{\cdot \ell} = -N_{ji}^{\cdot\cdot \ell}F_\ell^{\cdot h} = -N_{i\ell}^{\cdot\cdot h}F_j^{\cdot \ell},$$ [5)]

(9.14) $$N_{ji}^{\cdot\cdot h} + F_j^{\cdot \ell}F_i^{\cdot k}N_{\ell k}^{\cdot\cdot h} = 0.$$

2) Cf. A. Lichnerowicz: Généralisations de la géométrie kählerienne globale. Coll. de Géom. Diff.Louvain, (1951), 99-122.

3) B. Eckmann and A. Fröhlicher: Sur l'intégrabilité de structures presque complexes. C.R., 232 (1951), 22 4-2286;
E. Calabi and D.C. Spencer: Completely integrable almost complex manifolds. Bull. Amer.Math.Soc., 57(1951), 254-255.

4) A. Nijenhuis: X_{n-1}-forming sets of eigenvectors, Indagationes Mathematicae, 8(1951), 200-212;
J.A. Schouten: Sur les tenseurs de V_n aux direction principales V_{n-1}-normales. Coll.de Géom.Diff.Louvain. (1951), 67-70.

5) B. Eckmann: Sur les structures complexes et presque complexes. Géom. Diff.Coll.Inter.de C.N.R.S., Strasbourg (1953), 151-159.

An almost complex structure which needs not being of class C^ω is called a pseudo-complex structure if $N_{ji}^{\cdot\cdot h} = 0$. A space with a pseudo-complex structure is called a pseudo-complex space.

Is is always possible to introduce in an almost complex manifold a linear connexion Γ_{ji}^h such that $\nabla_j F_i^h = 0$. If $\overset{\circ}{\Gamma}{}_{ji}^h$ is an arbitrary symmetric connexion and $T_{ji}^{\cdot\cdot h} \overset{\text{def}}{=} \Gamma_{ji}^h - \overset{\circ}{\Gamma}{}_{ji}^h$, we have

$$0 = \nabla_j F_i^h = \overset{\circ}{\nabla}_j F_i^h + T_{jm}^{\cdot\cdot h} F_i^m - T_{ji}^{\cdot\cdot \ell} F_\ell^h,$$

from which

$$\tfrac{1}{2}(\overset{\circ}{\nabla}_j F_i^\ell) F_\ell^h = -\tfrac{1}{2} T_{ji}^{\cdot\cdot h} - \tfrac{1}{2} T_{jm}^{\cdot\cdot \ell} F_i^m F_\ell^h =$$
$$= -\tfrac{1}{2}(A_i^m A_\ell^h + F_i^m F_\ell^h) T_{jm}^{\cdot\cdot \ell}.$$

The operator ${}^*O = \frac{1}{2}(AA + FF)$ is idempotent but not reversible and from this it follows that there are more solutions and that

(9.15) $$T_{ji}^{\cdot\cdot h} = -\tfrac{1}{2}(\overset{\circ}{\nabla}_j F_i^\ell) F_\ell^h$$

is one of them (compare with 9.24).

To this solution every term can be added that is made zero by the operator ${}^*O$, for instance, $+\frac{1}{2}(\overset{\circ}{\nabla}_{[j} F_{i]}^{\ell}) F_\ell^h + \frac{1}{2}(\overset{\circ}{\nabla}_{[j} F_{\ell]}^{h}) F_i^\ell$.

Then we get the solution

(9.16) $$T_{ji}^{\cdot\cdot h} = -\tfrac{1}{2}(\overset{\circ}{\nabla}_{(j} F_{i)}^{\ell}) F_\ell^h + \tfrac{1}{2}(\overset{\circ}{\nabla}_{[j} F_{\ell]}^{h}) F_i^\ell.$$ [5)]

On the other hand, the Nijenhuis tensor $N_{ji}^{\cdot\cdot h}$ can be written also in the form

(9.17) $$N_{ji}^{\cdot\cdot h} = 2 F_{[j}^{\ell}(\nabla_{|\ell|} F_{i]}^h - \nabla_{i]} F_\ell^h) +$$
$$+ 2(S_{ji}^{\cdot\cdot h} - F_j^\ell F_i^k S_{\ell k}^{\cdot\cdot h} + F_j^\ell F_k^h S_{\ell i}^{\cdot\cdot k} - F_i^\ell F_k^h S_{\ell j}^{\cdot\cdot k}),$$

where ∇_j denotes the covariant differentiation with respect to an arbitrary linear connexion and $S_{ji}^{\cdot\cdot h}$ its torsion tensor.

Thus, if the space is pseudo-complex and if we introduce a linear connexion such that $\nabla_j F_i^h = 0$, then the torsion tensor satisfies

(9.18) $$S_{ji}^{\cdot\cdot h} - F_j^\ell F_i^k S_{\ell k}^{\cdot\cdot h} + F_j^\ell F_\ell^h S_{\ell i}^{\cdot\cdot k} - F_i^\ell F_k^h S_{\ell j}^{\cdot\cdot k} = 0.$$

Conversely if we can introduce, in an almost complex space a linear connexion such that $\nabla_j F_i^h = 0$ and (9.18) holds, then the space is pseudo-complex. Thus we have [6)]

Theorem 9.3 In order that an almost complex space be a pseudo-complex space, it is necessary and sufficient that we can introduce in it a linear connexion such that $\nabla_j F_i^h = 0$ and that (9.18) holds.

Furthermore, if the space is pseudo-complex, we can introduce a symmetric linear connexion such that $\nabla_j F_i^h = 0$, because the linear

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6) K. Yano and I. Mogi: Sur les variétés pseudo-kählériennes à courbure holomorphique constante. C.R. 237(195?), 962-96?.

connexion (9.16) satisfies

$$S_{ji}{}^{h} = -\tfrac{1}{4} N_{ji}{}^{h} = 0. \tag{9.19}$$

Conversely if we can introduce in an almost complex space, a symmetric linear connexion such that $\nabla_j F_i{}^h = 0$, then $N_{ji}{}^h = 0$ and the space is pseudo-complex. Thus we have

Theorem 9.4 [7] In order that an almost complex space be a pseudo-complex space, it is necessary and sufficient that we can introduce in it a symmetric linear connexion such that $\nabla_j F_i{}^h = 0$.

If an almost (pseudo-) complex space has a positive definite Riemannian metric $ds^2 = g_{ji}\, d\xi^j d\xi^i$ which satisfies

$$F_j{}^l F_i{}^k g_{lk} = g_{ji}, \tag{9.20}$$

the space is called an almost (pseudo-) Hermitian space. In this case the tensor $F_{ih} \overset{\text{def}}{=} F_i{}^l g_{lh}$ is antisymmetric in i and h. A. Lichnerowicz [8] has proved

Theorem 9.5 In an almost complex space, it is always possible to define a Hermitian metric.

In fact, let a_{ji} be a tensor which defines a positive definite Riemannian metric to an almost complex space and let

$$g_{ji} = \tfrac{1}{2}(a_{ji} + F_j{}^l F_i{}^k a_{lk}), \tag{9.21}$$

then g_{ji} defines another positive definite Riemannian metric and satisfies (9.20).

The equation (9.20) and the antisymmetry of the tensor F_{ih} show that the transformation $v^h \to F_i{}^h v^i$ changes a vector v^h into a vector orthogonal to it and does not change its length.

Moreover, we can easily see that

$$F_{jih} \overset{\text{def}}{=} 3\partial_{[j} F_{ih]} \tag{9.22}$$

are components of an antisymmetric tensor.

If an almost (pseudo-) Hermitian space satisfies $F_{jih} = 0$, the space is called an almost (pseudo-) Kählerian space.

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7) B. Eckmann: loc.cit. in 5).
W.V.D. Hodge: Structure problems for complex manifolds. Rend.Mat.Ser.V, 11(1952), 101-110.
E.M. Patterson: A characterisation of Kähler manifolds in terms of parallel fields of planes. J. London Math.Soc., 28 (1953), 260-269.

8) A. Lichnerowicz: Un théorème sur les espaces homogènes complexes. Arch.der Math.,5(1954), 207-215.

The relations between these spaces may be seen in the following diagram:

$$\begin{tikzcd}[column sep=large]
\begin{array}{c}\text{Almost complex}\\ \text{structure}\\ \left(\begin{array}{c}X_{2n} \text{ of class } C^{\tau} \text{ with}\\ F_i{}^j F_j{}^h = -A_i^h\end{array}\right)\end{array} \arrow[r, "\begin{array}{c}g_{ih} \text{ with}\\ F_i{}^l F_h{}^k g_{lk} = g_{ih}\end{array}"] \arrow[d, "N_{ji}{}^h = 0"] & \begin{array}{c}\text{Almost Hermitian}\\ \text{structure}\end{array} \arrow[r, "\partial_{[j} F_{ih]} = 0"] \arrow[d, "N_{ji}{}^h = 0"] & \begin{array}{c}\text{Almost Kählerian}\\ \text{structure}\end{array} \arrow[d, "N_{ji}{}^h = 0"] \\
\begin{array}{c}\text{Pseudo-complex}\\ \text{structure}\end{array} \arrow[r] \arrow[d, "\tau = \omega"] & \begin{array}{c}\text{Pseudo-Hermitian}\\ \text{structure}\end{array} \arrow[r] \arrow[d, "\tau = \omega"] & \begin{array}{c}\text{Pseudo-Kählerian}\\ \text{structure}\end{array} \arrow[d, "\tau = \omega"] \\
\text{Complex structure} \arrow[r] & \text{Hermitian structure} \arrow[r] & \text{Kählerian structure}
\end{tikzcd}$$

In an almost Hermitian space, we denote by $\mathring{\nabla}$ the covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop j\,i} \right\}$. If $\mathring{\nabla}_j F_{ih}$ vanishes, then the tensors $N_{ji}{}^h$ and F_{jih} vanish too, and consequently the space is pseudo-Kählerian.

Conversely, since the Nijenhuis tensor can be written also in the form

$$N_{jih} = -2\left(F_{[j}{}^{l} F_{i]hl} - F_j{}^l \mathring{\nabla}_h F_{il} \right), \tag{9.23}$$

$\mathring{\nabla}_j F_{ih}$ vanishes if the tensors $N_{ji}{}^h$ and F_{jih} vanish.

Thus we have

<u>Theorem 9.6</u> [9] In order that an almost Hermitian space be pseudo-Kählerian, it is necessary and sufficient that $\mathring{\nabla}_j F_{ih}$ vanish.

In an almost Hermitian space, the four following connexions occur in literature:

(9.24) (I)[10] $$\overset{1}{\Gamma}{}_{ji}^{h} = \left\{ {h \atop j\,i} \right\} - \tfrac{1}{2}(\mathring{\nabla}_j F_{il}) F^{lh},$$

(9.25) (II)[11] $$\overset{2}{\Gamma}{}_{ji}^{h} = \left\{ {h \atop j\,i} \right\} - \tfrac{1}{2}(\mathring{\nabla}_j F_{il} + \mathring{\nabla}_i F_{jl} + \mathring{\nabla}_l F_{ji}) F^{lh},$$

(9.26) (III)[11] $$\overset{3}{\Gamma}{}_{ji}^{h} = \left\{ {h \atop j\,i} \right\} - \tfrac{1}{2}(\mathring{\nabla}_j F_{il} - \mathring{\nabla}_i F_{jl} - \mathring{\nabla}_l F_{ji}) F^{lh},$$

(9.27) (IV)[12] $$\overset{4}{\Gamma}{}_{ji}^{h} = \left\{ {h \atop j\,i} \right\} - \tfrac{1}{2}(\mathring{\nabla}_j F_{il} + \mathring{\nabla}_i F_{jl}) F^{lh} + \tfrac{1}{2} F_{jl}(\mathring{\nabla}_k F_i{}^l) g^{kh} - \tfrac{1}{4} N_{hij}.$$

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9) B. Eckmann: loc.cit. in 5).
K. Yano: Quelques remarques sur les variétés presque complexes. Bull.Soc.Math.France, to appear.
K. Yano and I. Mogi: loc.cit. in 6).

10) A. Lichnerowicz: loc.cit. in 8).

11) K. Yano: On three remarkable affine connexions in almost Hermitian spaces. Indagationes Mathematicae, to appear.

12) J.A. Schouten and K. Yano: On an intruisic connexion in an X_{2n} with an almost Hermitian structure. Indagationes Mathematicae, to appear.

All these four connexions satisfy

(9.28) $$\nabla_j F_{ih} = 0.$$

We shall give a geometrical characterisation for each of these connexions. In the following, we put

(9.29) $$\Gamma_{ji}^h = \left\{ {h \atop j\,i} \right\} + T_{ji}^{\cdot\cdot h}.$$

The first connexion is characterized by the following property: Take an arbitrary contravariant vector u^h. We can associate with this contravariant vector a covariant vector defined by $F_{ih} u^h$. The hyperplane representing the covariant vector $F_{ih} u^h$ contains the direction representing the contravariant vector u^h. So this is a so-called null system. Now we transport the contravariant vector u^h parallelly with respect to the Riemannian connexion from the point ξ^h to the point $\xi^h + d\xi^h$. Then we get at $\xi^h + d\xi^h$

(9.30) $$u^h - d\xi^j \left\{ {h \atop j\,i} \right\} u^i.$$

Next we transport the covariant vector $F_{ih} u^h$ parallelly with respect to the linear connexion Γ_{ji}^h from the point ξ^h to $\xi^h + d\xi^h$. Then we get at $\xi^h + d\xi^h$

(9.31) $$F_{ih} u^h - d\xi^j \left(\left\{ {\ell \atop j\,i} \right\} + T_{ji}^{\cdot\cdot \ell} \right) F_{\ell h} u^h.$$

We assume that the hyperplane representing (9.31) contains the direction representing (9.30) for any vector u^h and for any displacement $d\xi^h$. This condition can be expressed as

(9.32) $$T_{ji}^{\cdot\cdot \ell} F_{\ell h} + T_{jh}^{\cdot\cdot \ell} F_{\ell i} = 0.$$

From the equations (9.28) and (9.32), we get (9.24).

