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notes by

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Introduction

by J. de Groot

Although there have been topological characterizations of metrizability for half a century, a real understanding of this problem has only obtained in the fifties through the work of J. Nagata, Smirnov, Bing and others. This development depended on the fundamental notion of paracompactness (Dieudonné, A.H. Stone, etc.). Similarly, if we ask for a topological characterization of complete metrizability (topological completeness), several characterizations are known. However, in my opinion, they don't reach the core of the matter.

First, this is shown very clearly in the status of the Baire category theorem. Locally compact Hausdorff spaces and completely metrizable spaces are Baire spaces. A nice unifying theorem (Čech) says that any G_δ-subspace of a compact Hausdorff space is a Baire space. But this is not an "intrinsic" theorem. Also it does not tell us anything really new about complete metrizability from the topological point of view.

Secondly, completely metrizable spaces find their proper generalization - or better: many mathematicians think this to be the case - in the theory of "complete uniform" spaces. However, as has been asked by A. Weil in the first edition of a historical note in one of the Bourbaki volumes, what is the status of the Baire category theorem? The answer is simple (Dieudonné and in particular G. Choquet, C.R. Acad. Sci. Paris 232 (1951), 2281-2283): there is no theorem! The rationals, e.g., are complete in a suitable uniform structure, but they constitute the prime example of a space which is not a Baire space. So, again, what does complete metrizability mean topologically?

Satisfactory and by no means simple answer to this question is given by the statement: (1) a metrizable space is topologically complete iff it is cocompact.

Now there are several closely related notions of cocompactness (see Indag. Math. 25 (1963), 761-767 for the also related notion of subcompactness). Let us look at the space of irrationals M. This space is cocompact. What does this mean? If we take all closed intervals in M (with irrational endpoints) and we consider "nests" of such intervals (that is a family satisfying the finite intersection property), and we maximalize such nests (taking maximal families as indicated) we obtain "ultra-nests" which can be considered as points of a new space. So we find the real line R, by the "usual completion" of M. This is nothing new. But now, we proceed just a bit differently. We do not take <u>all</u> closed intervals in M to start with but only certain suitable ones, but still so many that their interiors form an open base for M (that is, they still determine the topology of M). Furthermore, by careful arrangement, one can define this family $\mathcal{F}$ of closed intervals in such a way that <u>every</u> nest of intervals has a non-empty intersection in M; in other words, the irrationals are cocompact, that is compact relative to the family $\mathcal{F}$. The irrationals are also topologically complete, but the rationals are neither topologically complete nor cocompact.

In general a regular space $S = (X,\mathcal{T})$ is <u>cocompact</u> (that is complementary compact) if there exists a family $\mathcal{F} = \{F\}$ of closed sets F for which the interiors form an open base of $\mathcal{T}$, and for which the <u>cospace</u> $S^* = (X,\mathcal{T}^*)$ (where $\mathcal{T}^*$ is that topology on X generated by the complementary sets $\{X \setminus F\}$ as an open subbase) is compact. (S^* is compact T_1; it appears to be rather unimportant, contrary to expectations, whether S^* is Hausdorff.) There are various cospaces S^*. Also various definitions (not mentioned here) of $\mathcal{F}$ define various types of cospaces S^*. However, the topology of a cospace is always weaker than that of the given space. A regular space is always a cospace of itself. A compact Hausdorff space has only itself for a cospace (and conversely).

In all our definitions cocompactness means compactness of a <u>suitable</u> cospace and statement (1) above always holds.

Also, locally compact Hausdorff spaces are cocompact.

Cocompactness is a beautiful invariant: it is invariant for arbitrary topological unions, arbitrary products, even boxproducts, for open (but not for closed) subsets, and it is invariant under fitting mappings (closed continuous mappings onto for which the inverse image of every point is compact).

Using the earliest version of cocompactness, G. Strecker and G. Viglino (independently) obtained in their theses results in, and connections with, the theory of absolutely closed and minimal (Hausdorff or regular) spaces.

Cotopology may be roughly defined as that part of topology in which cospaces are used to study the properties of the given space. Indeed, apart from cocompactness one can introduce other co-properties, e.g. co-connectedness.

It appears to be useful to consider a whole category of spaces and maps and to look at the corresponding cospaces and comappings.

E.g., a (not necessarily continuous) onto map f: X $\rightarrow$ Y is cocontinuous if there is a commutative scheme

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \gamma\downarrow & & \downarrow\lambda \\ X^* & \xrightarrow{f^*} & Y^* \end{array}$$

in which X^* and Y^* are suitable cospaces of X and Y (and γ and λ the induced natural maps, the so called compression maps), such that f^* is continuous onto.

A cocontinuous map admits a certain but limited amount of discontinuity.

A significant application is the following. In the class S of separable metrizable spaces we have: the continuous images of the Cantor space are exactly the compact spaces. This is a classical result, and nowadays we have almost trivial proofs of this theorem. Now the following generalization holds in S: the cocontinuous images of the Cantor space are exactly the completely metrizable spaces. The proof is intricate and by no means a simple generalization. We really obtain a better insight in the area.

An interesting application which arose out of cotopology is the theory of antispaces. I.e., if we consider the real line $\mathbb{R}$ and the cospace over the family $\mathcal{F}$ of all compact subsets of $\mathbb{R}$ we obtain an $\mathbb{R}^*$ which is a compact T_1-space, but which conversely determines $\mathbb{R}$, although - paradoxically - its topology is strictly weaker. A major part of mathematics could be based on $\mathbb{R}^*$ instead of $\mathbb{R}$ (!); one loses the Hausdorff property but gains compactness. Also $\mathbb{R}$ is superconnected (i.e., every open subset is connected!).

Cotopology is still very much underdeveloped. There is a general background, including Strecker's and Viglino's results; also there are some specific, rather deep applications in the theory of metric spaces, as mentioned above. However, there are also grave unsolved problems. Finally, we mention a note by P.C. Baayen and A.B. Paalman-de Miranda, Bohr-compactifications are cocompactifications (Math. Centre, WN 16).

Acknowledgements

After initiating the cotopology concept in several lectures early 1964 my collaborator Dr. J.M. Aarts and I decided to have a seminar on this subject during the academic year 1964-1965 at the Mathematical Centre. Since I was not able to carry on after November 1964, Dr. Aarts conducted this seminar himself.
He not only wrote the notes and filled the gaps in my proof of theorem (1) above, but gave a completely new and better approach to this fundamental theorem (see 2.2 and 2.3). Also the notion of cocontinuity as given here, is due to him. 2.4 and chapter 3 have been added later on to these notes. The first part of 2.4 is due to Dr. Aarts and Dr. R.H. Mac Dowell; the second part of 2.4 (beginning at lemma 1) and chapter 3 is due to myself. Again the final text has been written by Dr. Aarts.
Several participants contributed essentially to this seminar. We mention Dr. P.C. Baayen, P. van Emde Boas, Dr. A.B. Paalman-de Miranda, and J. van der Slot.

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Colloquium Co-topologie

Conventions and Notations

If X is a set and $\mathcal{A}$ a family of subsets of X, then $\mathcal{A}$ is an *additive* (*multiplicative*) *semigroup* iff $\mathcal{A}$ is closed under finite unions (intersections). If $\mathcal{A}$ is closed under finite intersections and finite unions, $\mathcal{A}$ is called a *semiring*.
If X is a topological space and $\mathcal{G}$ a system of closed subsets of X, then $\mathcal{G}$ is a *ring* iff $\mathcal{G}$ is a semiring and $\overline{G_1 \setminus G_2}$, the *difference* of G_1 and G_2, is contained in $\mathcal{G}$ for every pair G_1, G_2 of $\mathcal{G}$. A ring which contains X, is called an *algebra*.

The additive semigroup generated by a family $\mathcal{G}$ is denoted by $\mathcal{S}^+(\mathcal{G})$; the multiplicative semigroup generated by a family $\mathcal{G}$ is denoted by $\mathcal{S}^{\cdot}(\mathcal{G})$. $\mathcal{S}(\mathcal{G})$, resp. $\mathcal{R}(\mathcal{G})$, resp. $\mathcal{A}(\mathcal{G})$ will denote the semiring, resp. ring resp. algebra which are generated by $\mathcal{G}$.

Let X be a topological space. A family $\mathcal{B}$ of open subsets of X is an *open base* for X iff each open subset of X is the union of members of $\mathcal{B}$. A family $\mathcal{S}$ of open subsets of X is called a *subbase for the open subsets* iff $\mathcal{S}^{\circ}(\mathcal{S})$ is an open base for X. A family $\mathcal{B}$ of closed subsets of X is a *base for the closed subsets* iff each closed subset of X is the intersection of members of $\mathcal{B}$. A family $\mathcal{S}$ of closed subsets of X is a *subbase for the closed subsets* iff $\mathcal{S}^+(\mathcal{S})$ is a base for the closed subsets.

Chapter 1

COSPACES

1.1. Definition of a cospace

A *(closed) base* for a topological space T is a family $\mathcal{B}$ of closed subsets of T such that for each point p of T and each open subset O of T which contains p, there is a member B of $\mathcal{B}$ satisfying

$$p \in \mathrm{int}(B) \subset O \quad , \quad p \in B \subset \bar{O}.$$

A (closed) probase for a topological space T is a family $\mathcal{P}$ of closed subsets such that $\mathcal{S}^+(\mathcal{P})$ is a base.

Let T be a topological space defined on a set X and let $\mathcal{P}$ be a probase for T. The cotopology (of T) relative $\mathcal{P}$ is obtained by taking $\mathcal{P}$ for a subbase for the closed sets. Or equivalently, the cotopology (of T) relative $\mathcal{P}$ is the topology on X which has the family $\{X \setminus P \mid P \in \mathcal{P}\}$ for a subbase for its open sets. The space on X defined by the cotopology relative $\mathcal{P}$ is the cospace (of T) relative $\mathcal{P}$. This space will be denoted by $T^*(\mathcal{P})$; and, if no confusion is to be feared, by T^*.

Notation. To distinguish between objects of a space and objects of one of its cospaces we will attach an asterisk to each object of a cospace. So, e.g. if x is a point of the space we write x^* if we wish to consider x as a point of T^*.
For better typography we attach the asterisk on the upper right side although the construction of the cospace is of covariant nature.

The map $c : T \to T^*(\mathcal{P})$, defined by $c(x) = x^*$, is called the compression map relative $\mathcal{P}$. The map $e = c^{-1} : T^*(\mathcal{P}) \to T$, defined by $e(x^*) = x$, is called the expansion map rel. $\mathcal{P}$.
Obviously, c is a continuous map. So the topology of T^* is coarser than the topology of T.

Caution: The space $T^*(\mathcal{P})$ is completely determined by the family $\mathcal{P}$. Soon we will prove that a space is not necessarily determined by a probase for it. As a matter of fact, given a topological space T and a probase $\mathcal{P}$ for it, one can find a probase $\mathcal{G}$ for T such that $\mathcal{G}^*$ is probase for $T^*(\mathcal{P})$. (1.2. Proposition 2)
If E is a topological property, then T has co-E relative $\mathcal{P}$ iff $T^*(\mathcal{P})$ has E. T has co-E iff there is a probasis $\mathcal{P}$ of T such that T has co-E relative $\mathcal{P}$.

1.2. Some simple properties of cospaces

Proposition 1: If T is a space and $\mathcal{P}$ a probase for T, and if $\mathcal{G}^*$ is the family of all closed sets of $T^*(\mathcal{P})$, then $T^*(\mathcal{P}) = T^*(\mathcal{S}^+(\mathcal{P})) = T^*(\mathcal{S}(\mathcal{P})) = T^*(\mathcal{G})$.

Proposition 2: If T is a space and if $\mathcal{G}^*$ is the family of all closed subsets of a cospace $T^*(\mathcal{P})$ of T, then $\mathcal{G}^*$ is a probase for $T^*(\mathcal{P})$ and $\mathcal{G}$ is a probase of T.

Proposition 3: Let $f:T \to S$ be a one-to-one continuous map. If there exists a probase $\mathcal{P}$ of T such that f(P) is closed for each $P \in \mathcal{P}$, then S is a cospace of T (relative the family of inverse images of all closed sets in S).

Proposition 4: If $f:T \to S$ is a one-to-one continuous map on a locally compact Hausdorff space T onto a Hausdorff space S, then S is a cospace of T relative the probase consisting of all compact subsets of T.

Proof: Apply proposition 3.

1.3. Examples

Remark: In this section we make use of a theorem which will be proved in the next section stating that a space T is cocompact relative a probase $\mathcal{P}$ iff each centered system of $\mathcal{P}$ has non-empty intersection.

Ex. 1: If T is a T_1-space on a set V and if $\mathcal{U} = \{U / U$ is a closed subset of T$\}$, then $\mathcal{U}$ is a probase of the discrete space D on V and, consequently, T is a cospace of D relative $\mathcal{U}$.

Particularly, the space of the rationals is a cospace of the countable discrete space.

Ex. 2: Let D be the discrete space on a set V. V endowed with the Zariski-topology (a set is closed iff it is finite) is a compact cospace of D relative the family of all one-point sets.

Proposition 1: A locally compact space is cocompact relative the family of all compact closed subsets.

Proof: If O is an open set and if $p \in O$, choose an open neighbourhood V of p whose closure is compact. Then $p \in O \cap V$, and $O \cap \overline{V}$ is compact, closed and contained in $\overline{O}$. So the family of closed compact subsets constitutes a base. From a well-known compactness-criterion it follows that each centered system of this base has non-empty intersection.

Excercise: A space is cocompact iff it is co-(locally compact).

Proposition 2: If T is a locally compact Hausdorff space, then there exists a cospace of T which is compact Hausdorff.

Proof: If $\alpha(T)$ denotes the one-point compactification of T, and if S is obtained from $\alpha(T)$ by identifying the point at infinity and an arbitrary point of T, proposition 1.2.4 shows that S is a cospace of T.

Ex. 3: As follows from proposition 1, the space $\mathbb{R}$ of the real numbers is cocompact. $\mathbb{R}$ is also cocompact relative the following probase $\mathcal{P}$.
$\mathcal{P} = \bigcup \{ \mathcal{A}_n | n = 1,2,\dots\}$, $\mathcal{A}_n = \{ A^n_m | m \in \mathbb{Z}\}$, and $A^n_m =$
$= \{ x | m\, 2^{-n} \leq x \leq (m+1)2^{-n}\}$.

Ex. 4: The space I of the irrationals is cocompact.

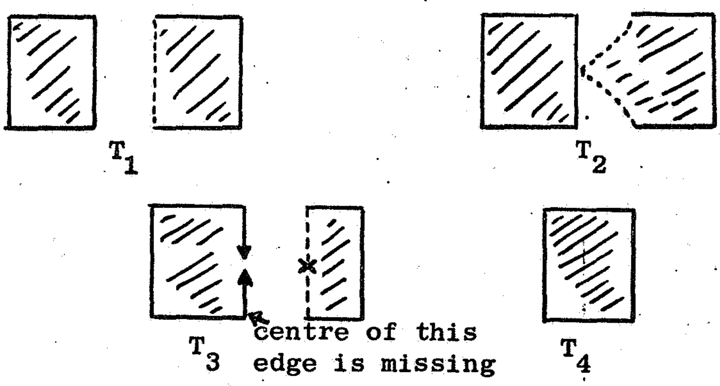
Proof: Let $r_1, r_2, \dots$ be a counting of the rationals.
Let $\mathcal{A}_n$, n=1,2,... be defined as in ex. 3.
Put $\mathcal{B}_n = \{ A | A \in \mathcal{A}_n, r_k \notin A \text{ for } k = 1,2,\dots,n\}$. Let $\mathcal{B}'_n$ be a system of closed intervals filling up those $A \in \mathcal{A}_n$ which are not contained in $\mathcal{B}_n$, but avoiding the points r_k with $k \leq n$, and having pairwise at most one point in common.
Observe that any centered system in $\mathcal{B}_n \cup \mathcal{B}'_n$ is finite.
Let $\mathcal{B} = \bigcup \{ \mathcal{B}_n \cup \mathcal{B}^1_n | n=1,2,\dots\}$.
If $\mathcal{C}_n = \{ C | C = I \cap B, B \in \mathcal{B}_n \cup \mathcal{B}'_n\}$ and if

$\mathcal{C} = \cup \{ \mathcal{C}_n | n=1,2,\dots \}$, then $\mathcal{C}$ is a probase of I (easy excercise) and $I^*(\mathcal{C})$ is compact.
For, if $\mathcal{F}$ is a centered system in $\mathcal{C}$, then $\mathcal{F}' = \{ F | F \in \mathcal{B}, F \cap I \in \mathcal{F} \}$ is also a centered system. From the observation above it follows that $\mathcal{F}'$ contains arbitrary small sets (i.e. $\mathcal{F}'$ is a Cauchy-filter) and consequently, $\cap \mathcal{F}'$ consist of just one point, because $\mathbb{R}$ is a complete metric space. Because r_k is not contained in any element of $\mathcal{B}_n \cup \mathcal{B}'_n$ if $n \geq k$, $\cap \mathcal{F}'$ is contained in I and it follows that $\cap \mathcal{F} = \cap \mathcal{F}'$ equals a point of I.

Excercise: Proof that the space of the rationals is not cocompact.

Ex. 5: Let T be the subspace of E^2 the pointset of which is $\{(x,y) | y=0$ and x is an irrational number, or, $x=p/q$ with $(p,q)=1$ and $y=1/q \}$.
T is cocompact relative the probase consisting of all isolated points (x,y) with $y \neq 0$ and of all intersections of T and a square having a set of $\mathcal{B}$ (ex. 4) as an edge.

Ex. 6: Consider the following subspaces of E^2.



Using proposition 1.2.3 one easily shows that T_4 is a cospace of T_1 and T_2. However, it is not a cospace of T_3.
From the pictures we see that an expansion map tears up the space ($T_4 \to T_1$, $T_4 \to T_2$). However not every tearing up is an expansion map ($T_4 \to T_3$).

Ex. 7: Let A $\{x \mid x$ is a rational number and $0 \leq x \leq 1\}$ and $B = \{x \mid x$ is an irrational number and $2 < x < 3\}$. The unit interval is not a co-space of the union of A and B, although there is a (very natural) one-to-one map of $A \cup B$ onto the unit interval (see excercise after ex. 4).

Ex. 8: The map $f(n) = \exp(2\pi i \alpha n)$, α irrational, is a continuous map of the integers $\mathbb{Z}$ into the circle C in the complex plane.
This map is a homomorphism into if $\mathbb{Z}$ and C are taken to be topological groups. $f(\mathbb{Z})$ is a cospace of $\mathbb{Z}$.

Ex. 9: Let T be a torus which is obtained from the plane vector group by reducing mod one in both the x and y directions.
The line $y = \alpha x$, α irrational, is mapped into T by reduction mod one. This reduction is a continuous homomorphism. The image of the line is a cospace of it.

1.4. A compact Hausdorff space coincides with each of its cospaces

Two points p and q of a space T are separated by open sets if there are open subsets U en V of T satisfying

$$p \in U,\ q \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Lemma 1: If T is a space and $\mathcal{P}$ a probase, then the points p and q are separated by open sets in $T^*(\mathcal{P})$ iff there is finite cover of T the elements of which are taken from $\mathcal{P}$ such that any element of the cover does not contain both p and q.

Proof: p and q are separated by open sets in $T^*(\mathcal{P})$ iff p and q are separated by elements of an open base of $T^*(\mathcal{P})$ iff there are elements $Q_i \in \{T \setminus P \mid P \in \mathcal{P}\}$, $i=1,\dots,m$, satisfying $\bigcap_{i=1}^{m} Q_i = \emptyset$,
$p^* \in \bigcap_{i=1}^{n} Q_i$, $q^* \in \bigcap_{i=n+1}^{m} Q_i$.

Iff (putting $P_i = T \setminus Q_i$)
$\bigcup_{i=1}^{m} P_i = T$, $p \notin \bigcup_{i=1}^{n} P_i$, $q \notin \bigcup_{i=n+1}^{m} P_i$.

Remark: Separation-axioms and related questions are treated in section 1.7.

Theorem 1: In a regular T_1-space T the following properties are equivalent:

a) If $\mathcal{P}$ is some probase, then $T^*(\mathcal{P})=T$,

b) If $\mathcal{P}$ is some probase, then $T^*(\mathcal{P})$ is a Hausdorff space,

c) If $\mathcal{P}$ is some probase, then in $T^*(\mathcal{P})$ there are points p and q which are separated by open sets, unless T has only one point,

d) T is compact,

e) T is cocompact relative a base which is an algebra,

f) T is cocompact relative a base $\mathcal{B}$ satisfying $T \in \mathcal{B}$ and if $B \in \mathcal{B}$, then $\overline{T \setminus B} \in \mathcal{B}$.

Proof: a) $\rightarrow$ b) $\rightarrow$ c) is obvious.

c) $\rightarrow$ d) Let $\mathcal{O}$ be a cover of T no finite subset of which is a cover. Choose a base $\mathcal{B}$ which refines $\mathcal{O}$.

Proposition c) and lemma 1 imply the existence of a finite cover of T by means of elements from $\mathcal{B}$. Hence $\mathcal{O}$ has a finite subcover, contradicting the hypothesis.

d) $\rightarrow$ a). It suffices to prove that the expansion-map e: $T^*(\mathcal{P}) \rightarrow T$ is continuous. Let $p \in O \subset T$.

For each $q \in T \setminus O$ choose an open neighbourhood U_q the closure of which avoids p, and choose $B_q \in \mathcal{B} = \mathcal{S}^+(\mathcal{P})$ such that $q \in B_q^o$, $B_q \subset \bar{U}_q$. By compactness of $T \setminus O$, there is a finite cover of $T \setminus O$ by means of elements of $\{B_q | q \in T \setminus O\}$. The union of this cover, B say, is element of $\mathcal{B}$. So $T^* \setminus B^*$ is an open set containing p^*. $e(T^* \setminus B^*) \subset T \setminus B$. So, continuity of e follows.

d) $\rightarrow$ e): Take the algebra of all closed sets.

e) $\rightarrow$ f): trivial.

f) $\rightarrow$ d): It suffices to prove that the expansion map $e: T^*(\mathcal{B}) \rightarrow T$ is continuous. If $p \in O \subset T$, then choose a member B of $\mathcal{B}$ such that $p \in B^o \subset B \subset O$ (regularity!). It follows that $\overline{T \setminus B} \in \mathcal{B}$ and $(T \setminus (\overline{T \setminus B}))^*$ is open in T^*.

$e((T \setminus \overline{(T \setminus B)})^*) = T \setminus \overline{(T \setminus B)} = B^o \subset O$. So, e is continuous.

Theorem 2: In a regular T_2-space T the following properties are equivalent:

a) T is locally compact,

b) T is cocompact relative a base which is a ring,

c) T is cocompact relative a base $\mathcal{B}$ which is closed for taking differences.

Proof: a) → b) → c) is obvious.

c) → a). If B_o is any member of $\mathcal{B}$, then B_o is cocompact relative the base $\mathcal{B}_o = \{ B_o \cap B \mid B \in \mathcal{B}\}$. If $B \in \mathcal{B}$, then $\overline{B_o \setminus B}^{(B_o)} = \overline{B_o \setminus B} = \overline{B_o \setminus B} \cap B_o \in \mathcal{B}_o$. So, $\mathcal{B}_o$ satisfies f) of Theorem 1, and, consequently, B_o is compact.

1.5. Cocompactness

Proposition 1: Let T be a space and $\mathcal{P}$ a probase of T and let $\mathcal{G}^*$ denote the family of all closed subsets of $T^*(\mathcal{P})$.

Then, T is cocompact rel. $\mathcal{P}$, iff T is cocompact rel. $\mathcal{S}^+(\mathcal{P})$, iff T is cocompact rel. $\mathcal{S}(\mathcal{P})$, iff T is cocompact rel. $\mathcal{R}(\mathcal{P})$, iff T is cocompact rel. $\mathcal{G}$.

Proof: Apply proposition 1.2.1.

Proposition 2: A space T is cocompact relative a probase $\mathcal{P}$ iff each centered system of $\mathcal{P}$ has non-void intersection.

Proof: "only if" is trivial. The "if-part" of the proposition is essentially the lemma of Alexander stating that if $\mathcal{S}$ is a subbase for the open sets of a space T such that every cover of T by members of $\mathcal{S}$ has a finite subcover, then T is compact. Each centered system of $\mathcal{P}$ has non-void intersection iff each cover the elements of which are taken from $\{ X \setminus P \mid P \in \mathcal{P}\}$ has a finite subcover.

Recall that the family $\{X \setminus P \mid P \in \mathcal{P}\}$ serves as a subbase for $T^*(\mathcal{P})$, so the lemma of Alexander yields the compactness of $T^*(\mathcal{P})$.

Problem 1: Let T be a cocompact space Give a survey of all compact cospaces of T

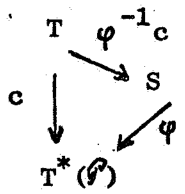
The following proposition gives some information of the family of all compact cospaces of a given space.

Lemma 1: If T is a Hausdorff space and if G^* is a compact subset of $T^*(\mathcal{P})$, then G is a closed subset of T.

Proof: Suppose $p \in \bar{G}$, $p \notin G$. If $\mathcal{U}$ denotes the family of all neighbourhoods of p which belong to $\mathcal{S}^+(\mathcal{P})$, then the trace of $\mathcal{U}$ on G, $\mathcal{U}_G$ say, is a centered system the intersection of which is empty for T is a Hausdorff space and $p \notin G$. Consequently, $\mathcal{U}^*_G$ is a centered system of closed sets on G^* the intersection of which is empty, contradicting the compactness of G^*.

Proposition 3: If T is a Hausdorff space and if S is a compact space which can be mapped one-to-one onto $T^*(\mathcal{P})$, then S is a cospace of T.

Proof:



If G is a closed subset of S, then G is compact and so is $\varphi(G)$. By lemma 1 this implies that $c^{-1}\varphi(G)$ is closed. So $\varphi^{-1}c$ is continuous.

If $P \in \mathcal{P}$, then c(P) is closed by the definition of $T^*(\mathcal{P})$ and $\varphi^{-1}c(P)$ is closed by continuity of φ. From proposition 1.2.3 it follows that S is a cospace of T.

If a space T is co(countably compact) relative $\mathcal{P}$, then each countable centered system of $\mathcal{P}$ has non-void intersection.

R.H. Mc.Dowell has constructed an example which shows that the converse of this statement does not hold.

Problem 2: If $\mathcal{P}$ is a probase of T each countable centered system of which has non-empty intersection, does there exist a probase $\mathcal{P}'$ of T such $T^*(\mathcal{P}')$ is co(countably compact).

Cocompactness does not imply any separation, as the following examples illustrate.

Ex. 1: Let T be a space the pointset of which is the set of real numbers.
Take the family of all open intervals together with the set of the rationals as a subbase for the open sets of T. T is a Hausdorff space, but T is not a regular space.
T is cocompact and the family of all compact intervals serves as a base relative which T is cocompact.

Ex. 2: (The Tychonoff plank). Let Ω' be the set of ordinal numbers not greater than the first uncountable ordinal Ω, and let ω' be the set of ordinals not greater than the first infinite ordinal, ω, each with the order topology. The product $\Omega' \times \omega'$ is called the Tychonoff plank. The subspace $P = \Omega' \times \omega' \setminus \{(\Omega, \omega)\}$ is locally compact, so cocompact.
P is a completely regular space which is not a normal one.

Ex. 3: (The Tychonoff ladder). For each $n \in \mathbb{Z}$ define P_n, Ω'_n, etc., as in ex. 2. Let $P = \bigcup \{ P_n \mid n \in \mathbb{Z} \}$ be the topological union of $\{ P_n \mid n \in \mathbb{Z} \}$. (We suppose $P_n \cap P_m = \emptyset$ if $n \neq m$).
Identify the points $(\Omega, y)_{2n-1}$ and $(\Omega, y)_{2n}$, and identify the points $(x, \omega)_{2n}$ and $(x, \omega)_{2n+1}$ for each y, x and n, and denote the de - composition space by P^*, and the natural project of P onto P^* by φ.
We obtain the Tychohoff ladder T by adding two points, ξ and η say, to P^* and by taking as a base for the neighbourhoodsystem the family of open sets of P^* together with the sets $V_n(\xi) = \xi \cup \varphi(\bigcup_{k=n}^{\infty} P_k)$ and $V_n(\eta) = \eta \cup \varphi(\bigcup_{k=n}^{\infty} P_{-k})$, $n=1,2,\ldots$.
If f is any continuous real valued function on L, then $f(\xi) = f(\eta)$ (hard excercise), so L is not completely regular.
It is not hard to show that L is a regular space.
If $\mathcal{B} = \{\text{compact sets of } L\} \cup \{V_n(\eta)\}_{n \in N} \cup \{V_n(\xi)\}_{n \in N}$, then $\mathcal{B}$ is a base for L and L is cocompact relative $\mathcal{B}$.

Remark:

The Tychonoff ladder L cannot be mapped one-to-one and continuously onto a compact Hausdorff space. For, if so, points of L could be separated by continuous functions.

So, the Tychonoff ladder has no cospace which is at the same time compact and Hausdorff. Shortly, the Tychonoff ladder is cocompact but not co-(compact Hausdorff).

1.6. Constructing new cocompact spaces from old

Proposition 1: For regular spaces cocompactness is an invariant for the taking of open subsets.

Proof: Suppose T is cocompact relative a prebase $\mathcal{B}$. If O is an open subset of T, then $\mathcal{B}' = \{B \mid B \in \mathcal{B}, B \subset O\}$ is a prebase for the relative topology on O, because T is regular. Obviously, every centered system of $\mathcal{B}'$ has a non-void intersection, so $O^*(\mathcal{B}')$ is compact.

Remark: In general cocompactness is not an invariant for the taking of closed subsets; see ex. 2 below.
However, in locally compact spaces and in metrizable spaces cocompactness is an invariant for the taking of closed subsets. In case of locally compact spaces this follows from the fact that a closed subset is locally compact. For metric spaces we will prove the invariance in the following chapter.

Proposition 2: Cocompactness is an invariant for the forming of topological unions.

Proof: Take a base which is a union of the bases relative to which each summand is cocompact.