With respect to the first connexion, we have

(9.33) $$\overset{1}{\nabla}_j g_{ih} = 0,$$

(9.34) $$\overset{1}{T}_{jih} = \frac{1}{4} N_{hij} - \frac{1}{2} F_{[h}^{\ \ \ell} F_{i]j\ell}.$$

In the Hermitian case, this connexion reduces to the connexion of Lichnerowicz mentioned in § 8.

The second connexion is characterized by the following property: Consider a contravariant vector u^h and a covariant vector $F_{ih} u^h$ which contains the vector u^h. We transport the covariant vector $F_{ih} u^h$ parallelly with respect to the Riemannian connexion, and with respect to the linear connexion Γ_{ji}^h respectively from the point ξ^h to the point $\xi^h + u^h \varepsilon$ where ε is an infinitesimal. Then we get at $\xi^h + u^h \varepsilon$

$$F_{ih} u^h - u^j \varepsilon \left\{ {\ell \atop j\,i} \right\} F_{\ell h} u^h$$

and

$$F_{ih} u^h - u^j \varepsilon \left[\left\{ {\ell \atop j\,i} \right\} + T_{ji}^{\cdot\cdot \ell} \right] F_{\ell h} u^h.$$

We assume that these two vectors coincide for any vector u^h. This condition can be expressed as

(9.35) $$T_{ji}{}^{\ell} F_{\ell h} + T_{hi}{}^{\ell} F_{\ell j} = 0.$$

From the equations (9.28) and (9.35), we obtain (9.25).

With respect to the second connexion, we have

(9.36) $$\overset{2}{T}_{jih} = \tfrac{1}{2} N_{hji} - \tfrac{1}{2} F_j{}^{\ell} F_{ih\ell}.$$

Thus in a pseudo-Hermitian space we have $\overset{2}{T}_{jih} = -\frac{1}{2} F_j{}^{\ell} F_{ih\ell}$ and consequently

(9.37) $$\overset{2}{\nabla}_j g_{ih} = 0.$$

In a Hermitian space, this connexion reduces to the connexion of Schouten-Van Dantzig mentioned in § 8.

The third connexion is characterized by the following property: We assume that the geodesics with respect to Γ^h_{ji} coincides with the geodesics of the Riemannian connexion. Then we have $\Gamma^h_{(ji)} = \left\{ {h \atop j\,i} \right\}$,which can be expressed as

(9.38) $$T_{ji}{}^{\ell} F_{\ell h} + T_{ij}{}^{\ell} F_{\ell h} = 0$$

From the equations (9.28) and (9.38), we get (9.26).

With respect to the third connexion, we have

(9.39) $$\overset{3}{T}_{jih} = \tfrac{1}{2} N_{jih} - 3 F_{[j}{}^{\ell} \overset{\circ}{\nabla}_{|\ell|} F_{ih]} =$$
$$= \tfrac{1}{2} N_{jih} + \tfrac{1}{2} F_j{}^m F_i{}^{\ell} F_h{}^k F_{m\ell k}.$$

Thus in a pseudo-Hermitian space, we have $\overset{3}{T}_{jih} = \frac{1}{2} F_j{}^m F_i{}^{\ell} F_h{}^k F_{m\ell k}$ and consequently

(9.40) $$\overset{3}{\nabla}_j g_{ih} = 0.$$

In a Hermitian space, this connexion reduces to the third connexion discussed in § 8.

The fourth connexion is characterized by the following property: We first assume that $\overset{4}{\nabla}_j g_{ih} = 0$, $\overset{4}{\nabla}_j F_{ih} = 0$. Next we assume that, in two directions u^h and $F_i{}^h u^i$, there exists an infinitesimal parallelogram. The last condition is equivalent to

(9.41) $$S_{j\ell}{}^{h} F_i{}^{\ell} + S_{i\ell}{}^{h} F_j{}^{\ell} = 0.$$

From these conditions we get (9.27). In a Hermitian space, this connexion reduces to the connexion of Schouten-Van Dantzig discussed in § 8.

For this fourth connexion, we get instead of (9.17) the simpler formulae

$$O^{m\ell}_{ji} S_{m\ell}{}^{h} = S_{ji}{}^{h},$$
$$N_{ji}{}^{h} = 8\, {}^*O^{mh}_{j\ell} S_{jm}{}^{\ell}.$$

where

$$O^{m\ell}_{ji} = \tfrac{1}{2}(A^m_j A^{\ell}_i - F_j{}^m F_i{}^{\ell}),$$
$${}^*O^{mh}_{j\ell} = \tfrac{1}{2}(A^m_j A^h_{\ell} + F_j{}^m F_{\ell}{}^h).$$

In a pseudo-Kählerian space, we have $\overset{\circ}{\nabla}_j g_{ih}=0$, $\overset{\circ}{\nabla}_j F_{ih}=0$. Applying the Ricci formula to $F_i^{\cdot h}$, we get

(9.42) $$K_{kjl}^{\cdot\cdot\cdot h} F_i^{\cdot l} = K_{kji}^{\cdot\cdot\cdot l} F_l^{\cdot h},$$

(9.43) $$K_{kjil} F_h^{\cdot l} = K_{kjhl} F_i^{\cdot l},$$

(9.44) $$K_{kjih} = K_{kjml} F_i^{\cdot m} F_h^{\cdot l}, \text{ or } O_{ih}^{ml} K_{kjml} = 0.$$

Transvecting (9.42) with g^{ji}, we find

$$K_k^{\cdot l} F_l^{\cdot h} = K_{kml}^{\cdot\cdot\cdot h} F^{ml} = \tfrac{1}{2}(K_{kml}^{\cdot\cdot\cdot h} - K_{klm}^{\cdot\cdot\cdot h}) F^{ml}$$

from which

(9.45) $$K_k^{\cdot l} F_l^{\cdot h} = -\tfrac{1}{2} K_{mlk}^{\cdot\cdot\cdot h} F^{ml}$$

Thus

$$K_k^{\cdot l} F_{lh} + K_h^{\cdot l} F_{lk} = 0,$$

from which

(9.46) $$K_j^{\cdot i} = - K_m^{\cdot l} F_j^{\cdot m} F_l^{\cdot i},$$

(9.47) $$K_{ji} = K_{ml} F_j^{\cdot m} F_i^{\cdot l}, \text{ or } O_{ji}^{ml} K_{ml} = 0.$$

Using these relations, we can prove

Theorem 9.7 [13]. If a pseudo-Kählerian space is of constant curvature, then it is of zero curvature.

Theorem 9.8 [14]. If a pseudo-Kählerian space is conformally Euclidian, it is of zero curvature.

Theorem 9.9 [15]. A projective correspondence between two pseudo-Kählerian spaces is necessarily affine.

Theorem 9.10 [16]. A conformal correspondence between two pseudo-Kählerian spaces is necessarily a trivial one.

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13). S.Bochner: Curvature in Hermitian manifolds. Bull.Amer.Math.Soc. 53(1947), 179-195.
14). K.Yano and I.Mogi: On real representations of Kählerian manifolds. Ann.of Math., 61(1955), 170-189.
15). S.Bochner: see 13).
W.J.Westlake: Hermitian spaces in geodesic correspondence. Proc.Amer.Math.Soc., 56(1954), 301-306.
K.Yano: Sur les correspondence projective entre deux espaces pseudo-hermitiens. C.R. 239(1954).
16). W.J.Westlake: Conformally Kähler manifolds. Proc.Cambridge Philos.Soc. 50(1954), 16-19.

Theorem 9.11[17]. A necessary and sufficient condition that a 2n-dimensional pseudo-Hermitian space be conformal to a pseudo-Kählerian space is that, for $2n > 4$,

$$C_{jih} \overset{\text{def}}{=} F_{jih} - \frac{1}{2(n-1)}\left(F_{ji}F_h + F_{ih}F_j + F_{hj}F_i \right) = 0 \tag{9.48}$$

and for $2n = 4$

$$C_{ji} \overset{\text{def}}{=} 2\,\partial_{[j}F_{i]} = 0, \tag{9.49}$$

where $F_j = F_{jih}F^{ih}$.

Now we put

$$H_{kj} \overset{\text{def}}{=} K_{kjih}F^{ih} = -2\,K_k^{\;\;l}F_{lj}. \tag{9.50}$$

We see that the H_{kj} is zero if and only if $K_{ji} = 0$ and that

$$F^{kj}H_{kj} = -2K. \tag{9.51}$$

Moreover from the Bianchi identity, we have

$$\nabla_{[l}H_{kj]} = 0. \tag{9.52}$$

On the othèr hand, we have

$$\begin{aligned} g^{lk}\nabla_l H_{kj} &= g^{lk}\nabla_l(2\,K_{km}F_j^{\;\;m}) = 2\nabla_l K_m^{\;\;l}F_j^{\;\;m} \\ &= (\nabla_m K)F_j^{\;\;m}. \end{aligned} \tag{9.53}$$

Thus:

Theorem 9.12. The tensor H_{kj} is harmonic if and only if $K = \text{const.}$, and it is effective (that is, $F^{kj}H_{kj} = 0$) if and only if $K = 0$.

Theorem 9.13. If, in a compact pseudo-Kählerian space, $K_{ji} \neq 0$, $K = 0$, then $B_2 \geq 2$. (See the theorem of Hodge at $\mathfrak{P}8$).

Theorem 9.14. If, in a compact pseudo-Kählerian space $K_{ji} \neq 0$, $K = \text{const.} \neq 0$ and $B_2 = 1$, then $K_{ji} = \frac{K}{2n}\,g_{ji}$.

In a pseudo-Kählerian space, we call a field v_i satisfying

$$F_j^{\;\;l}\nabla_i v_l - F_i^{\;\;l}\nabla_l v_j = 0. \tag{9.54}$$

a covariant pseudo-analytic vector field. From (9.54) we can deduce

$$g^{ji}\nabla_j\nabla_i v_h - K_h^{\;\;l}v_l = 0, \tag{9.55}$$

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17) W.J.Westlake: loc.cit. in 16)
K.Yano: Geometria conforme in varietà quasi hermetiane. To appear.

which is a necessary and sufficient condition that a vector field v_h in a compact orientable Riemannian space be harmonic.[18]

Conversely if v_h is harmonic, then we have (9.55), from which

$$g^{ji}\nabla_j\nabla_i F_m^{\cdot h}v_h - K_m^{\cdot h}F_h^{\cdot \ell}v_\ell = 0 \tag{9.56}$$

by virtue of $F_m^{\cdot h}K_h^{\cdot \ell} = K_m^{\cdot h}F_h^{\cdot \ell}$. It follows from this that $F_m^{\cdot h}v_h$ is also harmonic.

Thus from

$$\nabla_j v_\ell = \nabla_\ell v_j \qquad \text{and} \qquad \nabla_i(F_j^{\cdot \ell}v_\ell) - \nabla_j(F_i^{\cdot \ell}v_\ell) = 0$$

we get (9.54), from which

Theorem 9.15. In order that a vector field in a compact pseudo-Kählerian space be covariant pseudo-analytic, it is necessary and sufficient that the vector be harmonic.

Thus applying a theorem of Bochner[19], we obtain

Theorem 9.16. If the Ricci curvature of a compact pseudo-Kählerian space is positive definite, there does not exist a covariant pseudo-analytic vector field.

In a pseudo-Kählerian space, we call a field v^h satisfying

$$\underset{v}{\mathcal{L}} F_i^{\cdot h} = -F_i^{\cdot \ell}\nabla_\ell v^h + F_\ell^{\cdot h}\nabla_i v^\ell = 0 \tag{9.57}$$

a contravariant pseudo-analytic vector field, where $\underset{v}{\mathcal{L}}$ denotes the Lie derivative with respect to v^h. From (9.57) we see that if v^h is contravariant pseudo-analytic, then $F_i^{\cdot h}v^i$ is also contravariant pseudo-analytic. Moreover, if u^h and v^h are both contravariant pseudo-analytic, then denoting by $\underset{u}{\mathcal{L}}$ and $\underset{v}{\mathcal{L}}$ the Lie derivatives with respect to u^h and v^h respectively, we have $\underset{u}{\mathcal{L}} F_i^{\cdot h} = 0$ and $\underset{v}{\mathcal{L}} F_i^{\cdot h} = 0$ from which $(\underset{v}{\mathcal{L}}\underset{u}{\mathcal{L}}) F_i^{\cdot h} = 0$, where $(\underset{v}{\mathcal{L}}\underset{u}{\mathcal{L}})$ denotes the Lie derivative with respect to the vector $\underset{v}{\mathcal{L}} u^h$. Thus we have

Theorem 9.17. If u^h and v^h are both contravariant pseudo-analytic vector fields in a pseudo-Kählerian space, then

$$F_i^{\cdot h}u^i,\ F_i^{\cdot h}v^i,\ \underset{v}{\mathcal{L}} u^h,\ \underset{Fv}{\mathcal{L}} u^h,\ \underset{v}{\mathcal{L}} F_i^{\cdot h}u^i,\ \underset{Fv}{\mathcal{L}} F_i^{\cdot h}u^i$$

are all contravariant pseudo-analytic vectors.

In an almost complex space, we have the following identity

$$\underset{v}{\mathcal{L}} u^h + F_i^{\cdot h}\underset{v}{\mathcal{L}} F_j^{\cdot i}u^j + F_i^{\cdot h}\underset{Fv}{\mathcal{L}} u^i - \underset{Fv}{\mathcal{L}} F_i^{\cdot h}v^i = N_{ji}^{\cdot\cdot h}u^j v^i \tag{9.58}$$

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18). K.Yano and S.Bochner: Curvature and Betti numbers. (1953).

19). S.Bochner: Vector fields and Ricci curvature. Bull.Amer.Math.Soc. 52(1946), 776-797.

from which

Theorem 9.18.[20] In order that an almost complex space be pseudo-complex, it is necessary and sufficient that the left hand side of (9.58) vanish for any vectors u^h and v^h.

If we put

$$w_{ji} = 2\partial_{[j} w_{i]}, \quad u_i = F_i{}^h w_h, \quad u_{ji} = 2\partial_{[j} u_{i]},$$

we have also the following identity

$$2\,O_{ji}^{lk} w_{lk} = N_{ji}{}^h w_h + 2 F_j{}^h O_{ih}^{lk} u_{lk}, \tag{9.59}$$

where

$$2\,O_{ji}^{lk} = A_j^l A_i^k - F_j{}^l F_i{}^k. \tag{9.60}$$

A tensor w_{ji} satisfying $O_{ji}^{lk} w_{lk} = 0$ is said to be hybrid.

Thus we have

Theorem 9.19[21]. A vector field with a hybrid rotation is always transformed by $F_i{}^h$ into a vector field whose rotation is also hybrid, if and only if $N_{ji}{}^h = 0$.

Now from (9.57) we have

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i = 0, \tag{9.61}$$

from which we get the following theorems which hold in a compact orientable pseudo-Kählerian space.

Theorem 9.20. A contravariant pseudo-analytic vector field v^h satisfying $\nabla_i v^i = 0$ is a killing vector.

Theorem 9.21. If the space has a negative definite Ricci curvature there does not exist a contravariant pseudo-analytic vector field other than zero vector.

Theorem 9.22. If a vector is at the same time contravariant and covariant pseudo-analytic, then it is a covariant constant vector field.

Theorem 9.23. If a contravariant pseudo-analytic vector field v^h satisfies $F^{ih} \nabla_i v_h = 0$, then $F_i{}^h v^i$ is a killing vector field.

Theorem 9.24. If a contravariant pseudo-analytic vector field v^h satisfies $g^{ih} \nabla_i v_h = 0$, $F^{ih} \nabla_i v_h = 0$, it is a covariant constant field.

We call a sectional curvature determined by two orthogonal vectors u^h and $F_i{}^h u^i$ the holomorphic sectional curvature.

We can prove [22]

20). B.Eckmann: loc.cit. in 5).
21). J.A.Schouten and K.Yano: loc.cit. in 12).
22). K.Yano and I.Mogi: loc.cit. in 6).

Theorem 9.25. If a pseudo-Kählerian space has a constant holomorphic sectional curvature at any point, then the curvature tensor of the space is of the form

(9.62) $$K_{kjih} = \frac{k}{4}\left[(g_{kh}g_{ji} - g_{jh}g_{ki}) + (F_{kh}F_{ji} - F_{jh}F_{ki}) - 2F_{kj}F_{ih}\right],$$

where k is a constant.

Theorem 9.26. In a pseudo-Kählerian space of constant holomorphic curvature, a general sectional curvature determined by two orthogonal unit vectors u^h and v^h is given by

(9.63) $$K = \frac{k}{4}(1+3a^2),$$

where $a = F_{ih}u^i v^h$ is the cosine of the angle between two unit vectors $F_i{}^h u^i$ and v^h and consequently $a^2 \leqq 1$. Thus

(9.64) $$\frac{k}{4} \leqq K \leqq k \quad \text{for} \quad k > 0$$ $$k \leqq K \leqq \frac{k}{4} \quad \text{for} \quad k < 0.$$

Theorem 9.27. In order that, in a pseudo-Kählerian space, there exist always a two-dimensional totally geodesic subspace passing through an arbitrarily given point and being tangent to an arbitrarily given holomorphic plane at this point, that is, to a plane determined by an arbitrarily given vector u^h and $F_i{}^h u^i$, it is necessary and sufficient that the space be of constant holomorphic curvature.

Theorem 9.28. In order that a pseudo-Kählerian space admit a group of motions which carry any two vectors u^h and $F_i{}^h u^i$ at a point P to any two vectors $'u^h$ and $'F_i{}^h\,'u^i$ at any point $'P$, it is necessary and sufficient that the space be of constant holomorphic curvature.

Theorem 9.29. In a pseudo-Kählerian space of constant holomorphic curvature $k > 0$, the distance between two consecutive conjugate points on a geodesic is constant and is equal to $\frac{2}{\sqrt{k}}\pi$.

§ 10. Groups of transformations.

1°. The Lie derivative of a geometric object.[1)]

In an n-dimensional space of class C^{τ} covered by a set of neighbourhoods with coordinate systems (κ), a geometric object at a point ξ^{κ} is defined as a correspondence between the allowable coordinate systems in regions containing ξ^{κ} and the ordered sets of N numbers satisfying the following conditions:

1) To every coordinate system (κ) there belongs one and only one ordered set of N numbers.
2) If Ω^{Λ}; $\Lambda = 1,2,\dots,N$ corresponds to (κ) and $\Omega^{\Lambda'}$; $\Lambda' = 1', 2', \dots, N'$ to (κ'), the $\Omega^{\Lambda'}$ are given by the equations of the form

$$(10.1)\qquad \Omega^{\Lambda'} = F^{\Lambda}(\Omega^{\Sigma}, \xi^{\kappa}, \xi^{\kappa'}, A^{\kappa'}_{\kappa}, \partial_{\lambda} A^{\kappa'}_{\kappa}, \dots, \partial_{\lambda_p} \dots_{\lambda_2} A^{\kappa'}_{\lambda_1}).$$

3) The functions F^{Λ}, denoted by $F^{\Lambda}(\Omega, \xi, \xi')$ for the sake of shortness, satisfy

$$(10.2)\qquad \begin{cases} F^{\Lambda}(F(\Omega,\xi,\xi'),\xi',\xi'') = F^{\Lambda}(\Omega,\xi,\xi''), \\ F^{\Lambda}(\Omega,\xi,\xi) = \Omega^{\Lambda}, \\ F^{\Lambda}(F(\Omega,\xi,\xi'),\xi',\xi) = \Omega^{\Lambda}. \end{cases}$$

If the functions F^{Λ} contain the partial derivatives of $\xi^{\kappa'}$ with respect to ξ^{κ} of the maximum order p, the object is said to be of class p $(p \leqq \tau)$.

We now consider a point transformation

$$(10.3)\qquad {}'\xi^{\kappa} = f^{\kappa}(\xi^{\nu})\,; \qquad \mathrm{Det}(\partial_{\lambda} f^{\kappa}) \neq 0,$$

which establishes a one-to-one correspondence of class C^{τ} between the points of a region R and those of some other region R'.

The transformation (10.3) associates a point ${}'\xi^{\kappa}$ of R' with a point ξ^{κ} of R. Thus the transformation ${}'\xi \to \xi$ inverse to (10.3) associates a point ξ^{κ} of R with a point ${}'\xi^{\kappa}$ of R'. We now introduce another coordinate system (κ') such that the transform in R by ${}'\xi \to \xi$ has the same coordinates with respect to (κ') as the original point in R' had with respect to (κ). Hence

$$(10.4)\qquad \xi^{\kappa'} = {}'\xi^{\kappa} = f^{\kappa}(\xi^{\nu}).$$

Then we say that the coordinate system (κ) is dragged along by the point transformation ${}'\xi \to \xi$ and we call (κ') the coordinate system dragged along by ${}'\xi \to \xi$.

If a field of a geometric object $\Omega^{\Lambda}(\xi)$ is given with respect to a coordinate system (κ), we define a new field of a geometric object ${}'\Omega^{\Lambda}(\xi)$ by the equation

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1) J.A. Schouten, Ricci Calculus (1954), P.67, P.102.

(10.5) $$'\Omega^{\wedge'}(\xi) = \Omega^{\wedge}('\xi)$$

with respect to the coordinate system (κ'). The difference

$$'\Omega^{\wedge'}(\xi) - \Omega^{\wedge'}(\xi) = \Omega^{\wedge}('\xi) - \Omega^{\wedge'}(\xi)$$

with respect to (κ'), or

(10.6) $$'\Omega^{\wedge}(\xi) - \Omega^{\wedge}(\xi)$$

with respect to (κ) is called the <u>Lie difference of the geometric object $\Omega^{\wedge}(\xi)$ with respect to the point transformation (10.3)</u>. When this Lie difference is zero, we say that the object $\Omega^{\wedge}(\xi)$ is invariant with respect to (10.3).

When the point transformation (10.3) is infinitesimal

(10.7) $$'\xi^{\kappa} = \xi^{\kappa} + v^{\kappa}(\xi)dt,$$

the Lie difference of $\Omega^{\wedge}$ takes the form

(10.8) $$'\Omega^{\wedge}(\xi) - \Omega^{\wedge}(\xi) = \underset{v}{\mathcal{L}}\Omega^{\wedge}dt$$

and is called the <u>Lie differential</u> of $\Omega^{\wedge}$ with respect to (10.7), or <u>with respect to the contravariant vector</u> $v^{\kappa}dt$.

The $\underset{v}{\mathcal{L}}\Omega^{\wedge}$ is called the <u>Lie derivative</u> of $\Omega^{\wedge}$ <u>with respect to</u> v^{κ} and is given by the formula

(10.9) $$\underset{v}{\mathcal{L}}\Omega^{\wedge} = v^{\mu}\partial_{\mu}\Omega^{\wedge} - \sum_{\tau=0}^{p} F^{\wedge\lambda_\tau \cdots \lambda_1}_{\cdot\,\kappa}\,\partial_{\lambda_\tau \cdots \lambda_1}v^{\kappa},$$

where

(10.10) $$F^{\wedge\lambda_\tau \cdots \lambda_1}_{\cdot\,\kappa} = \left[\frac{\partial F^{\wedge}}{\partial(\partial_{\lambda_\tau \cdots \lambda_2}A^{\kappa'}_{\lambda_1})}\right]_{\xi^{\kappa'}=\xi^{\kappa}}$$

2°. <u>The invariance group of a geometric object</u>.

All the transformations which leave invariant a geometric object form a group. This group is called an <u>invariance group</u> of the geometric object. We shall give some well-known examples.

If, in a Riemannian space V_n with the fundamental tensor $g_{\lambda\kappa}$, a point transformation does not change the metric, the transformation is called a <u>motion</u> in a Riemannian space.

<u>Theorem 10.1</u>. In order that an infinitesimal point transformation (10.7) be a motion in a V_n, it is necessary and sufficient that

(10.11) $$\underset{v}{\mathcal{L}}g_{\lambda\kappa} = \nabla_{\lambda}v_{\kappa} + \nabla_{\kappa}v_{\lambda} = 0.$$

This equation is called the <u>Killing equation</u>. A vector satisfying the Killing equation is called a <u>Killing vector</u>.

<u>Theorem 10.2</u> [2] Any closed group of motions in a V_n of class C^{τ} $(\tau \geq 2)$ is a Lie group of motions.

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2) S.B. Myers and N.E. Steenrod: The group of isometrics of a Riemannian manifold. Ann. of Math., 40 (1939), 400-416.

If, in a space L_n with a linear connexion $\Gamma^{\kappa}_{\mu\lambda}$, a point transformation does not change the linear connexion, the transformation is called an affine motion in the L_n.

Theorem 10.3. In order that an infinitesimal point transformation (10.7) be an affine motion in an L_n, it is necessary and sufficient that

(10.12) $$\underset{v}{\mathcal{L}}\Gamma^{\kappa}_{\mu\lambda} = \nabla_{\mu}(\nabla_{\lambda}v^{\kappa} + 2S_{\rho\lambda}{}^{\kappa}v^{\rho}) + R_{\nu\mu\lambda}{}^{\kappa}v^{\nu} = 0,$$

where the $S_{\mu\lambda}{}^{\kappa}$ and the $R_{\nu\mu\lambda}{}^{\kappa}$ are the torsion tensor and the curvature tensor respectively of the space.

Theorem 10.4.[3] The group of all affine motions in a complete L_n of class C^{∞} is a Lie group.

An L_n is said to be complete if and only if every geodesic can be extended for any large value of the affine parameter on it.

If, in a space A_n with a linear symmetric connexion $\Gamma^{\kappa}_{\mu\lambda}$, a point transformation changes any geodesic into a geodesic, the transformation is called a projective motion in the A_n.

Theorem 10.5. In order that an infinitesimal point transformation (10.7) be a projective motion in an A_n, it is necessary and sufficient that

(10.13) $$\underset{v}{\mathcal{L}}\Gamma^{\kappa}_{\mu\lambda} = p_{\mu}A^{\kappa}_{\lambda} + p_{\lambda}A^{\kappa}_{\mu},$$

where p_{μ} is a certain covariant vector.

If, in a Riemannian space V_n, a point transformation does not change its angular metric, the transformation is called a conformal motion.

Theorem 10.6. In order that an infinitesimal point transformation (10.7) be a conformal motion in a V_n, it is necessary and sufficient that

(10.14) $$\underset{v}{\mathcal{L}} g_{\lambda\kappa} = 2\phi g_{\lambda\kappa},$$

where the ϕ is a certain scalar.

Theorem 10.7.[4] A group of transformations in a space of C^{∞} which leave invariant a linear, projective or conformal connexion is a Lie group.

When a field of a geometric object $\Omega^{\wedge}(\xi)$ is given in an n-dimensional space, we call the complete invariance group the largest group of transformations which leave invariant the geometric object $\Omega^{\wedge}(\xi)$.

Studying the integrability conditions of (10.11), we obtain

Theorem 10.8. In a V_n, the maximum order of the complete group of motions is $\frac{1}{2}n(n+1)$ and if a V_n admits such a group, the V_n is of constant curvature.

3) K. Nomizu: A group of affine transformations of an affinely connected space. Proc. Amer.Math.Soc., 4(1953), 816-823.

4) S. Kobayashi: Le groupe des transformations qui laissent invariant le parallélisme. Colloque de Topologie de Strasbourg. (1954).

Studying the integrability conditions of (10.12), (10.13) and (10.14), one can prove

Theorem 10.9. In an n-dimensional space, the maximum orders of complete groups of affine, projective and conformal motions are respectively n^2+n, n^2+2n and $\frac{1}{2}(n+1)(n+2)$.[5] If a space admits such a group, the space is respectively affinely, projectively and conformally Euclidean.