The cocompactness of a product of cocompact spaces follows from the next theorem. In fact, cocompactness is an invariant for the forming of mixed-product which we now define.
If for each α from an index set A, T_α is a space on X_α and if π_α denotes the natural projection of the cartesian product $\prod\{X_\alpha \mid \alpha \in A\}$ onto X_α and if $\mathcal{C}$ is a subset of the family 2^A of all subsets of A, which contains all one-element sets of A, then the mixed-product $P_\mathcal{C}$ of the family $\{T_\alpha \mid \alpha \in A\}$ is a space on the cartesian product of $\{X_\alpha \mid \alpha \in A\}$ for which the family $\{\bigcap\{\pi_\gamma^{-1}(O_\gamma) \mid \gamma \in C\} \mid C \in \mathcal{C}, O_\gamma$ open in $T_\gamma\}$ serves as a subbase for the open sets.
So, the usual topological product is a mixed product (take for $\mathcal{C}$ the family of one-element subsets of A) and every mixed product $P_\mathcal{C}$ is finer than the usual topological product. If $\mathcal{C}$ equals 2^A, then $P_\mathcal{C}$ is the box-product of the family $\{T_\alpha \mid \alpha \in A\}$, and every mixed product is coarser than the box product. Obviously, in $P_\mathcal{C}$ the natural

projection π_α is both open and continuous for every $\alpha \in A$.
The usual topological product of a family $\{T_\alpha \mid \alpha \in A\}$ is denoted by $\prod\{T_\alpha \mid \alpha \in A\}$.

Lemma 1: If for each $\alpha \in A$ T_α is a space and $\mathcal{U}_\alpha$ a base of T_α and if T is a mixed product of the family $\{T_\alpha \mid \alpha \in A\}$, then $\prod\{T^*_\alpha(\mathcal{U}_\alpha) \mid \alpha \in A\}$ is a cospace of T.
Proof: First, observe that if Y_α is a subset of T_α for each α, then the closure of $\prod\{Y_\alpha \mid \alpha \in A\}$ in T equals $\prod\{Y^-_\alpha \mid \alpha \in A\}$. For, if π_α denotes the natural projection onto T_α, then the closure of $\prod\{Y_\alpha \mid \alpha \in A\}$ is contained in $\pi_\alpha^{-1}(Y^-_\alpha)$ for each α, so, contained in $\bigcap\{\pi_\alpha^{-1}(Y^-_\alpha) \mid \alpha \in A\} = \prod\{Y^-_\alpha \mid \alpha \in A\}$. Conversely, if $q \in \prod\{Y^-_\alpha \mid \alpha \in A\}$ and if β is a fixed member of A, then $\pi_\beta(q) \in Y^-_\beta$ and consequently $V_\beta \cap Y_\beta \neq \emptyset$ for every neighbourhood V_β of $\pi_\beta(q)$. So, if $\prod\{V_\alpha \mid \alpha \in A\}$ is a neighbourhood of q in T, then $\prod\{Y_\alpha \mid \alpha \in A\} \cap \prod\{V_\alpha \mid \alpha \in A\} = \prod\{Y_\alpha \cap V_\alpha \mid \alpha \in A\} \neq \emptyset$ and therefore q belongs to the closure of $\prod\{Y_\alpha \mid \alpha \in A\}$.

Now, let $\mathcal{C}$ be a subset of 2^A such that the family of subsets of the form $\bigcap\{\pi_\gamma^{-1}(O_\gamma) \mid \gamma \in C\}$ ($C \in \mathcal{C}$, O_γ open in T_γ) is a base for the open sets of T. Then, the family $\mathcal{U}$ of subsets of T of the form $\bigcap\{\pi_\gamma^{-1}(U_\gamma) \mid \gamma \in C\}$ ($C \in \mathcal{C}$, $U_\gamma \in \mathcal{U}_\gamma$) is a base for T. For, if p is a point of an open subset O of T, then choose an element of the base for the open sets $\bigcap\{\pi_\gamma^{-1}(O_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$ which is contained in O. For each $\gamma \in C_o$ select an element U_γ of the base $\mathcal{U}_\gamma$ satisfying $\pi_\gamma(p) \in U^o_\gamma$ and $U_\gamma \subset O^-_\gamma$. Then, $\bigcap\{\pi_\gamma^{-1}(U_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$ is contained in the set $\bigcap\{\pi_\gamma^{-1}(O^-_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$ which equals the closure of $\bigcap\{\pi_\gamma^{-1}(O_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$ as follows from the observation above. $\bigcap\{\pi_\gamma^{-1}(U^o_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$ is a member of the base for the open sets, so p is contained in the interior of $\bigcap\{\pi_\gamma^{-1}(U_\gamma) \mid \gamma \in C_o \in \mathcal{C}\}$.

Let T^* denote the cospace of T relative the family $\mathcal{U}$. Then $\mathcal{U}$ is a subbase for the closed subsets of T^*. Because each element of $\mathcal{U}$ is the intersection of sets of the form $\pi_\gamma^{-1}(U_\gamma)$ ($\gamma \in A, U_\gamma \in \mathcal{U}_\gamma$), we conclude that $\{\pi_\gamma^{-1}(U_\gamma) \mid \gamma \in A, U_\gamma \in \mathcal{U}_\gamma\}$ is a subbase for the closed subsets of T^*. But $\{\pi_\gamma^{-1}(U_\gamma) \mid U_\gamma \in \mathcal{U}_\gamma, \gamma \in A\}$ is a subbase which defines the topological product of $\{T^*_\alpha(\mathcal{U}_\alpha) \mid \alpha \in A\}$.

Corollary: Any two mixed products of a family $\{T_\alpha \mid \alpha \in A\}$ have cospaces which are homeomorphic.

Theorem 1: If T is a mixed product of a family of cocompact spaces, then T is cocompact.

Proof: If T is a mixed product of the family $\{T_\alpha \mid \alpha \in A\}$, then for each T_α choose a base $\mathcal{U}_\alpha$ such that $T^*(\mathcal{U}_\alpha)$ is compact. Applying lemma 1 and the Tychonoff product theorem, we deduce that $\prod\{T^*(\mathcal{U}_\alpha) \mid \alpha \in A\}$ is a compact cospace of T.

Corollary: If T is a mixed product of a family of co-(compact hausdorff) spaces, then T is co-(compact hausdorff).

Ex.1: An interesting application of the lemma above is obtained by the observation that the space of the irrationals I is homeomorphic to the product space $\prod_{i=1}^{\infty} \mathbb{Z}_i$, $\mathbb{Z}_i$ being the space of the integers. For each factor $\mathbb{Z}_i$ take a cospace as described in the proof of proposition 1, 3,2 and denote this cospace by $\mathbb{Z}_i^*$. $\mathbb{Z}_i^*$ is compact, metric and zero-dimensional, so $\prod_{i=1}^{\infty} \mathbb{Z}_i^*$ is. Furthermore $\prod_{i=1}^{\infty} \mathbb{Z}_i^*$ is dense in itself, so it is homeomorphic to the Cantor-set as follows from the characterization of the Cantor-set: any compact zerodimensional metric space which is dense in itself is homeomorphic to the Cantor-set.
So, the Cantor-set is a cospace of the space of the irrationals.

Now, we start the construction of an example of a cocompact space which has closed non-cocompact subspaces.

Proposition 3: Every separable metric space is homeomorphic to a closed subset of $\mathbb{R}^{\underline{c}}$.

Proof: If M is a separable metrizable space, then by the Urysohn-embedding-theorem M is homeomorphic to a closed subset of a countable product of copies of the real line. So we suppose $M \subset \mathbb{R}^a$. Let $A = \mathbb{R}^a \setminus M$. For each $x \in A$: $f_x : M \to \mathbb{R}$ is defined by $f_x(y) = (\rho(x,y))^{-1}$, ρ being a metric of $\mathbb{R}^a$. Clearly, f_x is continuous. Let $f : M \to \mathbb{R}^{\underline{c}} = \prod\{\mathbb{R}_x \mid x \in A\}$ be defined by $f(y) = (f_x(y))_{x \in A}$, then f is continuous. Consider the graph G of f in $M \times \mathbb{R}^{\underline{c}}$. Because of the continuity of f, G is a closed subset of $M \times \mathbb{R}^{\underline{c}}$ and G is homeomorphic to M. Now, we prove that G is a closed subset of $\mathbb{R}^{\underline{a}} \times \mathbb{R}^{\underline{c}}$. For, if not, let z be an accumulation point of G which is not contained in G. Put $z = (z_1, z_2)$, $z_1 \in \mathbb{R}^{\underline{a}}$, $z_2 \in \mathbb{R}^{\underline{c}}$ and let π denote the natural projection of $\mathbb{R}^{\underline{a}} \times \mathbb{R}^{\underline{c}}$ onto $\mathbb{R}^{a}$. $z_1 \notin M$, for G is a closed subset of $M \times \mathbb{R}^{\underline{c}}$. Clearly, z_1 is an accumulation point of M. Because $f_{z_1}(y)$ tends to infinity if y tends to z_1, for every real number k there is a neighbourhood U of z_1 such that if $w \in G$, $w \in \pi^{-1}(U(z_1))$, then $\pi_{z_1}(w) > k$, π_{z_1} denoting the natural projection into $\mathbb{R}_{z_1}$. Now, putting $k = \pi_{z_1}^{-1}(z_2) + 1$, we obtain a neighbourhood $\pi^{-1}(U_{z_1}) \cap \pi_{z_1}^{-1}(z_2-1, z_2+1)^1$ of (z_1, z_2) which is disjoint from G, contradicting the assumption that z is an accumulation point of G.

As an application of proposition 3 we obtain:

Ex.2: The space of the rational numbers is homeomorphic to a closed subset of $\mathbb{R}^{\underline{c}}$. $\mathbb{R}^{\underline{c}}$ is cocompact by Theorem 1 and the rationals are not (section 1.3, exercise).
So cocompactness is not an invariant for the taking of closed subsets. For later use we observe that $\mathbb{R}^{\underline{c}}$ is not a G_δ-subset of any compact hausdorff space (see chapter 2).

We finish this section by formulating some problems related to the invariance of cocompactness.
Problem 1: Is a G_δ-subset of a cocompact space cocompact?
Is a G_δ-subset of a compact hausdorff space cocompact?

Problem 2: From proposition 1 and 2 it follows that the topological union of a family of spaces is cocompact iff each summand is cocompact. Does the cocompactness of the product of a family imply the cocompactness of every factor?

Problem 3: Let $T = A \cup B$, A open, B open, A and B cocompact. Is T cocompact?

Problem 4: Let $T = A \cup B$, A and B cocompact. Is $A \cap B$ cocompact?

Illustrating problem 3 and 4 we have the following example.

Ex.3: The cocompactness of a space T relative a prebase U and relative a prebase V does not imply the cocompactness of T relative the prebase $U \cup V$. For, let T be the subspace of the plane the point set of which is $[0,1] \times [0,1] \setminus \{0,0\}$. For each natural number n let $V_n = \{(x,y) \mid x \leq 1/n\}$ and $U_n = \{(x,y) \mid y \leq 1/n\}$. If $V = \{C \mid C$ is compact$\} \cup \{V_n\}_{n=1}^{\infty}$ and $U = \{C \mid C$ is compact$\} \cup \{U_n\}_{n=1}^{\infty}$, then T is cocompact relative U and V, but T is not cocompact relative $U \cup V$.

Problem 5: Let T be a space which is

1. separable metric
2. zerodimensional
3. cocompact (or, equivalently (see chapter 2), complete in a suitable metric)
4. nowhere locally compact.

Is T homeomorphic to the space of the irrational numbers? (Cf.: ex.1.)

1.7. Separation axioms

In this section we give a characterization of separation axioms for a topological space by means of properties of a suitable (sub)base for the closed sets. Its usefulness for the study of cospaces is evident.

The characterization of complete regularity seems to be new and is rather surprising (theorem 1).

If T is a space and if $\mathcal{O}$ is a family of open sets of T then two subsets A_1 and A_2 are separated by $\mathcal{O}$ if there are elements O_1 and O_2 of $\mathcal{O}$ such that $A_1 \subset O_1$, $A_2 \subset O_2$ and $O_1 \cap O_2 = \emptyset$.

Dually, if T is a space and if $\mathcal{G}$ is a family of closed sets of T then two subsets A_1 and A_2 are separated by $\mathcal{G}$ if there are elements G_1 and G_2 of $\mathcal{G}$ such that $G_1 \cap A_1 = \emptyset$, $G_2 \cap A_2 = \emptyset$ and $G_1 \cup G_2 = T$ (and consequently $A_2 \subset G_1$, $A_1 \subset G_2$).

Proposition 1: A T_1-space T is hausdorff iff there is a base for the closed sets $\mathcal{B}$ such that any two points are separated by $\mathcal{B}$.

Verification of this proposition is straightforward.

Proposition 2: A T_1-space T is regular iff there is a base for the closed sets $\mathcal{B}$ such that, if p is a point of T and B an element of $\mathcal{B}$ not containing p, then p and B are separated by $\mathcal{B}$.

Proof: "only if": If T is regular, then the family of all closed sets satisfies the condition of the theorem.

"if": Let p be a point of T and G a closed set not containing p. Because $\mathcal{B}$ is a base, there is an element $B_1 \in \mathcal{B}$ such that $p_1 \notin B_1$ and $B_1 \supset G$.

p and B_1 can be separated by $\mathcal{B}$ and so there are B_2 and B_3 in $\mathcal{B}$ satisfying: $p \notin B_2$, $B_1 \cap B_3 = \emptyset$ and $B_2 \cup B_3 = T$.

The complement of B_2 resp. B_3 are neighbourhoods of p resp. G which are disjoint.

In the same way we obtain

Proposition 3: A T_1-space T is normal iff there is a base for the closed sets $\mathcal{B}$ such that any two disjoint closed subsets of T are separated by $\mathcal{B}$.

Theorem 1: A T_1-space T is completely regular iff there is a base $\mathcal{B}$ for the closed sets such that

1. any point p and any element B of $\mathcal{B}$ not containing p, are separated by $\mathcal{B}$,
2. any two disjoint elements of $\mathcal{B}$ are separated by $\mathcal{B}$.

Proof: "only if": If f is a continuous real-valued function on T, then the zero-set of f, Z(f), is defined by:

$Z(f) = \{x \mid x \in T,\ f(x) = 0\}$. Let $\mathcal{Z}$ denote the family of all zero-sets of T. Obviously, as T is completely regular, $\mathcal{Z}$ is a base for the closed sets. If $B \in \mathcal{Z}$, $p \notin B$, then, as it is easy to see, there is a zero-set Z_1 such that $p \in Z_1$ and $Z_1 \cap B = \emptyset$. So it remains to proof 2.

If $Z(f) \cap Z(g) = \emptyset$, let h be defined by $h(x) = \dfrac{|f(x)|}{|f(x)| + |g(x)|}$.

Then $h(Z(f)) = 0$ and $h(Z(g)) = 1$.

$Z_1 = Z(\sup(h,2/3) - 2/3) = \{x \mid h(x) \leq 2/3\}$ and $Z_2 = Z(\inf(h,1/3) - 1/3) = \{x \mid h(x) \geq 1/3\}$ satisfy $Z_1 \cap Z(g) = \emptyset$, $Z_2 \cap Z(f) = \emptyset$ and $Z_1 \cup Z_2 = T$.