3°. Groups of motions in a Riemannian space.

We first state a famous theorem of Fubini. [6]

Theorem 10.11. A V_n ; $n > 2$, cannot admit a complete group of motions of order $\frac{1}{2}n(n+1)-1$.

In 1947, H.C. Wang [7] proved the following theorem:

Theorem 10.12. If an n-dimensional Finsler space; $n>2$, $n\neq 4$, admits a group of motions of an order greater than $\frac{1}{2}n(n-1)+1$,the space is Riemannian and of constant curvature.

To prove this theorem, Wang used the first of the following two theorems of D. Montgomery and H. Samelson. [8]

Theorem 10.13. In an n-dimensional Euclidean space; $n\neq 4$, there does not exist a proper subgroup of the group of rotations of an order greater than $\frac{1}{2}(n-1)(n-2)$.

Theorem 10.14. In an n-dimensional Euclidean space; $n>2$, $n\neq 4$, $n\neq 8$, any subgroup of the order $\frac{1}{2}(n-1)(n-2)$ of the group of rotations leaves invariant one and only one direction.

According to the theorem 10.12 of Wang, we can state

Theorem 10.15. A V_n ; $n>2$, $n\neq 4$, which is not of constant curvature cannot admit a complete group of motions of an order greater than $\frac{1}{2}n(n-1)+1$.

On the other hand, I.P. Egorov [9] proved in 1947 the following two theorems.

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5) S.Sasaki: Geometry of the conformal connexion. Sci.Rep.Tôhoku Imp. Univ., 29 (1940), 219-267.

K. Yano: Groups of transformations in generalized spaces. Tokyo, (1949).

A.H. Taub: A characterization of conformally flat spaces. Bull.Amer. Math. Soc., 55 (1949), 85-89.

6) G. Fubini: Sugli spazi che amettono un gruppo continuo di movimenti: Annali di Mat., (3) 9 (1903), 39-81.

7) H.C. Wang: On Finsler spaces with completely integrable equations of Killing. Journ. of the London Math. Soc., 22 (1947), 5-9.

8) D. Montgomery and H. Samelson: Transformation groups of spheres. Ann. of Math., 44 (1943), 454-470.

9) I.P. Egorov: On a strengthening of Fubini's theorem in the order of the group of motions of a Riemannian space. Doklady Akad. Nauk SSSR (N.S.), 66 (1949), 793-796.

Theorem 10.16. The maximum order of the complete groups of motions of those V_n's which are not Einstein spaces is $\frac{1}{2}n(n-1)+1$.

Theorem 10.17. The order of the complete groups of motions of those V_n's which are different from spaces of constant curvature is not larger than $\frac{1}{2}n(n-1)+2$.

Now using the same method as was used by Wang, we can prove

Theorem 10.18. In a V_n ; $n \neq 4$, there does not exist a group of motions of the order τ such that

$$\frac{1}{2}n(n+1) > \tau > \frac{1}{2}n(n-1)+1 .$$

Thus it would be interesting to study the V_n's which admit a group of motions of the order $\tau = \frac{1}{2}n(n-1)+1$.

For $n=3$, we have

$$\frac{1}{2}n(n+1) = 6 , \quad \frac{1}{2}n(n-1)+1 = 4.$$

By the theorem of Fubini, there does not exist, in a V_3 , a complete group of motions of the order 5.

E. Cartan [10] studied the 3-dimensional Riemannian spaces admitting a group of motions of the order 4, and obtained

Theorem 10.19. The 3-dimensional complete, simply connected Riemannian spaces which admit a group of motions of the order 4, are homeomorphic to one of the following spaces:

Euclidean space, topological product of a straight line and a sphere,

Space of positive constant curvature.

For the general dimensional Riemannian spaces, I obtained the following

Theorem 10.20. [11] In order that an n-dimensional Riemannian space, $n>4$, $n \neq 8$, admit a group of motions of the order $\frac{1}{2}n(n-1)+1$, it is necessary and sufficient that the space be a product of a straight line and an $(n-1)$- dimensional Riemannian space of constant curvature or a space of negative constant curvature.

This theorem contains two exceptional cases $n=4$ and $n=8$.
S. Ishihara [12] studied the 4-dimensional homogeneous Riemannian spaces, that is, the 4-dimensional Riemannian spaces which admit a transitive group of motions and he obtained

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10) E. Cartan: Leçons sur la géométrie des espaces de Riemann, Gauthier-Villars, Paris (1951), p.305.

11) K. Yano: On n-dimensional Riemannian spaces admitting a group of motions of order $\frac{1}{2}n(n-1)+1$. Trans Amer. Math.Soc., 74 (1953), 260-279.

12) S. Ishihara: Homogeneous Riemannian spaces of four dimensions. Forth coming.

Theorem 10.21. The four-dimensional connected and simply connected homogeneous Riemannian spaces are homeomorphic to one of the following spaces.

4-dimensional Euclidean space.

4-dimensional sphere.

projective complex space of complex dimension 2.

product of two 2-dimensional spheres.

product of a straight line and a 3-dimensional sphere.

product of a Euclidean plane and a 2-dimensional sphere.

For $n=4$, we have

$$\tfrac{1}{2}n(n+1)=10, \qquad \tfrac{1}{2}n(n-1)+1=7.$$

Ishihara studied the V_4's which admit a group of motions of the order 9 or 8. He obtained

Theorem 10.22. In a V_4, there does not exist a G_9 of motions. If a V_4 admits a G_8 of motions, the group is transitive and the space is a Kählerian space of constant holomorphic curvature.

Ishihara proved also that the theorem 10.20 is valid for $n=4$.

M. Ohata [13] proved that the theorem 10.14 of Montgomery and Samelson is valid also for $n=8$, and consequently that the theorem 10.20 is valid also for $n=8$.

I. Mogi and myself studied the groups of motions in a pseudo-Kählerian space and obtained:

Theorem 10.23. If a pseudo-Kählerian space admits a group of motions which carry every holomorphic section at an arbitrary point into a holomorphic section at an arbitrary point, the space is of holomorphic constant curvature.

4°. Groups of affine and projective motions.

In 1947, I.P. Egorov [14] proved

Theorem 10.24. The maximum order of the complete group of affine motions of those A_n's which are not E_n is n^2 $(n \geqq 3)$.

We can easily verify that the theorem is also true for $n=2$.

He proved in 1949 [15] that the theorem is also true for an L_n and moreover

Theorem 10.25. An A_n which admits a complete group of affine motions of the order n^2 is projectively Euclidean.

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13) M. Ohata: On n-dimensional homogeneous spaces of Lie groups of dimension greater than $\frac{1}{2}n(n-1)$. Forthcoming.

14) I.P. Egorov: On the order of the group of motions of spaces with affine connexion. Doklady Akad.Nauk SSSR (N.S.), 57 (1947),867-870.

15) I.P. Egorov: On groups of motions of spaces with asymmetrical affine connexion. Ibidem, 64 (1949), 621-624.

Theorem 10.26. An L_n, $n \geq 4$, which admits a complete group of affine motions of the order n^2 has a semi-symmetric linear connexion.

A semi-symmetric linear connexion is a linear connexion whose torsion tensor has the form

$$2 S_{\mu\lambda}{}^{\kappa} = S_{\mu} A_{\lambda}^{\kappa} - S_{\lambda} A_{\mu}^{\kappa} \tag{10.15}$$

As to the groups of projective motions in an A_n, G. Vranceanu 16) and I.P. Egorov 17) proved the following interesting

Theorem 10.27. If an A_n admits a complete group of projective motions of an order $\tau > n^2 - 2n + 5$, the space is projectively flat. There exists always, for any given number of n, an A_n which admits a group of projective motions of the order $n^2 - 2n + 5$, and which is not projectively Euclidean. In this case, the group is always transitive.

5°. The A_n and L_n admitting a complete group of affine motions of an order $\tau > n^2 - n + 1$.

I.P. Egorov 18) has proved the following theorems.

Theorem 10.28. We consider an A_n which admits a complete group of affine motions leaving invariant a symmetric covariant tensor $H_{\mu\lambda}$ and we denote the rank of the matrix $H_{\mu\lambda}$ by m

Then we have

(1) The order τ of the group satisfies the inequality

$$\tau \leq n^2 + n - nm + \tfrac{1}{2} m(m-1) \tag{10.16}$$

(2) If in (10.16) the sign = holds, the group is transitive.

(3) There exists a space admitting a group of affine motions for which the sign = holds in (10.16).

Theorem 10.29. We consider an A_n which admits a complete group of affine motions leaving invariant an alternating covariant tensor $S_{\mu\lambda}$ and we denote the rank of the matrix $S_{\mu\lambda}$ by $2k$. Then we have:

(1) The order τ of the group satisfies the inequality

$$\tau \leq n^2 - (n-k)(2k-1) + 2k \tag{10.17}$$

(2) In particular, if the space is projectively Euclidean and $S_{\mu\lambda} = R_{[\mu\lambda]}$ we have

$$\tau \leq n^2 - (n-k)(2k-1) \tag{10.18}$$

(3) There exists a space for which the sign = holds in (10.18).

(4) If the sign = holds in (10.17) or (10.18), then the group is transitive.

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16) G. Vranceanu: Groupes de mouvements des espaces à connexion. Studii şi Cercetari Matematice, 2 (1951), 387-444.

17) I.P. Egorov: Collineations of projectively connected spaces. Doklady Akad. Nauk SSSR. (N.S.), 80 (1951), 709-712.

18) I.P. Egorov: A tensor characteristic of A_n of non zero curvature with maximum mobility. Doklady Akad.Nauk SSSR (N.S.), 84 (1952), 209-212.

Theorem 10.30. We consider an A_n which admits an τ-parameter complete group of motions.

(1) If $\tau > n^2 - n + 1$,

(a) The space is projectively Euclidean.

(b) The Ricci tensor has the form $R_{\mu\lambda} = -(n-1)\varepsilon w_\mu w_\lambda$, where $\varepsilon = \pm 1$ and where w_λ is a gradient vector, $w_\lambda = \partial_\lambda w$.

(c) The vector w_λ satisfies the relation

(10.19) $$\nabla_\mu w_\lambda = \sigma w_\mu w_\lambda,$$

where σ is a scalar which is a function of w.

(2) We assume that the conditions (a), (b), (c) are satisfied.

If $w_\lambda = 0$, the space is affinely Euclidean.

If $w_\lambda \neq 0$, $\sigma = \text{const.}$, we have $\tau = n^2$ and the group is transitive.

If $w_\lambda \neq 0$, $\sigma \neq \text{const.}$, we have $\tau = n^2 - 1$ and the group is intransitive.

(3) The conditions (a), (b), (c) are equivalent to the following ones which constitute a completely integrable system of partial differential equations:

In a certain coordinate system, the components of the connexion are given by

(10.20) $$(\alpha) \quad \Gamma^\kappa_{\mu\lambda} = p_\mu A^\kappa_\lambda + p_\lambda A^\kappa_\mu, \qquad p_\lambda = \partial_\lambda p.$$

(10.21) $$(\beta) \quad \partial_\mu p_\lambda = p_\mu p_\lambda + \varepsilon w_\mu w_\lambda,$$

(10.22) $$(\gamma) \quad \partial_\mu w_\lambda = \sigma(w) w_\mu w_\lambda + w_\mu p_\lambda + w_\lambda p_\mu. \quad w_\lambda = \partial_\lambda w.$$

From (1) (a) and (b) we see that the curvature $R_{\nu\mu\lambda}{}^\kappa$ of the space is of the form

(10.23) $$R_{\nu\mu\lambda}{}^\kappa = \varepsilon(w_\nu A^\kappa_\mu - w_\mu A^\kappa_\nu) w_\lambda,$$

and hence the $R_{\nu\mu\lambda}{}^\kappa$ satisfies

(10.24) $$\nabla_\omega R_{\nu\mu\lambda}{}^\kappa = 2\sigma w_\omega R_{\nu\mu\lambda}{}^\kappa.$$

This shows that the space is a so-called κ-space of the English School. The form (10.23) of the curvature tensor was also obtained by Y. Moto [19].

From (3) we see that there exists an A_n which admits a complete group of affine motions of the order n^2 or $n^2 - 1$.

Theorem 10.31. We consider an L_n with a semi-symmetric linear connexion $\Gamma^\kappa_{\mu\lambda}$,

(10.25) $$2 S_{\mu\lambda}{}^\kappa = S_\mu A^\kappa_\lambda - S_\lambda A^\kappa_\mu, \qquad S_\mu \neq 0$$

and denote the space with the symmetric linear connexion $\Gamma^\kappa_{(\mu\lambda)}$ by A_n.

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19) Y. Moto: On the affinely connected space admitting a group of affine motions. Proc. Japan Acad., 26 (1950), 107-110.

Y. Moto: On a curved affinely connected space admitting a group of affine motions of maximum order. Sci. Rep. Yokohama Nat. Univ., Sec. I, No.3 (1954), 1-12.

(1) In order that the L_n admit a complete group of affine motions of the order n^2, it is necessary and sufficient that

(a) the A_n be projectively flat,

(b) the Ricci tensor $R_{\mu\lambda}$ of the A_n be of the form

(10.26) $$R_{\mu\lambda} = -(n-1)\varepsilon s_\mu s_\lambda$$

(c) in which the vector s_λ satisfies the equation

(10.27) $$\tilde{\nabla}_\mu s_\lambda = c s_\mu s_\lambda,$$

where $\tilde{\nabla}_\mu$ is with respect to $\Gamma^\kappa_{(\mu\lambda)}$ and c is a constant.

(2) The conditions (a), (b), (c) are equivalent to the following completely integrable system of partial differential equations:

(10.28) (α) $$\Gamma^\kappa_{(\mu\lambda)} = p_\mu A^\kappa_\lambda + p_\lambda A^\kappa_\mu, \quad \Gamma^\kappa_{[\mu\lambda]} = s_\mu A^\kappa_\lambda - s_\lambda A^\kappa_\mu,$$

(10.29) (β) $$\partial_\mu p_\lambda = p_\mu p_\lambda + \varepsilon s_\mu s_\lambda; \quad \varepsilon = \pm 1,$$

(10.30) (γ) $$\partial_\mu s_\lambda = c s_\mu s_\lambda + s_\mu p_\lambda + s_\lambda p_\mu.$$

From (1) we can see that the curvature tensor $R_{\nu\mu\lambda}{}^{\kappa}$ of L_n is of the form

$$R_{\nu\mu\lambda}{}^{\kappa} = (\varepsilon - c - 1)(s_\nu A^\kappa_\mu - s_\mu A^\kappa_\nu) s_\lambda.$$

From (2) we see that there exists an L_n with a semi-symmetric linear connexion which admits a group of affine motions of the order n^2.

6^0. A_n's admitting a group of affine motions of the order $r \geqq n^2 - n + 5$.

We consider the group H_n formed by the non singular affine transformations

(10.31) $$'x^\kappa = a^\kappa_\lambda x^\lambda$$

and denote its subgroups by

H_n^+ : a^κ_λ ; $\mathrm{Det}(a^\kappa_\lambda) > 0$,

P_n : a^κ_λ ; $\mathrm{Det}(a^\kappa_\lambda) = 1$,

K : a^κ_λ ; $a^\kappa_\lambda = \alpha\delta^\kappa_\lambda$, $\alpha > 0$,

L : a^κ_λ ; $a^1_1 = 1$, $a^\alpha_1 = 0$, $\mathrm{Det}(a^\kappa_\lambda) = 1$, $\alpha = 2, \ldots, n$,

L' : a^κ_λ ; $a^1_1 = 1$, $a^1_\alpha = 0$, $\mathrm{Det}(a^\kappa_\lambda) = 1$, $\alpha = 2, \ldots, n$,

M : a^κ_λ ; $a^1_1 > 0$, $a^\alpha_1 = 0$, $\mathrm{Det}(a^\kappa_\lambda) = 1$, $\alpha = 2, \ldots, n$,

M' : a^κ_λ ; $a^1_1 > 0$, $a^1_\alpha = 0$, $\mathrm{Det}(a^\kappa_\lambda) = 1$, $\alpha = 2, \ldots, n$,

$$I(b): \qquad a_\lambda^\kappa = \begin{pmatrix} e^{(1+b)t} & 0 & \cdots & 0 \\ 0 & e^{bt} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{bt} \end{pmatrix}.$$

H.C. Wang and K. Yano [20] proved the following theorems.

Theorem 10.32. A closed and connected subgroup of the order $\geqq n^2-2n+5$ of H_n is, except for a coordinate transformation, equal to one of the group, H_n^+, P_n, $K\times M$, $K\times M'$, $K\times L$, $K\times L'$, $I(b)\times L$, $I(b)\times L'$, L and L'.

Theorem 10.33. If an A_n admits a group G of affine motions of the order $r \geqq n^2-n+5$, for the isotropy group $G(P)$ at a point P, the order of $G(P)$, the group G of affine motions, the order of G, and the structure of the space, we have only the following cases:

Isotropy group $G(P)$	Order of $G(P)$	Group G	Order of G	Structure of the space A_n
H_n	n^2	transitive	n^2+n	$R_{\nu\mu\lambda}{}^{\kappa}=0$,
H_n^+	"	"	"	"
P_n	n^2-1	"	n^2+n-1	"
$K\times M$	n^2-n+1	"	n^2+1	"
$K\times M'$	"	"	"	"
$K\times L$	n^2-n	"	n^2	"
$K\times L'$	"	"	"	"
$I(b)\times L$	"	"	"	"
$I(b)\times L'$	n^2-n	transitive	"	(i) $1+b\neq 0$, $R_{\nu\mu\lambda}{}^{\kappa}=0$ (ii) $1+b=0$, $R_{\nu\mu\lambda}{}^{\kappa}=k(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa)w_\lambda$ $\nabla_\mu w_\lambda = \sigma w_\mu w_\lambda$, k,σ : constants.
		intransitive	n^2-1	(i) $1+b\neq 0$, $R_{\nu\mu\lambda}{}^{\kappa}=0$, (ii) $1+b=0$, $R_{\nu\mu\lambda}{}^{\kappa}=k(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa)w_\lambda$ $\nabla_\mu w_\lambda = \sigma w_\mu w_\lambda$ ($w_\lambda=\partial_\lambda w$) k,σ : functions of w.

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20) H.C. Wang and K. Yano: A class of affinely connected spaces. To appear in Trans. Amer. Math.Soc.

L	n^2-n-1	transitive	n^2-1	$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}=0.$
L'	n^2-n-1	transitive	"	$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}=k(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa)w_\lambda,$ $\nabla_\mu w_\lambda = \sigma w_\mu w_\lambda;$ k, σ : constansts.
		intransitive	n^2-2	$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}=k(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa)w_\lambda,$ $\nabla_\mu w_\lambda = \sigma w_\mu w_\lambda,$ $(w_\lambda = \partial_\lambda w)$ k, σ : functions of w.

Y. Moto [21] proved, by a method quite different from that of Egorov, the following theorems:

Theorem 10.34. An A_n with non vanishing curvature tensor admits a complete group of affine motions of the maximum order if and only if the equations

$$(10.31)\qquad R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}=\varepsilon(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa)w_\lambda,\quad \nabla_\mu w_\lambda = a w_\mu w_\lambda;\quad \varepsilon=\pm 1,\quad a=\text{const.}$$

are satisfied. Then the order is n^2 and we can find a coordinate system with respect to which the components of the linear connexion are

$$\Gamma_{nn}^{\alpha}=-\varepsilon\xi^\alpha,\quad \Gamma_{nn}^{n}=-a;\quad \text{the other } \Gamma_{\mu\lambda}^{\kappa}=0;\quad \alpha=1,2,\dots,n-1;$$

and the finite equations of the group are given by

$$(10.32)\qquad \begin{cases} '\xi^\alpha = P_\beta^\alpha \xi^\beta + Q^\alpha e^{c_1\xi^n} + R^\alpha e^{c_2\xi^n}, \\ '\xi^n = \xi^n + S, \end{cases}$$

or

$$(10.33)\qquad \begin{cases} '\xi^\alpha = P_\beta^\alpha \xi^\beta + (Q^\alpha + R^\alpha \xi^n)e^{c\xi^n}, \\ '\xi^n = \xi^n + S \end{cases}$$

according as the roots c_1 and c_2 of the quadratic equation $(\xi)^2+a\xi-\varepsilon=0$ satisfy $c_1 \neq c_2$ or $c_1 = c_2 = c$; P, Q, R, S being constants.

Theorem 10.35.[22] In order that a projectively Euclidean [23] A_n with non vanishing curvature tensor admit a complete group of affine

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21) Y. Moto. loc. cit.

22) Y. Moto: On n-dimensional projectively flat spaces admitting a group of affine motions of order $r > n^2-n$. Sci.Rep. of Yokohama National Univ., Sec.I, No.4, In press.

23) For $n \geqq 5$, we have $r > n^2-n \geqq n^2-2n+5$. Consequently according to Theorem 10.27, we do not need this assumption.

motions G_r of order $r > n^2 - n$, it is necessary and sufficient that the curvature tensor belong to one of the following three types T_1, T_2 and T_3, and the vectors appearing in the expressions of the curvature tensors satisfy the associated equations. Such linear connexions and groups actually exist.

The curvature tensors are

(10.34) $$T_1 : R_{\nu\mu\lambda}{}^{\kappa} = \varepsilon(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa) w_\lambda, \quad \varepsilon = \pm 1, \quad w_\lambda \neq 0,$$

(10.35) $$\begin{aligned} T_2 : R_{\nu\mu\lambda}{}^{\kappa} = {} & \varepsilon(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa) w_\lambda \\ & + A_\nu^\kappa(w_\mu x_\lambda - w_\lambda x_\mu) - A_\mu^\kappa(w_\nu x_\lambda - w_\lambda x_\nu) \\ & - 2(w_\nu x_\mu - w_\mu x_\nu) A_\lambda^\kappa, \end{aligned}$$

$\varepsilon = \pm 1$; w_λ and x_λ are linearly independent.

(10.36) $$T_3 : R_{\nu\mu\lambda}{}^{\kappa} = \varepsilon_1(w_\nu A_\mu^\kappa - w_\mu A_\nu^\kappa) w_\lambda + \varepsilon_2(x_\nu A_\mu^\kappa - x_\mu A_\nu^\kappa) x_\lambda,$$

$\varepsilon_1, \varepsilon_2 = \pm 1$; w_λ and x_λ are linearly independent.

The associated equations are

T_1:

(10.37) $$\nabla_\mu w_\lambda = \alpha w_\mu w_\lambda \;; \quad \alpha = \alpha(w), \quad w_\lambda = \partial_\lambda w$$

T_2:

(10.38) $$\begin{cases} \nabla_\mu w_\lambda = \theta(2\varepsilon w_\mu w_\lambda - w_\mu x_\lambda + w_\lambda x_\mu) \\ \nabla_\mu x_\lambda = y_\mu x_\lambda + \theta(\varepsilon w_\mu x_\lambda - x_\mu x_\lambda) \end{cases}$$

(10.39) $$\begin{aligned} \nabla_\mu y_\lambda - \nabla_\lambda y_\mu = {} & \varepsilon\theta(y_\mu w_\lambda - y_\lambda w_\mu) \\ & - 2\theta(x_\mu y_\lambda - x_\lambda y_\mu) - \varepsilon(w_\mu x_\lambda - w_\lambda x_\mu) \;; \quad \varepsilon\theta^2 = -1 \end{aligned}$$

T_3:

(10.40) $$\nabla_\mu w_\lambda = -\varepsilon_1\varepsilon_2 y_\mu x_\lambda, \quad \nabla_\mu x_\lambda = y_\mu x_\lambda,$$

(10.41) $$\nabla_\mu y_\lambda - \nabla_\lambda y_\mu = \varepsilon(x_\mu w_\lambda - x_\lambda w_\mu).$$

The groups are

T_1:

(10.42) $$\mathcal{L} w_\lambda = 0, \quad \mathcal{L}\alpha = 0.$$

If α is a constant, then $r = n^2$ and the group is transitive.
If α is not a constant, then $r = n^2 - 1$ and the group is intransitive.

T_2:

(10.43) $$\mathcal{L} w_\lambda = 0, \qquad \mathcal{L} x_\lambda = \beta w_\lambda \;; \qquad \beta: \text{ a scalar}$$

$r = n^2 - n + 1$ and the group is transitive.

T_3 :

(10.44) $\quad \mathcal{L} w_\lambda = -\varepsilon_1 \varepsilon_2 \beta x_\lambda, \quad \mathcal{L} x_\lambda = \beta w_\lambda \; ; \quad \beta$: a scalar.

$\tau = n^2 - n + 1$ and the group is transitive.

§ 11. Motions in a compact orientable Riemannian space.

In this section, we consider an n-dimensional compact orientable Riemannian space with positive definite metric $ds^2 = g_{\mu\lambda} d\xi^\mu d\xi^\lambda$ and denote it by $\overset{*}{V}_n$.

We first remind of a theorem of Green:

Theorem 11.1. In a $\overset{*}{V}_n$, we have

$$\int_{\overset{*}{V}_n} \nabla_\mu v^\mu d\tau = 0,$$

for an arbitrary vector field v^κ, where

$$(11.2) \qquad d\tau \overset{\text{def}}{=} \sqrt{g}\, d\xi^1 \cdots d\xi^n > 0$$

is the volume element of the space.

Using this theorem, we can prove the following two theorems.

Theorem 11.2. In a $\overset{*}{V}_n$, we have

$$(11.3) \qquad \int_{\overset{*}{V}_n} \Delta f\, d\tau = 0$$

for an arbitrary scalar field f, where

$$(11.4) \qquad \Delta f \overset{\text{def}}{=} g^{\mu\lambda} \nabla_\mu \nabla_\lambda f$$

Theorem 11.3. If, in a $\overset{*}{V}_n$, we have $\Delta f \geqq 0$ everywhere, then $\Delta f = 0$ and f is a constant.

For an arbitrary alternating tensor field $w_{\lambda_p \cdots \lambda_1}$, the rotation and the divergence are respectively defined by[1]

$$(11.5) \qquad \begin{cases} \mathrm{Rot}\, w: & (p+1)\,\partial_{[\mu} w_{\lambda_p \cdots \lambda_1]}, \\ \mathrm{Div}\, w: & \nabla_\mu w^{\mu\lambda_{p-1} \cdots \lambda_1} \end{cases}$$

when the w is of valence p, $\mathrm{Rot}w$ is of valence $p+1$ and $\mathrm{Div}w$ is of valence $p-1$.

For two arbitrary alternating tensors $u_{\lambda_p \cdots \lambda_1}$ and $v_{\lambda_p \cdots \lambda_1}$ of the same valence p, we define the global inner product (u,v) by

$$(11.6) \qquad (u,v) = \int_{\overset{*}{V}_n} u_{\lambda_p \cdots \lambda_1} v^{\lambda_p \cdots \lambda_1} d\tau .$$

Since the metric is positive definite, we have always

$$(11.7) \qquad (u,u) \geqq 0,$$

and $(u,u) = 0$ if and only if $u_{\lambda_p \cdots \lambda_1} = 0$

Consider two arbitrary alternating tensors $u_{\lambda_p \cdots \lambda_1}$ of valence p and $v_{\lambda_{p+1} \cdots \lambda_1}$ of valence $p+1$ and apply the theorem 11.1 to the vector $u_{\lambda_p \cdots \lambda_1} v^{\mu\lambda_p \cdots \lambda_1}$,

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1) J.A. Schouten: Ricci Calculus (1954), p.83.

then we obtain

Theorem 11.4. In a $\overset{*}{\nabla}_n$, we have, for two arbitrary alternating tensors $u_{\lambda_p \dots \lambda_1}$ and $v_{\lambda_{p+1} \dots \lambda_1}$,

(11.8) $$(\mathrm{Rot}\, u, v) + (p+1)(u, \mathrm{Div}\, v) = 0$$

An alternating tensor $w_{\lambda_p \dots \lambda_1}$ is called a harmonic tensor if it satisfies

(11.9) $$\mathrm{Rot}\, w = 0, \qquad \mathrm{Div}\, w = 0$$

Using the theorem 11.4, we can prove

Theorem 11.5. In order that an alternating tensor $w_{\lambda_p \dots \lambda_1}$ in a $\overset{*}{\nabla}_n$ be harmonic, it is necessary and sufficient that

(11.10) $$\Delta w \overset{\mathrm{def}}{=} \mathrm{Div}\,\mathrm{Rot}\, w + \mathrm{Rot}\,\mathrm{Div}\, w = 0$$

Using also the theorem 11.4, one can prove

Theorem 11.6. A harmonic tensor which is the rotation of an alternating tensor is identically zero.