"if": We will make use of the following lemma (see J.L. Kelley: General Topology pag.114):

<u>Lemma 1</u>: Suppose that for each member t of a dense subset D of the positive reals F_t is an open subset of a topological space X such that:

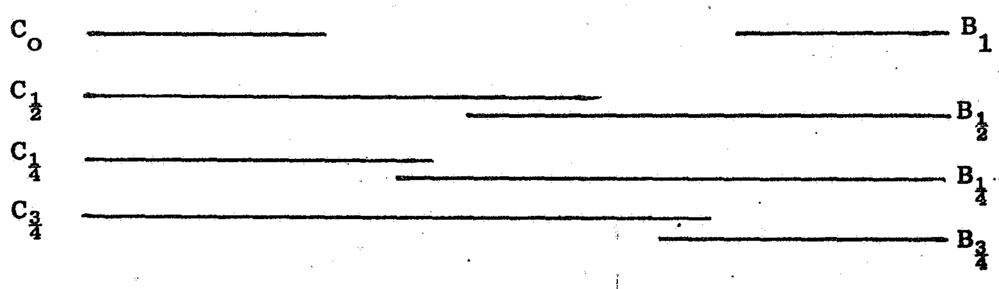
(a) if $t < s$, then the closure of F_t is a subset of F_s; and

(b) $\bigcup \{F_t \mid t \in D\} = X$.

Then the function f such that $f(x) = \inf \{t \mid x \in F_t\}$ is continuous.

To show the complete regularity of T we have to construct for every point p and every set G not containing p a continuous real-valued function f satisfying $f(T) \subset [0,1]$, $f(p) = 0$ and $f(G) = 1$.
Because of condition 1 it is sufficient to construct for any two disjoint elements C_o and B_1 of $\mathcal{B}$ a continuous real-valued function f such that $f(C_o) = 0$, $f(B_1) = 1$ and $f(T) \subset [0,1]$.
The construction resembles the usual one for a similar function for disjoint closed sets of a normal space (cf. J.L. Kelley, p.115).
Observe, that in view of proposition 3 our construction also yields for any pair of disjoint closed sets F and G in a normal space a continuous function to the interval $[0,1]$ such that f is zero on F and one on G.

Let D be the set of positive numbers of the form $p.2^{-q}$, where p and q are positive integers. For t in D and $t > 1$ let $F(t) = T$ and let $F(1) = T \setminus B_1$. For t in D and $0 < t < 1$ write t in the form $t = (2m+1)2^{-n}$ $(n \geq 1,\ m \geq 0)$ and choose, inductively on n, two elements C(t) and B(t) from $\mathcal{B}$ such that $B((2m+2).2^{-n}) \cap C(t) = \emptyset$, $C(2m.2^{-n}) \cap B(t) = \emptyset$ and $C(t) \cup B(t) = T$.
This choise is possible because B satisfies condition <u>2</u>.



Now, define $F(t) = T \setminus B_t$ and $f(x) = \inf \{t \mid x \in F_t\}$ then f is continuous in view of lemma 1, $f(C_o) = 0$ and $f(B_1) = 1$.

To obtain analogous theorems for subbases we define:
If T is a space and if $\mathcal{G}$ is a family of closed sets, then two subsets A_1 and A_2 are screened by $\mathcal{G}$ if there is a finite cover of T by elements of $\mathcal{G}$ such that each element of the covering meets at most one of the sets A_1 and A_2.

Obviously we have the following:

Lemma 2: Let T be a space and let A_1 and A_2 be subsets of T. If $\mathcal{G}$ is a family of closed sets, then A_1 and A_2 are screened by $\mathcal{G}$ iff A_1 and A_2 are separated by $\mathcal{J}^+(\mathcal{G})$.

From this we obtain

Proposition 4: A T_1-space T is hausdorff iff there is a subbase for the closed sets $\mathcal{B}$ such that any two points are screened by $\mathcal{B}$.

Proposition 5: A T_1-space T is normal iff there is a subbase for the closed sets $\mathcal{B}$ such that any two disjoint closed subsets of T are screened by $\mathcal{B}$.

Proposition 6: A T_1-space T is regular iff there is a subbase for the closed sets $\mathcal{B}$ such that, if p is a point of T and B an element of $\mathcal{J}^+(\mathcal{B})$ not containing p, then p and B are screened by $\mathcal{B}$.

Proposition 7: A T_1-space T is completely regular iff there is a subbase for the closed sets $\mathcal{B}$ such that

1. any point p and any element B of $\mathcal{J}^+(\mathcal{B})$ not containing p, are screened by $\mathcal{B}$,
2. any two disjoint elements of $\mathcal{J}^+(\mathcal{B})$ are screened by $\mathcal{B}$.

The following proposition is a modification of proposition 7

Proposition 8: A T_1-space T is completely regular iff there is a subbase for the closed sets $\mathcal{B}$ such that

1. $\mathcal{B} = \mathcal{J}^*(\mathcal{B})$,
2. any point p and any element B of $\mathcal{B}$ not containing p, are screened by $\mathcal{B}$,
3. any two disjoint elements of $\mathcal{B}$ are screened by $\mathcal{B}$.

Proof: We proof that $\mathcal{S}^+(\mathcal{B})$ satisfies 1 and 2 of theorem 1.
We only verify condition 2, as condition 1 is verified similarly.

Let $C = \bigcup_{i=1}^{n} C_i$ and $D = \bigcup_{j=1}^{l} D_j$, $C_i \in \mathcal{B}$, $i=1,\ldots,n$, $D_j \in \mathcal{B}$, $j=1,\ldots,l$.

Let $C \cap D = \emptyset$.

For each pair (C_i, D_j) take a "screening cover" $\mathcal{C}_{ij}$ the elements of which are taken from $\mathcal{B}$.

For $i=1,\ldots,n$ and $j=1,\ldots,l$ put

$$A_{ij} = \bigcup \{ X \in \mathcal{C}_{ij} \mid X \cap C_i \neq \emptyset \}$$
$$B_{ij} = \bigcup \{ X \in \mathcal{C}_{ij} \mid X \cap C_i = \emptyset \}$$

and, finally,

$$A = \bigcup_{i=1}^{n} \bigcap_{j=1}^{l} A_{ij} \quad \text{and} \quad B = \bigcap_{i=1}^{n} \bigcup_{j=1}^{l} B_{ij} .$$

Clearly condition 1 implies that A and B belong to $\mathcal{S}^+(\mathcal{B})$. Furthermore, the construction insures that
$A \cup B = T$, $A \cap D = \emptyset$ and $B \cap C = \emptyset$.

Remark: In the proof of proposition 8 given above condition 1 cannot be omitted as the following example shows.
Let T be the unit interval and let $\mathcal{B} = \{ [0,x] \mid x \in [0,1] \} \cup \{ [x,1] \mid x \in [0,1] \} \cup \{ \{x\} \mid x \in [0,1] \}$.
$\mathcal{B}$ satisfies condition 2 and 3 of proposition 8, but it does not satisfy 1. The family $\mathcal{S}^+(\mathcal{B})$ does not satisfy condition 2 of theorem 1, as the reader may verify.

Problem 1: Can we omit condition 1 of proposition 8?

Problem 2: If T is a space and if there are bases for the closed sets of T $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1$ satisfies condition 1 of theorem 1 and $\mathcal{B}_2$ satisfies condition 2 of theorem 1, then is T completely regular?

1.8. Cocontinuity

If T and S are spaces and if f is a map from T to S, then f is cocontinuous (resp. cotopological) relative to prebases $\mathcal{U}$ and $\mathcal{V}$ if the induced map f_*: $T^*(\mathcal{U}) \to S^*(\mathcal{V})$ is continuous (resp. topological. f is cocontinuous (resp. cotopological) if there are prebases $\mathcal{U}$ and $\mathcal{V}$ such that f is cocontinuous (resp. cotopological) relative to $\mathcal{U}$ and $\mathcal{V}$

Obviously, f is cocontinuous iff there is a prebase $\mathcal{V}$ of S such that $f^{-1}(V)$ is closed in T whenever $V \in \mathcal{V}$

For if T is a space and $\mathcal{B}$ the family of all closed sets in T, then T is homeomorphic to $T^*(\mathcal{B})$

Further, if T (resp. S) is a compact Hausdorff space, then in view of 1.4 theorem 1 no specification of $\mathcal{U}$ (resp. $\mathcal{V}$) is needed.

From the definitions we immediately obtain a proof of proposition 1 up to proposition 4 inclusive.

Proposition 1: A map f: $T \to S$ is cotopological iff f^{-1} is cotopological. If f: $T \to S$ is cotopological, then f is cocontinuous. If f: $T \to S$, and if for any $V \in \mathcal{V}$ $f^{-1}(V) \in \mathcal{U}$, $\mathcal{U}$ resp. $\mathcal{V}$ being prebases of T resp. S, then f is cocontinuous relative to $\mathcal{U}$ and $\mathcal{V}$.

Propositon 2: If f: $T \to S$ is cocontinuous relative to $\mathcal{U}$ and $\mathcal{V}$ and if $\mathcal{U}' \supset \mathcal{U}$ and $\mathcal{V}' \subset \mathcal{V}$, $\mathcal{V}'$ being a prebase of S, then f is cocontinuous relative to $\mathcal{U}'$ and $\mathcal{V}'$

Proposition 3: If f: $T \to S$ is cocontinuous relative to $\mathcal{U}$ and $\mathcal{V}$ and if g: $S \to R$ is cocontinuous relative to $\mathcal{V}$ and $\mathcal{W}$, then gf: $T \to R$ is cocontinuous relative to $\mathcal{U}$ and $\mathcal{W}$.

Proposition 4: If T is cocompact relative to $\mathcal{U}$ and if f: $T \to S$ is cocontinuous relative to $\mathcal{U}$ and $\mathcal{V}$, then S is cocompact relative to $\mathcal{V}$. A cocontinuous image of a compact space is cocompact.

The relation between continuity and cocontinuity is given in proposition 5,6 and 7.

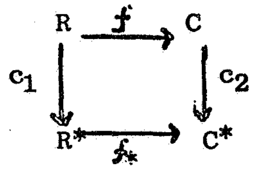
Proposition 5: A continuous (topological)map is cocontinuous (cotopological). If f: T → S is cocontinuous relative the base of all closed sets of S, then f is continuous.

Proposition 6: The compositon of a continuous map and a cocontinuous one is cocontinuous. (If f: T → S is continuous and g: S → R is cocontinuous, then gf is cocontinuous).

Proof: Use propositions 2 and 3.

Proposition 7: A cocontinuous map in a compact Hausdorff space is continuous.

Proof: Consider the following diagram.

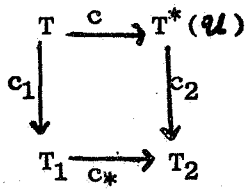


The condensation map c_1 is continuous; C^* is homeomorphic with C (section 1.4). So $f = c_1 f_* c_2^{-1}$ is continuous if f_* is continuous.

Properties of the condensation map and the expansion map are given in proposition 8 and 9.

Proposition 8: The condensation map and the expansion map are cotopological.

Proof: Consider the following diagram:



If c is a condensation map relative $\mathcal{U}$, then let c_1 be the condensation map relative to $\mathcal{U}$, so T_1 is homeomorphic with $T^*(\mathcal{U})$. If c_2 is the condensation map relative to the family of all closed sets of $T^*(\mathcal{U})$ then T_2 is homeorphic to $T^*(\mathcal{U})$. So c_* is a homeomorphism. Therefore, a condensation map is cotopological , and consequently in view of proposition 1 an expansion map is cotopological.

Proposition 9: A map f: T → S is a compression map iff f is one-to-one and onto and cotopological relative to the family of all closed sets in S.

Proof: "only if": see proposition 8.
"if": f is continuous as follows from proposition 5. So 1.2. proposition 3 applies.

Proposition 10: The following proberties of a space T are equivalent.

a) T is a cotopological image of a compact space

b) T is a cocontinuous image of a compact space

c) T is cocompact.

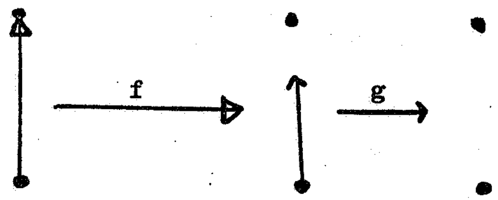
Proof: a) → b) see proposition 1) ; b) → c) see proposition 4) ; c) → a) see proposition 8)

Proposition 11: If f is an one-to-one map from S onto a Hausdorff space T, which is cocontinuous relative to $\mathcal{U}$ and $\mathcal{V}$, and if S is cocompact relative to $\mathcal{U}$, then f is cotopological.

An one-to-one cocontinuous map from a compact space onto a Hausdorff space is cotopological.

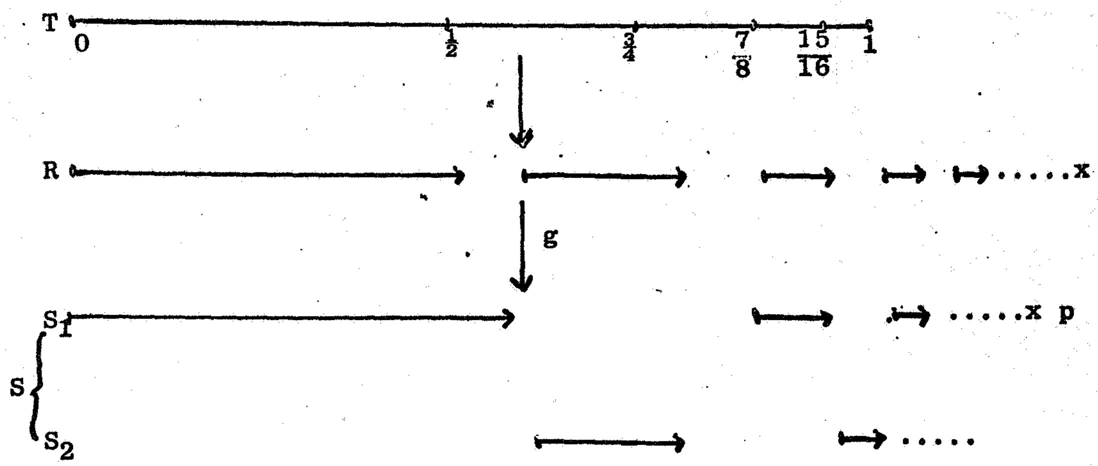
Proof: $S^*(\mathcal{U})$ is compact and $S^*(\mathcal{U})$ can be mapped continuously and one-to-one onto $T^*(\mathcal{V})$. In view of 1.5 proposition 3. $S^*(\mathcal{U})$ is a cospace of T.

Example 1: The order of the maps in proposition 6 is essential, as the following picture shows



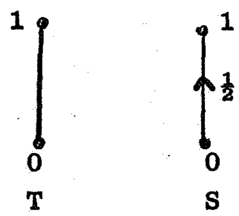
f is cocontinuous (so cotopological by proposition 11), g is continuous. If gf is cocontinuous, then by proposition 7 it is continuous . So gf is not cocontinuous.

Example 2: As the following picture indicates the composition of two cotopological mappings need not to be cotopological.



T is a cospace of R; R is a cospace of S (1.2 prop. 3). gf is not cocontinuous, for the inverse under gf of any neighbourhood of p which is contained in S_1 is not closed in T.
So, a cospace of a cospace need not to be a cospace.