We now suppose that the $\overset{*}{\nabla}_n$ admits a one-parameter group of motions generated by the infinitesimal transformation

(11.11) $$'\xi^\kappa = \xi^\kappa + v^\kappa(\xi)\, dt,$$

then we have

(11.12) $$\underset{v}{\mathcal{L}}\, g_{\lambda\kappa} = 0,$$

(11.13) $$\underset{v}{\mathcal{L}} \left\{ {\kappa \atop \mu\lambda} \right\} = 0$$

and the operators ∇_μ and $\underset{v}{\mathcal{L}}$ are commutative.

Suppose that there exists in the $\overset{*}{\nabla}_n$ a harmonic tensor $w_{\lambda_p \dots \lambda_1}$, then we have

$$\nabla_{[\mu} w_{\lambda_p \dots \lambda_1]} = 0 \quad \text{and} \quad g^{\nu\mu} \nabla_\nu w_{\mu\lambda_{p-1} \dots \lambda_1} = 0,$$

from which

$$\nabla_{[\mu} \underset{v}{\mathcal{L}}\, w_{\lambda_p \dots \lambda_1]} = 0 \quad \text{and} \quad g^{\nu\mu} \nabla_\nu \underset{v}{\mathcal{L}}\, w_{\mu\lambda_{p-1} \dots \lambda_1} = 0.$$

This means that $\underset{v}{\mathcal{L}}\, w_{\lambda_p \dots \lambda_1}$ is also a harmonic tensor. But we may easily verify that

(11.14) $$\underset{v}{\mathcal{L}}\, w_{\lambda_p \dots \lambda_1} = \mathrm{Rot}\, v^\mu w_{\mu\lambda_{p-1} \dots \lambda_1}.$$

Thus according to the theorem 11.6, we get

<u>Theorem 11.7.</u>[2] If a $\overset{*}{V}_n$ admits an infinitesimal motion, the Lie derivative of a harmonic tensor with respect to this motion vanishes identically.

Applying the theorem 11.7 to a harmonic vector w_λ and a Killing vector v^κ, we find

$$(11.15) \qquad 0 = \underset{v}{\mathcal{L}} w_\lambda = \nabla_\lambda (w_\mu v^\mu),$$

from which we have

<u>Theorem 11.8.</u>[3] In a $\overset{*}{V}_n$, the inner product of a harmonic vector and a Killing vector is constant.

Now, take an arbitrary vector field v^κ in the $\overset{*}{V}_n$. We can easily verify the following identity:

$$\nabla_\mu (v^\lambda \nabla_\lambda v^\mu) - \nabla_\lambda (v^\lambda \nabla_\mu v^\mu)$$
$$= (\nabla^\mu v^\lambda)(\nabla_\lambda v_\mu) - (\nabla_\mu v^\mu)(\nabla_\lambda v^\lambda) + K_{\mu\lambda} v^\mu v^\lambda,$$

from which

$$(11.16) \qquad \int_{V_n} \left[(\nabla^\mu v^\lambda)(\nabla_\lambda v_\mu) - (\nabla_\mu v^\mu)(\nabla_\lambda v^\lambda) + K_{\mu\lambda} v^\mu v^\lambda \right] d\tau = 0,$$

where $\nabla^\mu = g^{\mu\lambda} \nabla_\lambda$

Suppose that a vector field v^κ generates a one-parameter group of motions in the $\overset{*}{V}_n$, then we have

$$\underset{v}{\mathcal{L}} g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 0, \quad \nabla_\lambda v^\lambda = 0$$

Substituting these equations in (11.16) we find

$$(11.17) \qquad \int_{V_n} \left[(\nabla^\mu v^\lambda)(\nabla_\mu v_\lambda) - K_{\mu\lambda} v^\mu v^\lambda \right] d\tau = 0,$$

from which

<u>Theorem 11.9.</u>[4] In a $\overset{*}{V}_n$ whose Ricci curvature is negative semi-definite, a vector generating a one-parameter group of motions is a covariant constant field. In a $\overset{*}{V}_n$ whose Ricci curvature is negative definite, there does not exist a continuous group of motions.

Suppose that a $\overset{*}{V}_n$ with $K_{\mu\lambda} = 0$ admits a transitive group of motions, then by the theorem 11.9, all the vectors generating the transitive group of motions are covariant constant. This means that the $\overset{*}{V}_n$ admits more than n linearly independent covariant constant vector fields. Thus the $\overset{*}{V}_n$ is locally Euclidean.

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2) W.V.D. Hodge: The theory and applications of harmonic Integrals. Cambridge University Press (1952).

K. Yano: On harmonic and Killing vector fields. Ann.of Math.,55 (1952), 38-45.

3) S. Bochner: Vector fields on complex and real manifolds. Ann.of Math., 52 (1950), 642-649.

4) S. Bochner: Vector fields and Ricci curvature. Bull.of the Amer.Math. Math. Soc., 52 (1946), 776-797.

Theorem 11.10.[5] A $\overset{*}{V}_n$ with $K_{\mu\lambda}=0$ admitting a transitive group of motions is locally Euclidean.

Suppose that a vector v^κ generates a one-parameter group of conformal motions, then we have

$$(11.18) \qquad \underset{v}{\mathcal{L}} g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 2\phi g_{\mu\lambda}, \quad \nabla_\lambda v^\lambda = n\phi.$$

Substituting (11.18) in (11.16), we find

$$(11.19) \qquad \int_{\overset{*}{V}_n} \left[(\nabla^\mu v^\lambda)(\nabla_\mu v_\lambda) + n(n-2)\phi^2 - K_{\mu\lambda} v^\mu v^\lambda\right] d\tau = 0,$$

from which

Theorem 11.11.[6] In a $\overset{*}{V}_n$ whose Ricci curvature is negative semi-definite, a vector generating a one-parameter group of conformal motions is a covariant constant vector field. In a $\overset{*}{V}_n$ whose Ricci curvature is negative definite, there does not exist a one-parameter group of conformal motions.

Now apply the theorem 11.2 to the square $v_\kappa v^\kappa$ of the length of an arbitrary vector field v^κ, we obtain

$$(11.20) \qquad \int_{\overset{*}{V}_n} \left[v_\kappa g^{\mu\lambda} \nabla_\mu \nabla_\lambda v_\kappa + (\nabla^\mu v^\lambda)(\nabla_\mu v_\lambda)\right] d\tau = 0.$$

Adding (11.16) and (11.20), we find

$$(11.21) \qquad \int_{\overset{*}{V}_n} \left[v_\kappa (g^{\mu\lambda} \nabla_\mu \nabla_\lambda v^\kappa + K_\lambda^{\ \kappa} v^\lambda) + 2(\nabla^{(\mu} v^{\lambda)})(\nabla_{(\mu} v_{\lambda)}) - (\nabla_\mu v^\mu)(\nabla_\lambda v^\lambda)\right] d\tau = 0.$$

Suppose that a vector v^κ generates a one-parameter group of motions in a $\overset{*}{V}_n$, then from

$$\underset{v}{\mathcal{L}} g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 0,$$

$$\underset{v}{\mathcal{L}} \left\{ {\kappa \atop \mu\lambda} \right\} = \nabla_\mu \nabla_\lambda v^\kappa + K_{\nu\mu\lambda}{}^\kappa v^\nu = 0,$$

we obtain

$$(11.22) \qquad g^{\mu\lambda} \nabla_\mu \nabla_\lambda v^\kappa + K_\lambda^{\ \kappa} v^\lambda = 0, \quad \nabla_\mu v^\mu = 0.$$

Conversely suppose that a vector field v^κ in a $\overset{*}{V}_n$ satisfies (11.22), then substituting (11.22) in (11.21), we obtain

$$\int_{\overset{*}{V}_n} (\nabla^{(\mu} v^{\lambda)})(\nabla_{(\mu} v_{\lambda)}) d\tau = 0,$$

from which

$$\underset{v}{\mathcal{L}} g_{\mu\lambda} = 2\nabla_{(\mu} v_{\lambda)} = 0,$$

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5) A. Lichnerowicz: Un théorème sur les espaces homogènes complexes. Archiv der Math., 5 (1954), 207-215.

6) K. Yano: loc. cit.

that is, the vector v^κ generates a one-parameter group of motions. Thus we have

Theorem 11.12. In order that a vector v^κ generate a one-parameter group of motions in a $\overset{*}{\nabla}_n$, it is necessary and sufficient that v^κ satisfy (11.22).

Suppose that a $\overset{*}{\nabla}_n$ admits a one-parameter group of affine motions generated by a vector v^κ:

$$\underset{v}{\mathcal{L}}\left\{{}^{\kappa}_{\mu\lambda}\right\} = \nabla_\mu \nabla_\lambda v^\kappa + K_{\nu\mu\lambda}{}^{\kappa} v^\nu = 0, \tag{11.23}$$

from which we obtain

$$g^{\mu\lambda}\nabla_\mu \nabla_\lambda v^\kappa + K_\lambda{}^\kappa v^\lambda = 0, \tag{11.24}$$

$$\nabla_\mu \nabla_\lambda v^\lambda = 0 \tag{11.25}$$

From (11.25) we see that $\nabla_\lambda v^\lambda$ is a constant. But we have (11.1) and hence

$$\nabla_\lambda v^\lambda = 0. \tag{11.26}$$

Thus according to the theorem 11.12, we obtain

Theorem 11.13.[7] A one-parameter group of affine motions in a $\overset{*}{\nabla}_n$ is a group of motions.

A $\overset{*}{\nabla}_n$, symmetric in the sense of Cartan is characterized by the equation

$$\nabla_\omega K_{\nu\mu\lambda}{}^\kappa = 0, \tag{11.27}$$

from which we get

$$\begin{aligned} H_{\pi\omega\nu\mu\lambda}{}^\kappa \overset{\text{def}}{=} & K_{\pi\omega\rho}{}^\kappa K_{\nu\mu\lambda}{}^\rho - K_{\pi\omega\nu}{}^\rho K_{\rho\mu\lambda}{}^\kappa \\ & - K_{\pi\omega\mu}{}^\rho K_{\nu\rho\lambda}{}^\kappa - K_{\pi\omega\lambda}{}^\rho K_{\nu\mu\rho}{}^\kappa = 0, \end{aligned} \tag{11.28}$$

$$\nabla_\omega K_{\mu\lambda} = 0. \tag{11.29}$$

Conversely, suppose that (11.28) and (11.29) are satisfied in a $\overset{*}{\nabla}_n$. Then, from a general identity:

$$\begin{aligned} & \tfrac{1}{2}\Delta(K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa}) \\ & = 4(\nabla_\mu \nabla_\lambda K_{\nu\kappa})K^{\nu\mu\lambda\kappa} + 2H_{\pi\nu\lambda\kappa\mu}{}^\pi K^{\nu\mu\lambda\kappa} \\ & \qquad + (\nabla_\omega K_{\nu\mu\lambda\kappa})(\nabla^\omega K^{\nu\mu\lambda\kappa}), \end{aligned} \tag{11.30}$$

we get

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7) K. Yano: On harmonic and Killing vector fields. Ann. of Math., 55 (1952), 38-45.

(11.31) $$\frac{1}{2}\Delta(K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa}) = (\nabla_\omega K_{\nu\mu\lambda\kappa})(\nabla^\omega K^{\nu\mu\lambda\kappa}),$$

which is positive definite. Thus according to the theorem 11.3, we obtain

$$\nabla_\omega K_{\nu\mu\lambda\kappa} = 0.$$

Thus we can state

Theorem 11.14. [8] A $\overset{*}{V}_n$ satisfying $H_{\pi\omega\nu\mu\lambda}{}^\kappa = 0$ and $\nabla_\omega K_{\mu\lambda} = 0$ is symmetric in the sense of Cartan.

We know that a symmetric V_n admits a transitive group G of motions and that the linear isotropy group $\tilde{G}(P)$ at a point P contains the homogeneous holonomy group $\sigma(P)$ at P of the space as a subgroup.

Conversely, we assume that an irreducible [9] V_n admits a transitive group G of motions and that the linear isotropy group $\tilde{G}(P)$ at P contains the homogeneous holonomy group $\sigma(P)$ at P of the space as a subgroup for every point of the space.

Denoting by $\underset{v}{\mathcal{L}}$ the infinitesimal operator corresponding to one of the generators of the group $\tilde{G}(P)$, we obtain

(11.32) $$\underset{v}{\mathcal{L}} K_{\nu\mu\lambda}{}^\kappa = -K_{\nu\mu\lambda}{}^\rho \nabla_\rho v^\kappa + K_{\rho\mu\lambda}{}^\kappa \nabla_\nu v^\rho + K_{\nu\rho\lambda}{}^\kappa \nabla_\mu v^\rho + K_{\nu\mu\rho}{}^\kappa \nabla_\lambda v^\rho = 0.$$

But we assumed that $\tilde{G}(P)$ contains $\sigma(P)$ and consequently $K_{\nu\mu\lambda}{}^\kappa$ must be linear combinations of $\nabla_\lambda \underset{a}{v}{}^\kappa$ formed by all generators $\underset{a}{v}{}^\kappa$ of the group $\tilde{G}(P)$. Thus from (11.32), we get

(11.33) $$H_{\pi\omega\nu\mu\lambda}{}^\kappa = 0.$$

On the other hand, from (11.32), we find

(11.34) $$\underset{v}{\mathcal{L}} K_{\mu\lambda} = 0.$$

But we have assumed that $\sigma(P)$ is irreducible and consequently $\tilde{G}(P)$ is also irreducible. Thus we get from (11.34)

$$K_{\mu\lambda} = \frac{K}{n} g_{\mu\lambda},$$

from which

(11.35) $$\nabla_\omega K_{\mu\lambda} = 0.$$

The equations (11.30), (11.33) and (11.35) show that

(11.36) $$\frac{1}{2}\Delta(K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa}) = (\nabla_\omega K_{\nu\mu\lambda\kappa})(\nabla^\omega K^{\nu\mu\lambda\kappa}).$$

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8) A. Lichnerowicz: Courbure, nombres de Betti, et espaces symétriques. Proc. Intern. Congress of Math., 2 (1950), 216-223.

9) A V_n is said to be irreducible, when the holonomy group σ of the V_n is irreducible.

The group G of motions is transitive and consequently, from

$$\underset{v}{\mathcal{L}}(K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa})=0$$

we can conclude that

$$K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa}=\text{const.}$$

and hence

$$\Delta(K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa})=0 \tag{11.37}$$

From (11.36) and (11.37), we get

$$\nabla_\omega K_{\nu\mu\lambda\kappa}=0. \tag{11.38}$$

This proves the following theorem.

Theorem 11.15.[10] If an irreducible V_n (not necessarily compact and orientable) admits a transitive group of motions whose linear isotropy group at any point contains the homogeneous holonomy group at the point, the V_n is symmetric in the sense of Cartan.

On Geometric objects and Lie groups of transformations.

Nicolaas H. Kuiper and Kentaro Yano.

Preliminary report.

1. Introduction.

Ehresmann defined prolongations (prolongements [1,2]) of a C^s-manifold. A prolongation is a principle fibre bundle which is determined by the base space a C^s-manifold and a natural number $r \leq s$. In differential geometry one often is led to fibre bundles which are not principal, but of which the associated principal fibre bundle is of the kind defined by Ehresmann. Following Haantjes and Laman [3] these fibre bundles will be called geometric object bundles. Geometric objects or geometric object bundles have been defined by Golab, Haantjes, Laman, Nijenhuis, Schouten, Wagner and others.

In § 1 we give a definition of geometric object bundles. In § 2 we consider transformations in the base space of a geometric object bundle, and their prolongations in the bundle. In § 3 we obtain a main theorem on geometric objects on a Lie-group space with applications. § 4 deals with Lie groups of transformations of a manifold and the existence of invariant geometric objects. In § 5 we define Lie-derivatives of geometric object fields and give some applications.

§ 1. Geometric object bundles.

In this paper Λ will denote a fixed pseudo-group of homeomorphic mappings of differentiability class C^s, $s > 0$ in n-dimensional number space R^n, which contains the group of translations:

$$T \subset \Lambda .$$

Two homeomorphic mappings $\varphi_k : U_k \rightarrow V_k$ $k = i,j$ of neighbourhoods U_k of an n-manifold X into R^n are called Λ-compatible [1]) if for $z \in \varphi_i(U_i \cap U_j)$:

$$\lambda_{ji} = \varphi_j \varphi_i^{-1} \mid \varphi_i (U_i \cap U_j) \in \Lambda$$

1) Compare Veblen and Whitehead [4] p.

A manifold with a local Λ-structure or a <u>Λ-manifold</u> is a manifold covered by a complete set of mappings $\varphi_k : U_k \to V_k$ of the above kind, any two of which are Λ-compatible. The homeomorphic mappings are called Λ-coordinate systems, <u>Λ-reference systems</u> or just <u>reference systems</u>.

In the sequel X will be a Λ-manifold. Points of X will be indicated by x; points of R^n will be indicated by z; in particular the point $(0,0,..,0)$ by 0.

Two reference-systems φ_i and φ_j both covering $x \subset X$ are called <u>jet-equivalent</u> at x if their restrictions to some neighbourhood of x are identical. The jet-equivalence class of $\{x, \varphi_i\}$ is called <u>a jet of kind Λ</u> or <u>Λ-jet</u> and it will be denoted by $j(\varphi_i(x), x; \varphi_i) = j(\varphi_j(x), x; \varphi_j)$; x is called the <u>source of this jet</u>, $z=\varphi_i(x)$ is the <u>butt of this jet</u>.[2] If $f \subset \Lambda$, $z \in R^n$ is covered by f, then the jet determined by z and f is denoted by

$$j(z',z;f) = j(z',z) \text{ where } z' = f(z).$$

Jets of this kind will sometimes be called <u>auto-jets</u>. If the butt of a first jet coincides with the source of a second jet, then the product can be formed:

$$j(z_3,z_1;f_2f_1) = j(z_3,z_2;f_2)\cdot j(z_2,z_1;f_1)$$

$$j(z_2,x;f\varphi) = j(z_2,z_1;f)\cdot j(z_1,x;\varphi)$$

The jet with source $z_1 \in R^n$ and butt $z_2 \in R^n$ obtained from a (unique) translation $t \in \Lambda$ is denoted by

$$t(z_2,z_1)\ .$$

<u>Proposition 1</u>. The jets of the kind $j(0,0;f)$ form a group Δ. We introduce a non-Hausdorf topology in Δ with respect to a C^s-Λ-manifold by the definition:
a neighbourhood in Δ consists of all jets that can be represented by functions whose systems of derivatives up to the s^{th}, at the source of the jet, form a neighbourhood in the suitable number space.
<u>Proposition 2</u>. Any jet of the kind $j(z',z;f)$ admits a unique factorization as follows

$$t(z',0)\cdot j\left[0,0;\ t(0,z')f\cdot t(z,0)\right]\ .\ t(0,z)$$

short: $$j(z',z) = t(z',0)\cdot\ j(0,0)\cdot\ t(0,z)$$

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2) The notion and word jet was introduced by <u>Ehresmann</u> [1,2] .

The mapping $O_j : j(z',z;f) \rightarrow j[0,0;t(0,z').f.t(z,0)]$ is a homomorphism of the pseudo-group of auto-jets onto the group of jets with source = butt = 0.

Proposition 3. Two jets $j(z,x;\varphi_1)$ and $j(z',x;\varphi_2)$ with the same source $x \in X$ determine a unique auto-jet $j(z',z)$ by division:

$$j(z',x;\varphi_2) = j(z',z;\varphi_2\ \varphi_1^{-1}).\ j(z,x;\varphi_1)$$

Proposition 4. If $\varphi_1, \varphi_2, \varphi_3$ determine three jets with the same source x, then the quotient-auto-jets obey

$$j(z_3,z_1;\varphi_3\ \varphi_1^{-1}) = j(z_3,z_2;\varphi_3\ \varphi_2^{-1}).j(z_2,z_1;\varphi_2\ \varphi_1^{-1})$$

and this product rule also holds for the images of these jets under O_j, which we denote as follows:

$$j_{31}(0,0) = j_{32}(0,0).\ j_{21}(0,0)$$

or

$$j_{31} = j_{32}.j_{21} \qquad j_{ji} \in \Delta$$

Theorem 1. The entities X,Y,G,h defined below determine a unique fibre bundle B with base space X, fibre of the kind Y, group G, which is called the geometric object bundle over X of the kind (Y,G,h).

X is a C^s- Λ -manifold of dimension n.

Y is an analytic manifold.

G is a Lie group of analytic transformations of Y.

h is a continuous homomorphism of Δ onto G.

The bundle is defined as follows.

Let $\varphi_i : U_i \rightarrow V_i (U_i \subset X,\ V_i \subset R^n)$ be Λ-reference-systems covering X [3].

In the set of triples $(i, x \in U_i, y \in Y)$ we introduce the equivalence, called identification:

$(i,x,y) \sim (j, xg_{ji}y)$ for $x \in U_i \cap U_j$ in which

$$g_{ji} = h\ j_{ji}$$

$$j_{ji} = t(0,z_j).j(z_j,z_i;\varphi_j\ \varphi_i^{-1}).\ t(z_i,0) \in \Delta$$

$$z_i = \varphi_i(x) \qquad z_j = \varphi_j(x)$$

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3) If we require moreover that the set of Λ-reference systems is complete, that is, is not contained in a (Λ-compatible) bigger set, then the definitions are independent of the particular set of subsets $\{U_i\}$ of X.

An equivalence class is by definition <u>a point of the fibre bundle</u>. The equivalence classes with a fixed x form the fibre of the bundle at x. The bundle projection π is the mapping of the fibre at x, onto x. The fibre is homeomorphic with Y. The mapping:

$$\text{class of } (i,x,y) \longrightarrow (x,y)$$

is a homeomorphic mapping of

$$\pi^{-1}(U_i) \text{ onto } U_i \times Y \qquad (1).$$

The mapping:

$$\text{class of } (i,x,y) \longrightarrow y$$

will be denoted by φ_i^*. We then have for $b \in \pi^{-1}(U_i)$

$$\pi^{-1}(U_i) \xrightarrow{i} U_i \times Y \xrightarrow{\varphi_i} V_i \times Y$$

$$b \longrightarrow \pi b \times \varphi_i^* b \xrightarrow{\varphi_i} \varphi_i \pi b \times \varphi_i^* b.$$

If $b \in \pi^{-1}(U_i \cap U_j)$:

$$\varphi_j^* b = g_{ji} \varphi_i^* b$$

and, if $b \in \pi^{-1}(U_i \cap U_j \cap U_k)$ we obtain from

$$j(z_k, z_i; \varphi_k \varphi_i^{-1}) = j(z_k, z_j; \varphi_k \varphi_j^{-1}) . j(z_j, z_i, \varphi_j \varphi_i^{-1})$$

$$j_{ki} = j_{kj} \cdot j_{ji}$$

hence, because h is a continuous homomorphism:

$$g_{ki} = g_{kj} \cdot g_{ji}$$

Also g_{ji} is a continuous function of x.

The mappings (1) which fulfill all the conditions just mentioned define the structure of fibre bundle in the point set $\pi^{-1}(X)$. <u>Steenrod</u> [8]. The fibre bundle B so obtained is the object-bundle required in theorem i. If $Y' \subset Y$ is invariant under G, then X,Y',G,h determine a unique object-bundle B', which can be considered as imbedded in B. We call B' a subbundle of B.

A cross-section of B is called a <u>geometric object field</u> or <u>geometric object of the kind</u> (Y,G,h). One point of B:$b \in \pi^{-1}(x)$ is called <u>a geometric object</u> at x . $\pi^{-1}(x)$ will also be designated by Y_x.

<u>Example</u>:

Let Λ be the pseudo-group of all C^s reversible homeomorphisms in R^n. Γ be the invariant subgroup of Δ consisting of those jets that can be obtained from homeomorphisms of $f \subset \Delta$ which are expressed by functions

$$z'_k = z'_k(z_1, \ldots, z_n)$$

which have the same k-th derivatives for $k = 0, 1, \ldots, l \leqq s$ at the point 0, as the functions that express the identity homeomorphism:

$$z'_k = z_k$$

$$G = \Delta/\Gamma$$

$$h_j = \text{the class } j.\Gamma.$$

The object-bundle B so defined is a manifold of differentiability class s-1. A C^r-cross-section in B ($r \leqq s-1$) is called object (field) of object class and (of course) of differentiability class r.
Haantjes and Laman [3] determined all geometric object-bundles of object-class $l = 1$ and dimension n+1 (dimension $Y = 1$).
Tensor-bundles are bundles of class $l = 1$.
Affine connections (parallel displacement) can be defined in the bundle of tangent vectors of a C^2-manifold. The affine connections are themselves cross-sections in a bundle of object-class $l = 2$.
Every affine connection belongs to a class of projectively equivalent affine connections, which class determines a unique normal projective connection. E.Cartan [5] .Such a normal projective connection is a cross-section in a bundle of object-class $l = 2$.

A connection defined in a general way in a fibre-bundle, is not a geometric object. For example a projective connection in a fibre-bundle with fibre the projective n-space, without fined oblique cross-section. Ehresmann [6] Kuiper [7] .

Another example of a geometric object-bundle is obtained from the tensor-bundle of covariant tensors of kind $t_{\kappa\lambda}$ under the identification:

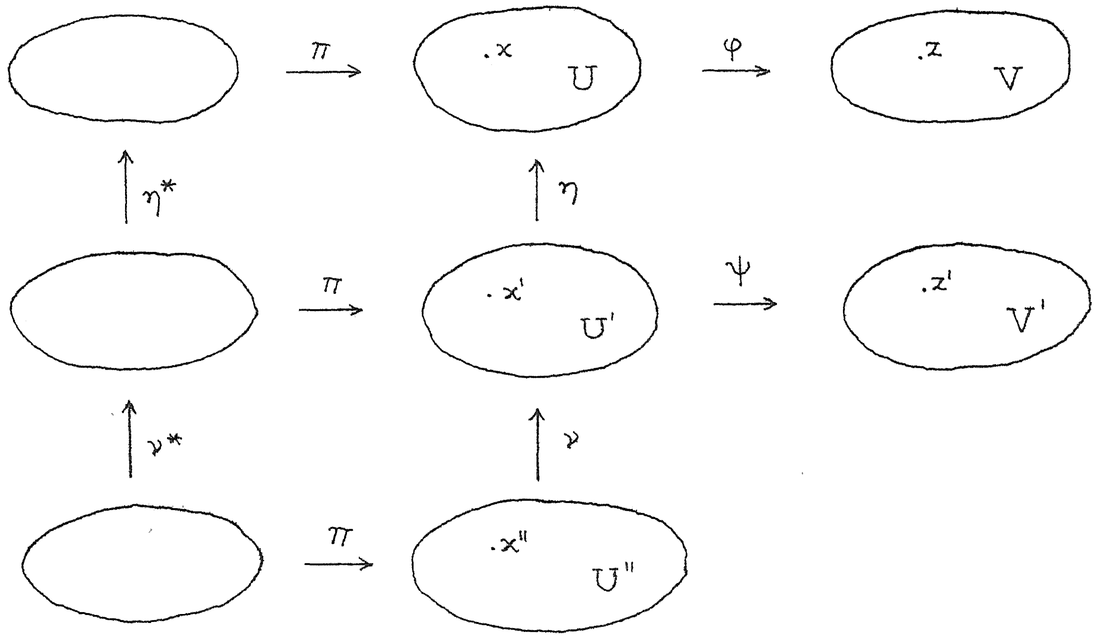
$$t_{\kappa\lambda} \sim \rho\, t_{\kappa\lambda} \qquad \rho > 0 .$$

If $t_{\kappa\lambda}$ is symmetric and positive definite then the geometric object is called a conformal metric. It is of class $l = 1$. The normal conformal connection determined by a conformal metric is a geometric object of class $l = 2$. A conformal connection defined in a general way in a fibre bundle with fibre a sphere and group the Moebius group, is not a geometric object.