Example 3: If f is a map from T to S, then the cocontinuity of f': T $\rightarrow$ f(T) (f' being induced by f) does not imply the cocontinuity of f.



There is a natural map f' from T onto $[0,\frac{1}{2}] \cup \{1\}$ (see ex. 1), which is cotopological: $f'(t) = \frac{1}{2}t$ if $t < 1$, $f'(1) = 1$. The induced map f: T $\rightarrow$ S is not cocontinuous (Consider base elements containing $\frac{1}{2}$).

Chapter 2

METRIC SPACES

The main purpose of this chapter is to prove, for metric spaces, the equivalence of topological completeness and cocompactness. The first section contains characterizations of topological completeness which are taken from the literature. In the second section we prove the equivalence for the separable case, and in the third section we attack the proof of the equivalence in the general case.
The fourth section treats the notion of co-(compact hausdorff)-ness and gives a characterization of cocompact spaces in the separable metric case.

2.1. Topological completeness

If (M,ρ) is a metric space and if $\{x_n\}$ is a sequence of points, then $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there is a natural number N such that $\rho(x_n,x_m) < \varepsilon$ whenever $m,n \geq N$; (M,ρ) is a complete (metric) space if every Cauchy sequence converges. A metric space (M,ρ) is topologically complete if there is an equivalent metric ρ' such that (M,ρ') is complete.

Theorem: For a metric space M the following properties are equivalent:

1. M is topologically complete,
2. M is a G_δ-subset of every hausdorff space H in which M is embedded as a dense subset,
3. M is a G_δ-subset of βM (Čech-Stone-compactification),
4. M is a G_δ-subset of every hausdorff compactification of M,
5. M is a G_δ-subset of some hausdorff compactification of M,
6. M is a G_δ-subset of every metric space in which M is embedded,
7. M is a G_δ-subset of every topologically complete metric space in which M is embedded,
8. M is a G_δ-subset of some topologically complete metric space in which M is embedded.

The characterization 8 is due to Alexandroff and Hausdorff; 6 and 7 to Sierpinski; 3 and 4 to Čech and 2 to Frolik.

Proof:

$1 \rightarrow 2$: Let ρ be a metric in which M is complete. If $\mathcal{U}_n = \{U \mid U$ open in M, $\operatorname{diam}_\rho(U) < \frac{1}{n}\}$ and if $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, then for each element $U \in \mathcal{U}$ choose an open subset U' of H such that $U' \cap M = U$. We will prove that if $V_n = \bigcup \{U' \mid U \in \mathcal{U}_n\}$, then $M = \bigcap_{n=1}^{\infty} V_n$. Trivially $M \subset \bigcap_{n=1}^{\infty} V_n$.

If $x \in \left(\bigcap_{n=1}^{\infty} V_n\right) \setminus M$ and if $\mathcal{N}$ denotes the neighbourhoodsystem of x, then $\mathcal{N}^* = \{N \cap M \mid N \in \mathcal{N}\}$ is a centered system, since M is dense in H.

Furthermore the assertion $x \in \left(\bigcap_{n=1}^{\infty} V_n\right) \setminus M$ implies that for each $n=1,2,\ldots$ there is an element $U_n \in \mathcal{U}_n$ which belongs to $\mathcal{N}^*$. Choose $x_n \in \bigcap_{i=1}^{n} U_i$, $n=1,2,\ldots$; clearly $\{x_n\}$ is a Cauchy sequence since $\rho(x_i,x_j) < \min(\frac{1}{i},\frac{1}{j})$ and converges to a point $y \in M$ because M is complete. Evidently $x \neq y$. Now $y \in \overline{N}^M$ for every $N \in \mathcal{N}^*$. For if not, choose i so large that $\frac{1}{i} < \rho(y,\overline{N}^M)$, then $U_i \cap \overline{N}^M = \emptyset$, which contradicts the fact that $\mathcal{N}^*$ is a centered system.

On the other hand, because H is a hausdorff space there is $A \in \mathcal{N}$ such that $y \notin \overline{A}^H$, so $y \notin \overline{A \cap M}^M$. Contradiction.

$2 \rightarrow 3$: Trivial.

$3 \rightarrow 4$: If γM is an arbitrary hausdorff compactification of M, then by the Čech-Stone-extension theorem, the identity map $i: M \rightarrow M$ can be extended to a continuous map $j: \beta M \rightarrow \gamma M$. First we will prove that $j(\beta M \setminus M) = \gamma M \setminus M$. For if $p \in \beta M \setminus M$ and if $j(p) \in M$, then the map $j \mid M \cup \{p\}: M \cup \{p\} \rightarrow M$ restricted to M is the identity on M, so equals the identity on $M \cup \{p\}$ since M is dense in the hausdorff space $M \cup \{p\}$. So $j(p) = p \in M$, which contradicts the fact that $p \in \beta M \setminus M$.

A compactness argument shows that $\gamma M = j(\beta M)$, so the statement follows. Now, if M is a G_δ-subset of βM, then $\beta M \setminus M$ is an F_σ-subset of βM. A compactness argument shows that $j(\beta M \setminus M) = \gamma M \setminus M$ is an F_σ-subset of γM.

$4 \to 5$: Trivial.

$5 \to 3$: If M is a G_δ-subset of some hausdorff compactification γM of M, then if $j: \beta M \to \gamma M$ is a continuous extension of the identity mapping $i: M \to M$ it is clear that $j^{-1}(M) = M$ is a G_δ-subset of βM.

$4 \to 6$: If M is embedded in a metric space M^*, then it suffices to prove that M is a G_δ-subset of $\overline{M}$ (closure in M^*), because $\overline{M}$ being a closed subset of the metric space M^* is a G_δ-subset of M^*.

If $\gamma\overline{M}$ is a hausdorff compactification of $\overline{M}$, then it is also a hausdorff compactification of M. As M is a G_δ-subset of $\gamma\overline{M}$, M is a G_δ-subset of $\overline{M}$.

$6 \to 7$: Trivial.

$7 \to 8$: Trivial.

$8 \to 1$: If N is some complete metric space, and $M \subset N$ such that $M = \bigcap_{n=1}^{\infty} G_n$, G_n open in N, then for each n=1,2,... define $f_n: G_n \to \mathbb{R}$ by $f_n(x) = \dfrac{1}{\rho(x, N \setminus G_n)}$.

Let $f: M \to \mathbb{R}^{\underline{a}} = \prod_{n=1}^{\infty} \mathbb{R}_n$ be defined by $(f(x))_n = f_n(x)$, n=1,2,..., then f is continuous. Consider the graph G of f in $N \times \mathbb{R}^{\underline{a}}$. G is homeomorphic to M and G is a closed subset of $N \times \mathbb{R}^{\underline{a}}$ (c.f. 1.6. proposition 3). $N \times \mathbb{R}^{\underline{a}}$ is topologically complete, so the closed subset G is, and the same statement holds for M.

<u>Corollary 1</u>: A separable metric space is topologically complete iff it is homeomorphic to a closed subset of $\mathbb{R}^{\underline{a}} = \prod_{n=1}^{\infty} \mathbb{R}_n$.

<u>Proof</u>: If M is a separable metric space, then by the Urysohn embedding theorem M can be embedded in $\mathbb{R}^{\underline{a}}$. Now reproduce $6 \to 1$ of the theorem.

For metric spaces we have:

A <u>hedgehog</u> (resp. <u>open hedgehog</u>) of spininess τ is obtained from a family of τ intervals $\{I_t \mid t \in T\}$, $I_t = \{x \mid 0 \leq x \leq 1\}$ (resp. $\{x \mid 0 \leq x < 1\}$) by identifying the zeros of the intervals and by defining $\rho(x,y) = |x-y|$ if x and y belong to the same spine and $\rho(x,y) = \sup(|x|,|y|)$ otherwise.

A <u>porcupine</u> (resp. <u>open porcupine</u>) is a countable product of hedgehogs (resp. open hedgehogs). Now by an embedding theorem of Kowalsky a metric space can be embedded in an (open) porcupine, so in the same way as above this yields:

<u>Corollary 2</u>: A metric space is topologically complete iff it is homeomorphic to a closed subset of an open porcupine.

2.2. The separable case

Theorem 1: If M is a separable metric space and if M is topologically complete, then M is cocompact.

Proof: By corollary 1 of section 1 M is homeomorphic to a closed subset M^* of $H = \prod_{i=1}^{\infty} \mathcal{R}_i$.

Now, we construct a family $\mathcal{P}$, which turns out to be a prebase for H, such that the intersection of an infinite centered system in $\mathcal{P}$ consists of one point.

For each i define

$$A^i_{n,m} = \{ x \mid x \in \mathcal{R}_i,\ m2^{-n} \leq x \leq (m+1)2^{-n} \}, \quad n=1,2,\dots,\ m \in \mathbb{Z},$$

$$\mathcal{A}^i_n = \{ A^i_{n,m} \}_{m \in \mathbb{Z}}, \quad n=1,2,\dots,$$

$$\mathcal{P}^i = \{ \bigcap_{k=1}^{i} \pi_k^{-1}(P^k) \mid P^k \in \mathcal{A}^k_i \}, \quad \pi_k : H \to \mathcal{R}_k \text{ denoting the natural projection,}$$

and finally

$$\mathcal{P} = \bigcup_{i=1}^{\infty} \mathcal{P}^i .$$

First we show that if $p \in O_1$, $O_1 = \bigcap_{j=1}^{k} \pi_j^{-1}(O_j)$, O_j open in $\mathcal{R}_j$, then there is a number n_o such that $n \geq n_o$, $p \in P$, $P \in \mathcal{P}^n$ implies $P \subset O_1$.

For each $j \leq k$ choose n_j such that $2^{-n_j} < \rho(\pi_j(p), O_j^c)$, ρ denoting the usual metric for $\mathcal{R}_j$.

Put $n_o = \max\{ \{n_j \mid j=1,\dots,k\} \cup \{k\} \}$. If $n \geq n_o$, $p \in P$, $P \in \mathcal{P}^n$ and if $q \in P \cap O_1^c$, then for some $j \leq k$ the inclusion $\pi_j(q) \in O_j^c$ holds. So $\rho(\pi_j(p), \pi_j(q)) > 2^{-n_j} \geq 2^{-n_o}$. On the other hand $p,q \in P$ implies $\rho(\pi_j(p), \pi_j(q)) < 2^{-n} \leq 2^{-n_o}$. This contradiction shows that n_o satisfies the desired condition.

If $\mathcal{F}$ is a centered system in $\mathcal{P}^k$ then $\mathcal{F}$ contains at most 2^k elements, for the centered system $\{ \pi_j F \mid F \in \mathcal{F} \}$ contains at most 2 elements for each j. So, if $\mathcal{F}$ is an infinite centered system in $\mathcal{P}$, then $\mathcal{F} \cap \mathcal{P}^k \neq \emptyset$ for an infinite number of indices $k_1, k_2, \dots$.

From this it follows that for each natural number j the intersection of $\{ \pi_j F \mid F \in \mathcal{F} \}$ contains one point. So, $\mathcal{F}$ has a one point intersection.

Now we show that $\mathcal{P}$ is a prebase for H. If p belongs to an open subset O of H, then choose an element of the product-base $O_1 = \bigcap_{j=1}^{k} \pi^{-1}(O_j)$ and a number n_o such that $n \geqslant n_o$, $p \in P$, $P \in \mathcal{P}^n$ implies $P \subset O_1$. If $B = \bigcup\{P \mid P \in \mathcal{P}^{n_o}, p \in P\}$, then one easily verifies that $p \in B^o \subset B \subset O_1 \subset O$. B being the union of a centered system in $\mathcal{P}^{n_o}$ is a finite union of P's, so $\mathcal{P}$ is a prebase for H.

Now, put $\mathcal{P}^* = \{P \cap M^* \mid P \in \mathcal{P}\}$.

$\mathcal{P}^*$ is a prebase for M^* and we will show that M^* is cocompact relative to $\mathcal{P}^*$. For if $\mathcal{F}^*$ is an infinite centered system in $\mathcal{P}^*$, then $\mathcal{F} = \{F \mid F \in \mathcal{P}, F \cap M^* \in \mathcal{F}^*\}$ is an infinite centered system in $\mathcal{P}$, so has one-point intersection p. We claim that $p \in M^*$. For, if not, choose $O_1 = \bigcap_{j=1}^{k} \pi_j^{-1}(O_j)$ such that $p \in O_1$, $O_1 \cap M^* = \emptyset$ (M^* is closed!). Choose n_o such that $n \geqslant n_o$, $p \in P$, $P \in \mathcal{P}^n$ implies $P \subset O_1$. Then there is an $F \in \mathcal{F}$ such that $F \in \mathcal{P}^n$ for some $n \geqslant n_o$, so $F \subset O_1$. So $F \cap M^* = \emptyset$. On the other hand by the definition of $\mathcal{F}$ we have $F \cap M^* \neq \emptyset$.

Contradiction.

<u>Theorem 2</u>: If M is a metric space and if M is cocompact then M is topologically complete.

<u>Proof</u>: Let $(\tilde{M},\rho)$ denote the (metric) completion of (M,ρ).

We will show that M is a G_δ-subset of $\tilde{M}$, which yields a proof of the theorem in view of section 1. Let M be cocompact relative to the base $\mathcal{B}$.

Define

$$\mathcal{B}_i = \{B \mid B \in \mathcal{B}, \text{ diameter of } B \leqslant 1/i\},$$

$$\mathcal{O}_i = \{O \mid O = B^{o(M)}, B \in \mathcal{B}_i\},$$

$$\mathcal{O} = \bigcup\{\mathcal{O}_i \mid i=1,2,\dots\}.$$

Observe that for each i $\mathcal{O}_i$ is an open base for M. For each $O \in \mathcal{O}$ let $\tilde{O}$ be an open subset of $\tilde{M}$ such that $\tilde{O} \cap M = O$. Since M is dense in $\tilde{M}$, the diameter of O and $\tilde{O}$ are equal.

If $\tilde{O}_i = \bigcup\{\tilde{O} \mid \tilde{O} \cap M \in \mathcal{O}_i\}$, then $M = \bigcap_{i=1}^{\infty} \tilde{O}_i$.

For trivially $M \subset \bigcap_{i=1}^{\infty} \tilde{O}_i$ and if $p \in \bigcap_{i=1}^{\infty} \tilde{O}_i$ then for each i there is a $\tilde{V}_i$ such that $\tilde{V}_i \cap M = V_i \in \mathcal{O}_i$ and $p \in \tilde{V}_i$. Since the diameters of

the V_i converge to zero if $i \to \infty$ the intersection $\bigcap \tilde{V}_i = \{p\}$. Hence if $p \notin M$ then $\bigcap V_i = \emptyset$. For each i choose a fixed B_i such that $V_i = B_i^{c(M)}$. Because M is dense in $\tilde{M}$, $\{V_i\}$ is a centered system, and so $\{B_i\}$ is. One easily verifies that $\bigcap V_i = \bigcap B_i$ and $\bigcap B_i \neq \emptyset$ because of the cocompactness of M relative to $\mathcal{B}$.
Contradiction.

Combining theorem 1 and theorem 2 we have

Theorem 3: If M is a separable metric space then M is cocompact iff M is topologically complete.

Remark: A proof of the equivalence of cocompactness and topological completeness in the general case could be based on corollary 2 of section 1 and a modification of theorem 1. An easier proof is given in the next section.