Other examples of geometric object-bundles are obtained by taking for Λ a subgroup of all reversible C^s($s = 1, 2, \ldots \infty$ or ω) homeomorphisms, for example consisting of all homeomorphisms that leave invariant a fibred structure or a complex structure in R^n.

§2. Prolongations of Λ-transformations in X.

Let b be a cross-section or a geometric object-field in the object-bundle (B,X,π,Y,G,h,Λ). $b(x)=Y_x \cap b$. A homeomorphism in X: $U' \to U$ is called a Λ-point-transformation if in case $\varphi : U \to V$ is a Λ-reference-system, the same is true for $\varphi\eta : U' \to V$. Let η be a Λ-point transformation.



η has a unique prolongation (Ehresmann [1])

η^* with $\pi\eta^* = \eta\pi$

$$\eta^* : \pi^{-1}(U') \xrightarrow{\text{onto}} \pi^{-1}(U)$$

defined by, if $b' \in \pi^{-1}(U')$,:

$$\eta^* b' = \pi^{-1}\eta\,\pi b' \cap (\varphi^*)^{-1}(\varphi\,\eta^*)b' \tag{4.1}$$

(4.1) can be understood as follows: If we use the reference systems φ and $\varphi\eta$ for U and U' then the prolongation η^* of η is expressed by the pair $\{\eta : U' \to U$ and identity in $Y\}$.

An expression in terms of two arbitrary reference systems $\varphi : U \to V = \varphi U$ and $\psi : U' \to V' = \psi U'$, instead of φ and $\varphi\eta$ respectively, is as follows $(x' = \pi b')$.

$$\eta^* b' = \pi^{-1}\eta\, b'\pi \cap (\varphi^*)^{-1} h\left[t(0,\varphi\eta x').j(\varphi\eta x', \psi x'; \varphi\eta\psi^{-1}).\right.$$
$$\left. .\,t(\psi x', 0)\right]\psi^* b' \tag{4.2}$$

Substituting $\psi = \varphi\eta$ in (4.2) we obtain (4.1). (4.2) is independent of the particular reference system φ for U. This can be seen from straightforward computation.

From (4.1) we have:

$$(\eta^{-1})^* = (\eta^*)^{-1}.$$

Therefore (4.2) is also independent of the reference system ψ for U', hence independent of reference systems used.

Now suppose we have two Λ-point transformations. $\eta : U' \to U$ and $\nu : U'' \to U'$, hence the product $\eta\nu : U'' \to U$. Using the reference systems $\varphi : U \to V$, $\varphi\eta : U' \to V$ and $\varphi\eta\nu : U'' \to V$, we observe that the prolongations η^*, ν^*, $(\eta\nu)^*$ are respectively expressed by:

$$(\eta, \text{identity})$$
$$(\nu, \text{identity})$$
$$(\eta\nu, \text{identity})$$

Hence $(\eta\nu)^* = \eta^*\nu^*$, and we have the

Theorem 2. Every Λ-transformation in X has a unique prolongation in B. The mapping which assigns to every Λ-transformation its prolongation is a group-isomorphism.

From the definitions we also have: Any subbundle $B' \subset B$ is invariant under the prolongations of any Λ-transformation in X.

§ 3. Geometric objects on Lie groups.

Theorem 3. Let $\mathcal{H}$ be a Lie group of transformations operating on the left on the group space H of $\mathcal{H}$. Any object bundle B with base space H and fibre space Y is an analytic product bundle $H \times Y$ with left invariant analytic cross-sections $H \times Y$ $(y \in Y)$.

Proof: Chose a fibre $Y_x \subset B$, and a point $b \in Y_x$. Let $\eta(t) \in \mathcal{H}$ be a transformation of X, $\eta^*(t) \in \mathcal{H}^*$ its prolongation in B, and t a point of the abstract analytic group of $\mathcal{H}$. The set $\mathcal{H}^*b$ consists of one point in each fibre and is an analytic cross-section because $\eta(t)$ is an analytic transformation, which depends also analytically on t. A point $b' \subset \mathcal{H}^*b$ is characterised by $b \in Y_x$ and $\pi(b') = x'$. The correspondence

$$B = \pi^{-1}(H) \to H \times Y_x$$

so obtained is 1 - 1, and $\eta^* \in \mathcal{H}^*$ the prolongation of $\eta \in \mathcal{H}$ is represented under this representation by

$$\eta^*(x \times y) = \eta x \times y .$$

Application:

Theorem: An n-dimensional Lie group has the following left invariant geometric object fields: an absolute parallellism; many affine connections among which symmetric connections; Riemannian metrics of any signature; for n even many almost-complex structures and almost-hermitian metrics; Finsler metrics.

In all these cases we define suitably the geometric object at one point of the group and then the required object-field consists of the images of this geometric object under the prolongations in the fibre-bundle of the transformations of the group.

§ 4. Geometric objects and transitive groups of transformations.

In this § H is a transitive group of $C^\infty - \Lambda$-point transformations of a C^∞-Λ-manifold X, base space of an object-bundle B. I_x is the subgroup of all transformations in H that leave $x \in X$ fixed. The prolongations in B of I_x and H are I_x^* and H^*. I_x and I_x^* are called a group of isotropy of $H(H^*)$ at x.

Theorem 4.1. The object bundle B over X admits a cross-section b invariant under all prolongations in H^*, if and only if the isotropy group I_x^* at x has a fixed point in Y_x. (for some $x \in X$, and then for any $x \in X$).

Proof: The necessity is obvious. To prove the sufficiency we consider a point $b_x \in Y_x$ invariant under I_x^*. For any two transformations f_1 and f_2 in H, which map x onto the same point x', the prolongations f_1^*, f_2^* obey:

$$(f_2^*)^{-1} f_1^* \in I_x^*$$
$$(f_2^*)^{-1} f_1^* \, b_x = b_x$$
$$f_1^* \, b_x = f_2^* \, b_x$$

Therefore the set of points $\{f^* b_x\}$, $f^* \in H^*$, contains exactly one point in the fibre Y_x, for any $x' \in X$. The group properties imply that this cross-section $\{f^* b_x\}$ is invariant under H^*.

In the applications it often occurs that the homomorphism-onto $I_x^* \to I_x^* \mid Y_x$ defined by restriction of the transformations of I_x to the fibre Y_x, is faithfull. This is the case when the restriction of an element η^* of I_x^* to Y_x uniquely determines η^*. In the proof of many theorems on groups of transformations leaving invariant some geometric object we therefore may restrict to considerations concerning one fibre Y_x. For example:

Theorem 4.2. {I} Let X^n be a space with
a) a Riemannian metric b) an affine connection c) a Kählerian metric d) an affine connection with an invariant almost complex structure, with an N-dimensional group of structure preserving transformations. {II} Let N^o be the dimension of the group of a) motions in a space of constant curvature b) affinities in the affine space c) motions in a Fubini-space d) complex-analytic affinities in complex affine space.
Then

$$N \leqq N^o,$$

and equality $N = N^o$ implies that X^n is of the kind mentioned under { II } .
$N^o_a = n(n+1)/2$, $N^o_b = n(n+1)$ and pitting $n = 2m$
$N^o_c = m^2 + 2\,m$, $N^o_d = 2m^2 + 2m$.

Proof: In a space with an affine connection, of which cases abcd are examples, an affine point transformation with fixed point x is determined by its prolongation restricted to the tangent space at x. This ensures the faithfulness of the representation mentioned above. The dimension of the isotropy-group obeys $N-n \leqq N^o - n$, hence $N \leqq N^o$. Next suppose $N = N^o$. In all cases abcd, there is an affine connection. Let Ω be the curvature tensor of this connection and S the (anti symmetric) torsion-tensor ($S = 0$ for the cases a and c), the vanishing of which charactorises the cases mentioned under {II}.

In cases b and d we find, among the prolongations of the point transformations in X with invariant point x, restricted to the tangent space Y_x, those which are geometrical multiplications of the tangent space Y_x. The curvature-tensor and the torsion-tensor must be invariant under the representation of these multiplications in the related tensor-spaces. These representations are also non-trivial geometrical multiplications. Hence $\Omega = 0$ $S = 0$. This proves b and d. In cases a(c) the (holomorphic) sectional curvature is invariant under all orthogonal (unitary) transformations in the tangent space at x.
As H is transitive the (holomorphic) sectional curvature is the same for all (holomorphic) sections at all points and the space is " of constant (holomorphic) curvature ". This proves cases a and c.

§ 5. Lie-derivatives.

Let b be a geometric object field in the bundle B over X. Let

N be a neighbourhood of the identity of a Lie group H, which operates as a group of Λ-transformations

$$\eta(t) : U \rightarrow U(t)$$

in X. $t \in N$ is a point in the group-space. $\eta(t)$ is the corresponding transformation in X. Suppose $x \in U(t)$ for all $t \in N$. Then a mapping of N into $\pi^{-1}(x) = Y_x$ is defined by:

$$L : t \rightarrow Y_x \cap [\eta^*(t).b] \qquad (5.1)$$

In case N = H, H acts as a Lie group of transformations on the image point set L(H) in Y_x.

Under the mapping (5.1) the tangent space at the unit-element of H is mapped into the tangent space of the point $b(x) = b \cap Y_x$ with respect to the fibre. This mapping is called the Lie-differential of the geometric object b, at x, with respect to the given Lie group. For any parametrised differentiable curve t(s), with the identity t(0), the image of the vector dt/ds under the Lie differential is called the Lie-derivative of b, at x, with respect to the parameter s. It is a tangent to a parametrised curve in Y_x.

Theorem 5.1. The Lie-derivative of a geometric object of differentiability-class ≥ 1 is a geometric object.

Proof: If X,Y,G,h are the entities which determine the fibre-bundle B, in which b is a C^r-cross-section $r \geq 1$, then the Lie-derivative $\mathcal{L}b$ is a C^{r-1} cross-section in the fibre-bundle determined in a unique way by: X, Y^1, G, h^1 where Y^1 is the space of all tangent vectors at all points of Y and h^1 is obtained from h by replacing any analytic transformation in Y by its prolongation in Y^1.

Theorem 5.2. A C^r-geometric object b in the bundle B over X, $r \geq 1$, is invariant under the prolongations of a connected Lie-group H of Λ-transformations of X, if and only if the Lie-differential of b at any point $x \in X$ with respect to H vanishes.

The necessity is obvious. The sufficiency is not equally obvious however (!). Suppose the Lie-differential of b at every point $x \in X$ with respect to H vanishes. Suppose, for a fixed x, that the set of points

$$Y_x \cap \eta^*(t).b \qquad t \in H$$

is not one point. Then a curve t(s) with a point $t(1) = t_1$ exists such that the tangent vector

$$\frac{d}{ds}\left\{ Y_x \cap \eta^*(t(s)).b \right\}$$

does not vanish for s = 1.

Let $x = \eta(t_1).x'$. The prolongation $\eta^*(t_1)$ maps $Y_{x'}$ onto Y_x and this mapping is under reference systems represented by an element of G operating in Y. Hence it carries non vanishing tangent vectors of $Y_{x'}$ onto such vectors of Y_x and vice versa. Therefore the curve with parameter s:

$$\left\{\eta^*(t_1)\right\}^{-1}.\left\{Y_x \cap \eta^*(t(s)).b\right\} =$$

$$Y_{x'} \cap \eta^*\left\{t_1^{-1}.t(s)\right\}.b$$

has a non-vanishing derivative for $s = 1$, that is at the point $Y_{x'} \cap b$. The Lie-differential of b at x' with respect to H is then not zero in contradiction with the assumptions.

References.

[1] Charles Ehresmann Les prolongements d'une variété differentiable. Attidel IV Congresso dell' Unione Matematico Italiana Taormina 25-31 Ott.1951. 9 pages.

[2] Charles Ehresmann Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie. Colloque international du C.N.R.S. Géométrie differentielle Strasbourg 1953 p.97-117.

[3] J.Haantjes and G.Laman. On the definition of geometric objects. Proc.Acad.Amsterdam,L VI, Series A= Indagationes Math.XV (1953), p. 208-222.

[4] O.Veblen and J.H.C.Whitehead. The foundations of differential geometry. Cambr.Tract. 29(1932) Ch.II § 8 and Ch.III.

[5] E.Cartan. Sur les variétés à connection projective. Bull.Soc.Math.France, 52 (1924), 205-241.

[6] Charles Ehresmann. Les connections infinitésimales dans un espace fibré différentiable. Colloque de topologie 1950 du C.B.R.M. p.29-55.

[7] N.H.Kuiper. Einstein spaces and connections Proc. Acad.Amsterdam L III, = Indagationes Math.XII (1950) p.504-521.

[8] N.Steenrod. The topology of fibre-bundles. Princeton University Press. (1951).