2.3. The general case

Recall that a family of subsets of X is point finite iff no point of X belongs to more than a finite number of members of the family. A family $\mathcal{A}$ of subsets of X is locally finite (discrete) iff each point of X has a neighbourhood which intersets only finitely many (at most one) members of $\mathcal{A}$.
A space is paracompact iff it is hausdorff and each open cover has an open locally finite refinement.
A theorem of A.H. Stone states that a metric space is paracompact.

Lemma 1: If $\mathcal{U} = \{U_\alpha\}_\alpha$ is a point finite open cover of a regular space X, and if $\overline{U}_\alpha$ is normal for each α, then there is an open cover $\{O_\alpha\}_\alpha$ of X such that $\overline{O}_\alpha \subset U_\alpha$ for each α.
Proof: Let $\mathcal{F}$ be the class of all functions F, defined in the following way: the domain of F is a subfamily of $\mathcal{U}$, F(U) is an open set whose closure is contained in U for each U in the domain of F and
(*) $\bigcup \{F(U) \mid U \in \text{domain } F\} \cup \{V \mid V \in \mathcal{U} \text{ and } V \notin \text{domain } F\} = X$.

If we define for each pair $F_1, F_2 \in \mathcal{F}$: $F_1 < F_2$ iff domain $F_1 \subset$ domain F_2 and $F_1(U) = F_2(U)$ for each $U \in$ domain F_1, then $<$ partially orders $\mathcal{F}$. In other words, $F_1 < F_2$ iff F_2 is an extension of F_1. If $\mathcal{F}_1$ is a chain in $\mathcal{F}$ (i.e. $\mathcal{F}_1$ is linearly ordered by $<$) then $\mathcal{F}_1$ has an upper bound F_1: $\sqrt{} \bigcup \{\text{domain } F \mid F \in \mathcal{F}_1\}$ and if $U \in$ domain F_1 and $U \in$ domain F_α for some $F_\alpha \in \mathcal{F}_1$ then $F_1(U) = F_\alpha(U)$. F_1 satisfies condition (*). For, if $p \in X$, $p \in U_{\alpha_i}$ for $i=1,\dots,n$, and if $p \notin \bigcup \{F_1(U) \mid U \in \text{domain } F_1\} \cup \{V \mid V \in \mathcal{U} \text{ and } V \notin \text{domain } F_1\}$, then clearly there exist $F_{\alpha_i} \in \mathcal{F}_1$ such that $U_{\alpha_i} \in$ domain F_{α_i} for each $i \leq n$. Take j such that $F_{\alpha_i} < F_{\alpha_j}$ for each $i \leq n$. Then $U_{\alpha_i} \in$ domain F_{α_j} and so F_{α_j} does not satisfy (*). Clearly, $F < F_1$ if $F \in \mathcal{F}_1$. Now, applying Zorn lemma we obtain a maximal element F_o of $\mathcal{F}$. To finish the proof, we will show that domain $F_o = \mathcal{U}$. For, if not, take a $V_o \in \mathcal{U}$ which is not contained in domain F_o. Let $Y = \bigcup \{F_o(U) \mid U \in \text{domain } F_o\} \cup \{V \mid V \in \mathcal{U}, V \in \text{domain } F_o, V \neq V_o\}$. If $Y = X$ then define $F_o^*(V_o)$ to be an open set, whose closure is contained in V_o and $F_o^*(U) = F_o(U)$ if $U \in$ domain F_o. Otherwise, take disjoint open neighbourhoods O_1 resp. O_2 of Y^c resp. $\overline{V}_o \setminus V_o$ in $\overline{V}_o$ and define $F_o^*(V_o) = O_1$ and $F_o^*(U) = F_o(U)$ if $U \in$ domain F_o. Then $F_o < F_o^*$, contradicting the maximality of F_o.
Observe that from the last part of the proof it also follows that $\mathcal{F}$ is not void.

<u>Lemma 2</u>: If $\{O_\alpha\}_{\alpha<\xi}$ is a well ordered locally finite open cover of a regular space X and if $\overline{O}_\alpha$ is normal for each $\alpha < \xi$, then there is an open cover $\{V_\alpha^n \mid \alpha < \xi,\ n = 1, 2, \dots\}$ of X satisfying:

1. Each centered system in $\{\overline{V}_\alpha^n \mid \alpha < \xi,\ n = 1, 2, \dots\}$ is finite,
2. $\{\overline{V}_\alpha^n \mid \alpha < \xi\}$ is a discrete family for each n,
3. $\overline{V}_\alpha^n \subset O_\alpha$.

<u>Proof</u>: Applying lemma 1, take an open cover $\{W_\alpha\}_{\alpha<\xi}$ such that $\overline{W}_\alpha \subset O_\alpha$ for every $\alpha < \xi$.
Using the normality of $\overline{O}_\alpha$, choose, inductively on n, open sets O_α^n satisfying

$\sqrt{}$ domain F_1 =

$$\overline{W}_\alpha \subset O^n_\alpha \subset \overline{O}^n_\alpha \subset O_\alpha ,$$
$$\overline{O}^n_\alpha \subset O^{n+1}_\alpha .$$
$$O^n_\alpha = \emptyset \quad \text{if} \quad W_\alpha = \emptyset \quad \text{and} \quad O^o_\alpha = O^{-1}_\alpha = \emptyset .$$

Define $U^n_\alpha = O^n_\alpha \setminus \overline{O^{n-2}_\alpha}$, $\alpha < \xi$, $n = 1, 2, \ldots$,

$$V^n_\alpha = U^n_\alpha \setminus \bigcup \{ \overline{O^{n+1}_\beta} \mid \beta < \alpha \} , \alpha < \xi , n = 1, 2, \ldots .$$

Now we have

a. $\overline{V^n_\alpha} \subset \overline{U^n_\alpha} \subset \overline{O^n_\alpha} \subset O^{n+1}_\alpha \subset O_\alpha$

b. $\{ V^n_\alpha \mid \alpha < \xi , n = 1, 2, \ldots \}$ is a cover of X.

For if $x \in X$, let $\alpha(x)$ be the smallest ordinal number such that $x \in O^n_{\alpha(x)}$ for some n. Let n(x) be the smallest number such that $x \in O^{n(x)}_{\alpha(x)}$. Then $x \in O^{n(x)}_{\alpha(x)}$ and $x \in V^{n(x)}_{\alpha(x)}$.

c. V^n_α is an open set, for U^n_α is an open set and $\{ O^{n+1}_\beta \mid \beta < \alpha \}$ is locally finite, so closure preserving.

d. If $|n - m| \geq 3$, then $\overline{V}^n_\alpha \cap \overline{V}^m_\alpha = \emptyset$.

For if $n - m \geq 3$, then $\overline{V}^m_\alpha \subset \overline{O}^m_\alpha \subset \overline{O}^{n-3}_\alpha \subset O^{n-2}_\alpha$ and $V^n_\alpha \cap O^{n-2}_\alpha = \emptyset$, so $\overline{V}^n_\alpha \cap O^{n-2}_\alpha = \emptyset$.

e. If $\alpha < \beta$ and $k \leq l$, then $\overline{V}^k_\alpha \cap \overline{V}^l_\beta = \emptyset$.

For $\overline{V}^k_\alpha \subset \overline{O}^k_\alpha \subset O^{k+1}_\alpha \subset O^{l+1}_\alpha \subset \overline{O}^{l+1} \subset \bigcup \{ \overline{O}^{l+1}_\alpha \mid \alpha < \beta \}$.

So $O^{l+1}_\alpha \cap V^l_\beta = \emptyset$ and consequently $O^{l+1}_\alpha \cap \overline{V}^l_\beta = \emptyset$.

So $\overline{V}^k_\alpha \cap \overline{V}^l_\beta = \emptyset$.

From b and c it follows that $\{ V^n_\alpha \mid \alpha < \xi , n = 1, 2, \ldots \}$ is an open cover of X. 3. follows from a.

To prove 2, observe that from e it follows that if $\alpha < \beta$ and $k = l$, then $\overline{V}^k_\alpha \cap \overline{V}^k_\beta = \emptyset$. The local finiteness of $\{ O_\alpha \}_{\alpha < \xi}$ implies the local finiteness of $\{ \overline{V}^k_\alpha \mid \alpha < \xi \}$. So this family is discrete. If $\mathcal{F}$ is a centered system in $\{ \overline{V}^n_\alpha \mid \alpha < \xi , n = 1, 2, \ldots \}$, then from d it follows that for each α there are at most three members of $\mathcal{F}$ of the form $\overline{V}^n_\alpha$.

Let α_o be the smallest ordinal number such that $V^{n_o}_{\alpha_o} \in \mathcal{F}$ for some n_o and let α_1 be the smallest ordinal number such that $\alpha_1 \neq \alpha_o$, $V^{n_1}_{\alpha_1} \in \mathcal{F}$ for some n_1. Now e implies $n_1 < n_o$. Let, inductively on i, α_i be the smallest ordinal number such that $\alpha_i \neq \alpha_j$ for $j = 0, 1, \ldots, i-1$, and $V_{\alpha_i}^{n_i} \in \mathcal{F}$ for some n_i. Then from e it follows that $n_o > n_1 > n_2 > \dots$ $\dots > n_i > n_{i+1} \dots$, so this sequence breaks off. So for not more than finitely many α the inclusion $V_\alpha^n \in \mathcal{F}$ holds. This proves 1.

Lemma 3: If $\mathcal{O}$ is an open cover of a paracompact space X, then $\mathcal{O}$ has an open refinement $\{V_\alpha\}$ such that each centered system of $\{\overline{V}_\alpha\}_\alpha$ is finite.

Proof: First, take a locally finite open refinement $\mathcal{U}$ of $\mathcal{O}$. Well order the set $\mathcal{U}$: $\mathcal{U} = \{U_\alpha \mid \alpha < \xi\}$. Since a paracompact space X is normal, we can apply lemma 2.

Theorem 1: If M is a metric space and if M is topologically complete, then M is cocompact.

Proof: Let ρ be a metric such that (M, ρ) is complete. For each i, let $\mathcal{U}^i = \{U \mid U \text{ open, diam } U < 1/i\}$. Since M is a metric space, by lemma 3 there is an open refinement $\mathcal{V}^i = \{V_\alpha^i\}_\alpha$ of the cover $\mathcal{U}^i$ such that each centered system in $\overline{\mathcal{V}}^i = \{\overline{V}_\alpha^i\}_\alpha$ is finite. Clearly $\bigcup\{\mathcal{V}^i \mid i = 1, 2, \dots\}$ is an open base for M, so $\overline{\mathcal{V}} = \bigcup\{\overline{\mathcal{V}}^i \mid i = 1, 2, \dots\}$ is a closed base for M.
M is cocompact relative to $\overline{\mathcal{V}}$. For if $\mathcal{F}$ is an infinite centered system in $\overline{\mathcal{V}}$, then $\mathcal{F} \cap \overline{\mathcal{V}}^i$ is finite for each i. So there is an infinite number of indices $i_1, i_2, \dots$ such that $\mathcal{F} \cap \overline{\mathcal{V}}^{i_k} \neq \emptyset$, $k = 1, 2, \dots$. So $\mathcal{F}$ contains elements of arbitrarily small diameter and consequently $\bigcap \mathcal{F} \neq \emptyset$, since (M, ρ) is complete.

From theorem 1 and § 2. theorem 2 we infer

Theorem 2: If M is a metric space, then M is topologically complete iff M is cocompact.

2.4. Co-(compact Hausdorff) spaces

Notation. cocH stands for co-(compact Hausdorff).
We start this section with collecting some propositions on cocH-spaces and some counterexamples.

Proposition 1: Let T^* be a cospace of a Hausdorff space T, and suppose there is a continuous map f from a compact Hausdorff space K onto T^*. Identify k_1 and k_2 in K if $f(k_1) = f(k_2)$, and denote the resulting quotient space by K'. Then K' is a cospace of X, lying between X and X^*.

Proof: The map φ in $K \xrightarrow{q} K' \xrightarrow{\varphi} T^*$ is obviously continuous, one-to-one and onto T^*. Apply proposition 1.5.3.

Corollary 1: If T is a Hausdorff space, then T is cocH iff T is a one-to-one cocontinuous image of a compact Hausdorff space.

Theorem 1: If T is the mixed product of a family of cocH spaces, then T is cocH.

Proof: Apply lemma 1.6.1.

Proposition 2: Every locally compact Hausdorff space is cocH.

Proof: Proposition 1.3.2.

Proposition 3: If T is a Hausdorff space which is locally compact except at a single point $p \in T$, then T is cocH.

Proof: If $\mathcal{U}(p)$ denotes a neighbourhoodbase at p, then T is cocompact relative the base $\mathcal{V} = \{V \mid V = U^-$ for some $U \in \mathcal{U}(p)$ or V is compact or V^{c-} is compact$\}$.
For any element of $\mathcal{V}$ which is not compact contains p. $T^*(\mathcal{V})$ is Hausdorff in view of lemma 1.4.1.

Remark: It is easy to prove (see the following proposition) that a regular space which is locally compact except at a finite pointset is cocompact.

Proposition 4: Let T be a regular space which is locally compact except at the points p and q, and let $\mathcal{U}'$ consist of all compact sets and all complements of interiors of compact sets. Then (1) $\mathcal{U}'$ can always be enlarged to a closed base $\mathcal{U}$ such that $T^*(\mathcal{U})$ is compact. (2) $\mathcal{U}'$ can be enlarged to a closed base $\mathcal{U}$ such that $T^*(\mathcal{U})$ is compact Hausdorff iff there is a compact set C in T separating p and q.

Proof: (1) Let U_1 and U_2 be disjoint closed neighbourhoods of p and q. Let $\mathcal{U}(p)$ (resp. $\mathcal{U}(q)$) be closed neighbourhoodbases of p (q) such that each element of $\mathcal{U}(p)$ ($\mathcal{U}(q)$) is contained in U_1 (U_2). Let $\mathcal{U} = \mathcal{U}' \cup \mathcal{U}(p) \cup \mathcal{U}(q)$.

(2) "if". Let $T \setminus C = O_1 \cup O_2$, $p \in O_1$, $q \in O_2$, $O_1 \cap O_2 = \emptyset$. Let $U_1 = O_1^c$ and $U_2 = O_2^c$. Let $\mathcal{U}(p)$ ($\mathcal{U}(q)$) be closed neighbourhoodbases of p (q) such that each element of $\mathcal{U}(p)$ ($\mathcal{U}(q)$) is contained in O_1 (O_2).
Let $\mathcal{U} = \mathcal{U}' \cup \mathcal{U}(p) \cup \mathcal{U}(q) \cup \{U_1\} \cup \{U_2\}$. It is easy to verify that $T^*(\mathcal{U})$ is compact. In view of lemma 1.4.1 $T^*(\mathcal{U})$ is Hausdorff.
"only if". Let $T^*(\mathcal{U})$ be a cospace of T which is compact Hausdorff and suppose $\mathcal{U} \supset \mathcal{U}'$. First, we will show that if $c: T \to T^*(\mathcal{U})$ denotes the compression map, then $c \mid T \setminus \{p,q\}$ is a homeomorphism. To show this we have only to verify that $(c|T \setminus \{p,q\})^{-1}$ is continuous.
Let $r \in T \setminus \{p,q\}$ and O be an arbitrary neighbourhood of r. Choose a neighbourhood O_1 such that $O_1^- \subset O$ and O_1^- is compact. Then $O_1^- \in \mathcal{U}'$, so $(O_1^{-c-c})^*$ is an open set of $T^*(\mathcal{U})$ and by $(c \mid T \setminus \{p,q\})^{-1}$ this set is mapped onto $O_1^{-c-c} \subset O$.
Now, if $T^*(\mathcal{U})$ is compact Hausdorff, then p^* and q^* can be separated in $T^*(\mathcal{U})$ by a compact set C^*, so C separates p and q in T. C is compact because $c \mid T \setminus \{p,q\}$ is a homeomorphism.

Example 1: We construct a space X with the following properties
(1) X is cocompact relative to some base $\mathcal{U}$.
(2) For no closed base $\mathcal{V} \supset \mathcal{U}$ is X cocH relative to $\mathcal{V}$.
(3) X is cocH.

Let $S = \{x \mid x \in E^3 \text{ and } \|x\| = 1\}$ and let ρ denote the usual metric of E^3 (i.e., $\rho(x,y) = \|x - y\|$). Let $n = (0,0,1)$ and $s = (0,0,-1)$. Define a metric σ on S as follows.
$\sigma(x,y) = \rho(x,y)$ if x and y lie on a circle through the points n and s. Otherwise $\sigma(x,y) = \inf(\rho(x,n) + \rho(n,y), \rho(x,s) + \rho(y,s))$. $X = (S,\sigma)$.
(1) and (2) follow from proposition 4, (3) follows from the observation that (S,ρ) is a cospace of X (proposition 1.2.3).

<u>Example 2</u>: We construct a space X with the following properties.

(1) X is cocompact

(2) X is metrizable

(3) X is not cocH. In fact, there is no one-to-one continuous map from X onto any compact Hausdorff space H.

For each n, let $I_n = \{(x,y) \mid (x,y) \in E^2, x = 1/n, 0 \leq y \leq 1\}$.
Let X be the subspace of E^2 defined by $X = \bigcup_{n=1}^{\infty} I_n \cup \{(0,1),(0,0)\}$.
To prove (1), observe that X is a G_δ-subset of a compact metric space. Now, to prove (3), suppose X^* is a compact Hausdorff space and f a continuous one-to-one map of X onto X^*. Denote $f(x) = x^*$ for $x \in X$, $p = (0,1)$, $q = (0,0)$.
If O^* is a clopen set of X^* containing p^*, then O is a clopen set containing p. Then clearly $q \in O$, so $q^* \in O^*$. From this it follows that the quasicomponent of p^* in X^* contains q^*. Since X^* is a compact Hausdorff space it follows that the component C^* of p^* contains q^*. Because f is one-to-one, C^* is a non-degenerated continuum. C is the countable union of pairwise disjoint compact sets. Because f is one-to-one C^* is the countable union of pairwise disjoint compact sets. However, this contradicts a theorem of Sierpinski: A compact connected space cannot be decomposed into countably many pairwise disjoint closed sets.

<u>Example 3</u>: Let X be the space from example 2. Let Y be the topological union of X and an open interval. It is easy to check that $\bigcup_{n=1}^{\infty} I_n \cup \{(0,y) \mid 0 \leq y \leq 1\}$ (see ex. 2) is a cospace of Y. So Y is cocH, but its <u>clopen</u> subset X is <u>not</u>.

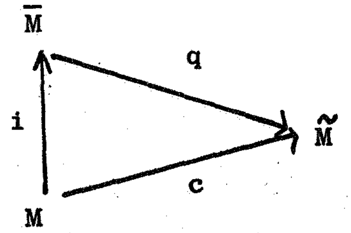
Example 4: Let $X = \bigcup_{n=1}^{\infty} I_n \cup \{(0,y) \mid y = 0,\ y = 1/n,\ n = 1, 2, \ldots\}$ (see Ex. 2). Let Y be the topological union of X and countably many open intervals. Then (1) Y is cocompact, (2) there is a one-to-one continuous mapping from Y onto a compact Hausdorff space, (3) Y is not cocH.

(1) and (2) are easy to check. Suppose Y is cocH relative a base $\mathcal{B}$. On account of the definition of a closed base, $\mathcal{B}$ has to contain a member B which satisfies: (a) the interior of B contains (0,0), and (b) $B \subset X$. From (a) it follows that B contains a "rectangular" subset of X. By the compression map this set is mapped one-to-one and onto a compact set. Applying the argument of example 2 we obtain a contradiction.

From the first part of this section it follows that a cocompact space need not to be cocH. However, for separable metrizable spaces we have that cocompact spaces coincide with the cocontinuous images of the Cantor set. Recall that for separable metrizable spaces the compact spaces coincide with the continuous images of the Cantor set. First we prove that for separable metrizable rimcompact spaces the notions cocompact and cocH coincide.

Lemma 1: If a metric space M can be compactified to a metric space $\overline{M}$ by adjoining countably many points, then M is co-(compact-metrizable). In particular M is cocH.

Proof: Let $\overline{M} = M \cup \{p_n\}$. Let ρ be a metric on $\overline{M}$. For each n, identify p_n with a point $x_n \in M$ in such a way that $\rho(x_n, p_n) < 1/n$ and $x_n \neq x_k$ for $k \neq n$. Call the resulting quotient space $\widetilde{M}$, and denote by q the quotient map of $\overline{M}$ onto $\widetilde{M}$. Let c denote the identity mapping of M onto $\widetilde{M}$ and i the inclusion map of M into $\overline{M}$.



Evidently, $\tilde{M}$ is compact, since it is a continuous image of $\overline{M}$. Clearly c is continuous. Now we will prove that

(1) $\tilde{M}$ is a normal T_1-space,

(2) $\tilde{M}$ is a cospace of M,

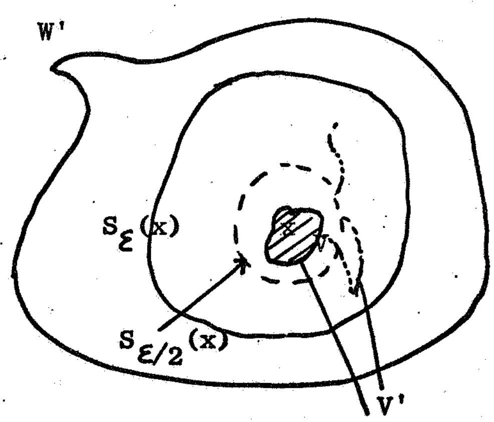
(3) $\tilde{M}$ is metrizable.

(1) We show that the decomposition $\mathcal{D} = \{q^{-1}(x) \mid x \in \tilde{M}\}$ of $\overline{M}$ is upper semi-continuous (for definition see Kelley, p. 99). The normality of $\tilde{M}$ then follows from the normality of $\overline{M}$. Let U be an open neighbourhood of some member D of $\mathcal{D}$. The set $\{x_k \mid x_k \in U,\ p_k \notin U\}$ is a closed subset of the subspace U.

For if s is any accumulation point of $\{x_k \mid x_k \in U,\ p_k \notin U\}$ then $\rho(x_k, s)$ is arbitrarily small for suitable x_k. Then $\rho(p_k, s)$ is also arbitrarily small for suitable p_k, since $\rho(x_n, p_n)$ tends to zero if n tends to infinity. Consequently $\rho(U, s) = 0$ and $\rho(U^c, s) = 0$. So s is an element of the boundary of U and a fortiori not contained in U.

In the same way it is proved that $\{p_k \mid p_k \in U,\ x_k \notin U\}$ is a closed subset of U. Now, it is easy to check that the set V defined by $V = U \setminus (\{x_k \mid x_k \in U,\ p_k \notin U\} \cup \{p_k \mid p_k \in U,\ x_k \notin U\}$ contains D, is open and is the union of members of $\mathcal{D}$.

(2) We construct a closed base $\mathcal{B}$ in M such that for each $B \in \mathcal{B}$, c(B) is closed in $\tilde{M}$ (c.f. proposition 1.2.3). Let $x \in M$ and W a neighbourhood of x in M. Let W' be an open set of $\overline{M}$ whose intersection with M equals W. Take an open ρ-sphere with radius ε and center x, $S_\varepsilon(x)$, which is contained in W'. Let $P = \{p_k \mid p_k \in S_\varepsilon(x),\ x_k \notin S_\varepsilon(x),\ \rho(x_k, p_k) > \frac{1}{2}\varepsilon\}$. Observe that P is finite, because $\rho(x_n, p_n)$ tends to zero. Take a closed neighbourhood V of x which is contained in $S_{\varepsilon/2}(x)$ and avoids P. Now, let $V' = V \cup \{x_k \mid p_k \in V,\ x_k \notin V\}$. $V' \subset S_\varepsilon(x)$. For if $x_k \notin S_\varepsilon(x)$ and $p_k \in V$, then $\rho(x_k, p_k) > \frac{1}{2}\varepsilon$, so $p_k \in P$.



Hence $p_k \notin V$ and consequently $x_k \notin V'$. Now, if s is any accumulation point of $\{x_k \mid p_k \in V,\ x_k \notin V\}$ in $\bar{M}$, then s is also an accumulation point of $\{p_k \mid p_k \in V,\ x_k \notin V\}$. Hence $s \in V$. From this it follows that V' is a closed neighbourhood of x in $\bar{M}$ which is contained in W'. The construction of V' insures that $q(V')$ equals $c(V' \cap M)$. So $c(V' \cap M)$ is a closed subset of $\tilde{M}$.

Hence $V' \cap M$ is a closed neighbourhood of x in M which is contained in W and which is mapped onto a closed subset $c(V' \cap M)$ of $\tilde{M}$. The collection of all such closed neighbourhoods is the required closed base for $\tilde{M}$.

(3) For compact spaces the weight of the image does not exceed the weight of the domain. So $\tilde{M}$ is second countable and hence metrizable (the normality of $\tilde{M}$ is proved in (1)).

Remark: For later use, observe that in the proof of (2) we have (an upper bar denotes closure in $\bar{M}$) $V' \cap M = (V' \cap M) \cup \{x_k \mid p_k \in (V' \cap M)^-\}$. For if $p_k \in (V' \cap M)^-$, then $p_k \in V'$, so $x_k \in V'$. Consequently $x_k \in V' \cap M$. Hence it follows that for each point $x \in M$ and each M-neighbourhood W of x, there is a closed M-neighbourhood V of x such that $V = V \cup \{x_k \mid p_k \in \text{closure of } V \text{ in } \bar{M}\}$.

A space is called rimcompact if each point has arbitrarily small neighbourhoods with compact boundary.

J. de Groot has proved that a rimcompact separable metric space X has a metric compactification $\tilde{X}$ such that $\dim(\tilde{X} \setminus X) = 0$ (J. de Groot, Topologische Studiën, Assen 1942). However, for a rimcompact topologically complete separable metric space X his construction yields a compactification $\tilde{X}$ such that $\tilde{X} \setminus X$ is countable.

The last result has also been obtained by L. Zippin (Amer. J. of Math. 57 (1935) p. 327-341).

Hence, the following theorem holds.

Theorem 2: A rimcompact cocompact separable metric space is co-(compact metrizable). In particular, it is co-(compact Hausdorff).

Corollary: A zero-dimensional cocompact separable metric space is co-(compact Hausdorff).[1)]

Lemma 2: Let M be a cocompact separable metrizable space. Then there is a zero-dimensional separable metric space C and a continuous map $\varphi: C \to M$ such that

1) pointinverses of φ are compact

2) φ is one-to-one on a dense G_δ-subset D of C

3) If $\{p_i\}$ converges to p, $p_i \in \varphi(D)$, then there is a subsequence $\{p_{i_k}\}$ of $\{p_i\}$ such that $\lim_{k\to\infty} \varphi^{-1}(p_{i_k}) = x$ and $\varphi(x) = p$.

Proof: Let $\overline{M}$ be a metric compactification of M.
Let $\{U_k\}$ be a finite collection of pairwise disjoint open sets in $\overline{M}$, each of diameter $< \frac{1}{2}$, such that $\bigcup \overline{U}_k = \overline{M}$ (an upper bar denotes closure in $\overline{M}$). Corresponding to each $\overline{U}_k$, let I_k be a closed subinterval of $[0,1]$ in such a way that the I_k are pairwise disjoint and of equal length $< \frac{1}{2}$. For each fixed k, we now regard $\overline{U}_k$ as a compact metric space and repeat the procedure. That is, we construct a finite collection of pairwise disjoint sets $\{U_{kl}\}$ open in $\overline{U}_k$ such that $\bigcup_l \overline{U}_{kl} = \overline{U}_k$, each of diameter $< 1/2^2$, and corresponding to each l let I_{kl} be a closed subinterval of I_k, with the I_{kl} pairwise disjoint and of equal length $< 1/2^2$. Proceeding inductively we define $U_{n_1 \dots n_k}$, $I_{n_1 \dots n_k}$ with each index restricted to a finite set, and the diameters of the $U_{n_1 \dots n_k}$, $I_{n_1 \dots n_k}$ $< \frac{1}{2}k$. Now if $n_1 n_2 \dots n_k \dots$ is an infinite sequence of integers, each chosen from a finite set $N_1, N_2, \dots$ etc. the sets $\overline{U}_{n_1 n_2 \dots} = \overline{U}_{n_1} \cap \overline{U}_{n_1 n_2} \cap \dots$ and $I_{n_1 n_2 \dots} = I_{n_1} \cap I_{n_1 n_2} \cap \dots$ each consists of a single point; let $I_{n_1 n_2 \dots} \equiv x_{n_1 n_2 \dots}$, $\overline{U}_{n_1 n_2 \dots} = p_{n_1 n_2 \dots}$; further, <u>each</u> $p \in \overline{M}$ is determined by such a sequence, while the set of all $x \in [0,1]$ so determined is a compact 0-dimensional set which we denote by C^*.
Define a map $\varphi^*: C^* \to \overline{M}$ by $f(x_{n_1 n_2 \dots}) = p_{n_1 n_2 \dots}$. The map φ^* is evidently continuous. Now if $p \in \overline{M}$ is not on the boundary of any of the sets $U_{n_1 \dots n_k}$, then p is the image of exactly <u>one</u> $x \in [0,1]$, since

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1) We can even prove that such a space is co (zerodimensional compact Hausdorff).

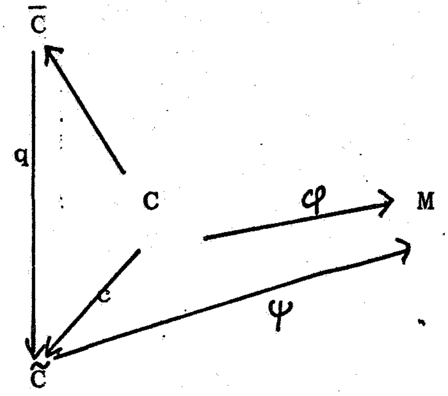
each index in the "expansion" of p is uniquely determined. The set S of all such p which lie in M is a G_δ set in M. S is a dense subset of M, so of $\overline{M}$, as follows from the Baire-category theorem.
If $D = \varphi^{*-1}(S)$, then D is a G_δ-subset in C^*. Let $\varphi^+ = \varphi^* | \overline{D}$, $\overline{D}$ denoting the closure of D in C^*. Then $\varphi^+(\overline{D}) = \overline{M}$. Let $\varphi^{+-1}(M) = C$. Then D is a dense G_δ-subset of C. Define $\varphi = \varphi^+ | C$, then 2) is clear. $\varphi^{-1}(x)$ is compact for each $x \in M$ since $\varphi^{-1}(x) = \varphi^{+-1}(x)$ is a closed subset of the compact space $\overline{D}$. This proves 1). To proof 3) observe that from the compactness of $\overline{D}$ it follows that $\{\varphi^{-1}(p_i) | i = 1, 2, \ldots\}$ has a subsequence $\{\varphi^{-1}(p_{i_k})\}_{k=1}^{\infty}$ which converges to a point $x \in \overline{D}$. Continuity of φ insures that $\varphi(x) = p$. Hence $x \in C$.

Now, we can prove the main theorem of this section.

<u>Theorem 3</u>: A separable metrizable space is cocompact (so completely metrizable) if and only if it is a cocontinuous image of the Cantor set.

This theorem generalizes a well known theorem stating that for separable metrizable spaces the compact spaces coincide with the continuous images of the Cantor set.

<u>Proof</u>: "if". Proposition 1.8.4.
"only if". Let M be a separable metrizable space which is cocompact. It suffices to show that M is a cocontinuous image of a compact space. For a compact space is a continuous image of the Cantor set and we can apply proposition 1.8.6. Embed M in a compact metric space $\overline{M}$ and construct a zerodimensional separable metric space C and a continuous map $\varphi: C \to M$, which is one-to-one on a dense set D as in lemma 2. Now C can be compactified by a countable set $\{p_k\}$ (see the remarks before theorem 2). Let $\overline{C}$ denote this compactification. Identify each p_k to an x_k in C in such a way that $\rho(x_k, p_k) < 1/n$ (here ρ is any metric on $\overline{C}$), $x_k \neq x_l$ if $k \neq l$, and $x_k = \varphi\varphi^{-1}(x_k)$ (i.e. $x_k \in D$; this is possible since φ is one-to-one on D and D is dense).



Denote the resulting quotient space by $\tilde{C}$. By lemma 1 $\tilde{C}$ is a compact metrizable cospace of C. In the diagram q indicates the quotient map, c the compression map and i the inclusion of C in $\bar{C}$. We shall see that $\psi = \varphi c^{-1}$ is the required cocontinuous map.

To show this we have to construct a closed base $\mathcal{U}$ in M such that if $U \in \mathcal{U}$ then $\psi^{-1}(U)$ is closed in $\tilde{C}$.

Recall that if G is any closed set in C and $G' = G \cup \{x_k \mid p_k \in$ closure of G in $\bar{C}\}$, then since $c(G') = q$ (closure of G in $\bar{C}$), $c(G')$ is closed in $\tilde{C}$. Therefore it suffices to find a closed base $\mathcal{U}$ in M such that for all $U \in \mathcal{U}$, $\varphi^{-1}(U) = [\varphi^{-1}(U)]'$.

Recall that for each point x of C and each C-neighbourhood W of x there is a closed C-neighbourhood V of x such that $V = V'$ (see the remark after theorem 1). Now we proceed as follows.

Let p be any point of M and W any neighbourhood of p. We shall construct a closed neighbourhood U of p such that $p \in U \subset W$ and $\varphi^{-1}(U) = [\varphi^{-1}(U)]'$. Then, the collection of all such U is our required base $\mathcal{U}$. $\varphi^{-1}(W)$ is a neighbourhood of $K = \varphi^{-1}(p)$ in C. K is compact as pointinverse of the map φ. For each $y \in K$ let V_y be a closed neighbourhood of y with $V_y' = V_y$ and $V_y \subset \varphi^{-1}(W)$. Since K is compact, $K \subset V_{y_1} \cup \ldots \cup V_{y_n} = V$, V being a closed neighbourhood of K, contained in $\varphi^{-1}(W)$ and satisfying $V = V'$. We show that there is a closed neighbourhood N of p such that $\varphi^{-1}(N) \subset V$.

Suppose there is not such a neighbourhood of p. Choose a sequence $\{q_i\}$ converging to p such that for each i we have $q_i \in \varphi(D)$ - which is a dense subset of M - and $\varphi^{-1}(q_i) \notin V$. By property 3) of lemma 2 there is a subsequence $\{q_{i_k}\}$ such that $\lim_{k\to\infty} \varphi^{-1}(q_{i_k}) = x$ and $\varphi(x) = p$. Hence $x \in K$. Consequently $\varphi^{-1}(q_{i_k}) \in V$ if k is sufficiently large, contradicting the choice of $\{q_i\}$. Let $N_1^k = \{\varphi(x_k) \mid p_k \in$ closure of $\varphi^{-1}(N)$ in $\bar{C}\}$.

Define $U = N \cup N_1$.

First observe that $\varphi^{-1}(U) = \varphi^{-1}(N) \cup \{x_k \mid p_k \in$ closure of $\varphi^{-1}(N)$ in $\bar{C}\} = [\varphi^{-1}(N)]' \subset V' = V \subset \varphi^{-1}(W)$, so $U \subset W$. Moreover $\rho(x_k, \varphi^{-1}(N))$ tends to zero if k tends to infinity. k ranging over the set $L = \{l \mid p_l \in$ closure of $\varphi^{-1}(N)$ in $\bar{C}\}$. From this it follows that $\varphi^{-1}(U)$ is closed, and that $\varphi^{-1}(U) = [[\varphi^{-1}(N)]']' = [\varphi^{-1}(U)]'$.
Moreover, U is a closed neighbourhood of x. We have only to show that each accumulation point of N_1 belongs to U. Suppose that $\{q_i\} \subset N_1$ converges to $y \notin U$. Using property 3) of lemma 2 there is a subsequence $\{q_{i_k}\}$ such that $\lim_{k\to\infty} \varphi^{-1}(q_{i_k}) = x$ and $\varphi(x) = y$. However $\{\varphi^{-1}(q_{i_k})\} \subset \varphi^{-1}(N_1)$, so $x \in \varphi^{-1}(U)$, whence $y \in U$.
Hence U satisfies all properties required.

Chapter 3

MISCELLANY

3.1. Another definition of a cospace

Here we present another definition of a cospace. Actually, this definition and the definition of 1.1 define the same notion as is easily seen. However, the definition given here is a more elegant one. Proposition 2 is new.

Let T be a topology on a set X. A topology $^*T$ on X is called a (topological) co-space of T if

(i) $^*T$ is weaker than T: $^*T < T$

(ii) For every point $p \in T$ and every neighbourhood [1)] of p in T, there exist a neighbourhood of p in T contained in it for which the closure in T is closed in $^*T$.

If P is a property, then by definition T has co-P (relative to $^*T$) if there exists a cospace $^*T$ of T which has property P.

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1) A neighbourhood need not to be open. Observe that the closure of the smaller neighbourhood in (ii) is the same in T and $^*T$.

${}^*T$ is also called a cotopology on T.
In particular T is a cotopology on T and T is a cospace of itself; if T has P, then T has co-P.
If ${}^*T$ is a cospace of T, then any topology V with

$${}^*T < V < T$$

is a cospace of T. On the other hand ${}^*T$ is not necessarily a cospace of V (c.f. 1.3 ex. 1). Neither is the cospace relation symmetric or transitive.
The identity map of X onto itself induces a one to one continuous map of T into ${}^*T$ (because of (i)) of a special kind (because of (ii)) which we will call a <u>compression map</u> c. The inverse map c^{-1} is called an <u>expansion map</u>.

If f: $T \to S$ is a mapping (transformation) of the space T onto S, we say that f has <u>co-P</u> (relative to ${}^*T$, ${}^*S$) if there exist cospaces ${}^*T$, ${}^*S$ such that ${}^*f$, defined by the commutativity of the diagram

$$\begin{array}{ccc} T & \xrightarrow{\quad f \quad} & S \\ {\scriptstyle c_T}\downarrow & & \downarrow{\scriptstyle c_S} \\ {}^*T & \xrightarrow{\quad {}^*f \quad} & {}^*S \end{array}$$

has property P.
In particular, f is cocontinuous, if there are cospaces ${}^*T$, ${}^*S$ such that ${}^*f$ is continuous.

If f has P, then f has co-P. In particular, every continuous or topological mapping is cocontinuous or cotopological.

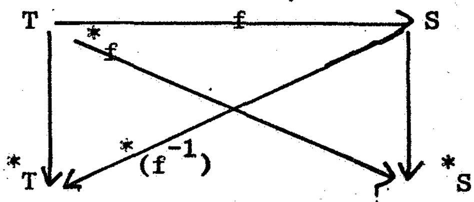
<u>Proposition 1</u>: f: $T \to S$ is cocontinuous iff there exists a cospace ${}^*S$ of S such that ${}^*f$ is ${}^*S$ continuous relative to T.

<u>Proof</u>: "if" is evident.
"only if" follows from the fact that in the diagram above ${}^*f \cdot c_T$ is continuous.

<u>Proposition 2</u>: f is cotopological iff it is one-to-one and both f and f^{-1} are cocontinuous.

<u>Proof</u>: "only if" is evident.

"if": Let T be a topology on X and S be a topology on Y and suppose that f is a one-to-one mapping of X onto Y such that both f and f^{-1} are cocontinuous. According to proposition 1 there exist cospaces $^*T$ and $^*S$ of T and S respectively such that $^*f$ is relative to T, $^*S$ continuous and $^*(f^{-1})$ is relative to S, $^*T$ continuous.



Let T_1 and S_1 respectively denote the topologies generated by $\{W \mid W \in {}^*T \text{ or } f(W) \in {}^*S\}$ and $\{W \mid W \in {}^*S \text{ or } f^{-1}(W) \in {}^*T\}$. We will show that T_1 and S_1 are cospaces of T and S respectively, such that the induced map $f_1: T_1 \to S_1$ is topological.

Obviously, $^*T < T_1 < T$ and $^*S < S_1 < S$. So, T_1 and S_1 are cospaces of T and S respectively. Let $W \subset Y$. If $W \in {}^*S$, then $f^{-1}(W) \in T_1$ since $f(f^{-1}(W)) = W \in {}^*S$. If $f^{-1}(W) \in {}^*T$, then $f^{-1}(W) \in T_1$. Since $\{W \mid W \in {}^*S \text{ or } f^{-1}(W) \in {}^*T\}$ is a subbase for S_1, f is cocontinuous relative T_1 and S_1. Hence, f_1 is continuous. The continuity of f^{-1} is proved similarly.

3.2. <u>Cocompactness as a mapping invariant</u>

According to Isbell (Uniform Spaces, p. 119) a proper map onto is called a <u>fitting map</u>. Recall that a proper map is a closed continuous map such that the preimage of each point is compact. A topological property P is called a <u>fitting property</u> if whenever $f: A \to B$ is a fitting map, both or neither of A and B must have P.

It is well known that in the metric case cocompactness is a fitting property. The following example shows that this does not hold in general.

<u>Example</u>: Let T_1 be the topological sum of $\mathbb{R}^c$ and the rationals Q and let T_2 be the space $\mathbb{R}^c$. Recall (1.6 ex. 2) that Q is homeomorphic to

a closed subset Q' of $\mathbb{R}^c$. Define f: $T_1 \to T_2$ as follows: f is the identity on T_1. On Q f is the natural map from Q onto Q'. f is fitting as is easily seen. However, T_1 is not cocompact, since its open subspace is Q, and T_2 is cocompact (cf. 1.6).

We have the following proposition:

<u>Proposition</u>: If f: $X \to Y$ is a fitting map and X is cocompact, then Y is cocompact. Briefly cocompactness is invariant under fitting maps.

<u>Proof</u>: Suppose X is cocompact relative to $^*X$. If C is a closed subset of $^*X$, then C is a closed subset of X. Consequently fC is closed in Y and the family $\{fC \mid C \text{ closed in } ^*X\}$ defines a topology $^*Y$ which is weaker than Y. The induced map $^*f$: $^*X \to {}^*Y$ is closed and continuous, as follows from the definition of $^*Y$. It follows that $^*Y$ is compact. We shall prove that $^*Y$ is a cospace of Y, which completes the proof. We only have to verify that for each point $y \in Y$ and each neighbourhood U of y in Y, there is a neighbourhood V of y in Y such that $cl_Y V = cl_{^*Y} V$. Each point $x \in f^{-1}(y)$ contains a neighbourhood W in X which is mapped into U such that $W^- = cl_X W$ is closed in $^*X$. Consequently, $f(W^-)$ is closed in $^*Y$.
Since $f^{-1}(y)$ is compact, it is covered by the interior of finitely many sets W of this form. Let R be the union of the corresponding W^-. Since f is closed continuous, V = fR is a closed neighbourhood of y in Y. It follows that $V = cl_Y V \subset U^-$ and is closed considered as a subset of $^*Y$.

The following problems are solved

p. 15 Problem 4: The answer is no (embed the rationals Q as a closed subset of $P = \mathbb{R}^c$; take a compact extension δP of P. The intersection of the cocompact subspaces $\overline{Q}^{\delta P}$ and P of δP is the non cocompact space Q).

p. 16 Problem 5: The answer is yes (this is a theorem of Alexandroff and Urysohn, cf. Math. Ann. **98** (1927), pp. 89).

Errata

	Instead of	Read
page 2, l. 1	$p \in int(B) \subset O$	$p \in int B \subset \overline{O}$
page 3, l. 14	of all compact subsets of T	of the preimages of all closed subsets of S
page 8, l. 2	T_2-space	T_1-space
page 8, l. 17	remove ", iff T is cocompact rel. $P(\mathcal{P})$, "	
page 9, l. 14	can be mapped one to one onto	can be mapped one to one continuously onto
page 12, l. 10	, in locally compact spaces	in locally compact Hausdorff spaces
page 14, l. 26	homeomorphic to a closed subset	homeomorphic to a subset
page 18, l. 8	and every set G	and every closed set G
page 21, l. 26	if $f: T \to S$ is	if $f: T \xrightarrow{onto} S$ is
page 22, l. 10, l. 13, l. 15, l. 18, l. 19, l. 20, l. 23	condensation map	compression map
page 23, l. 1	proberties	properties
page 24, l. 7	there is a natural map f^1 from T onto $[0,\frac{1}{2}] \cup \{1\}$	there is a natural map f^1 from T onto $(0,\frac{1}{2}] \cup \{1\}$
page 29, l. 16	if $p \in O_1$, $O_1 = \bigcap_{j=1}^{k} \pi_j^{-1}(O_j)$	if $p \in O'$, $O' = \bigcap_{j=1}^{k} \pi_j^{-1}(O_j)$
page 29, l. 17	implies $p \in O_1$	implies $p \in O'$
page 29, l. 21	$q \in P \cap O_1^c$	$q \in P \cap O'^c$
page 30, l. 2	$O_1 = \bigcap_{j=1}^{k} \pi^{-1}(O_j)$	$O' = \bigcap_{j=1}^{k} \pi_j^{-1}(O_j)$
page 30, l. 3	$P \subset O_1$	$P \subset O'$
page 30, l. 5	$p \in B^o \subset B \subset O_1 \subset O$	$p \in B^o \subset B \subset O' \subset O$
page 31, l. 1	$\bigcap \tilde{V}i = \{p\}$	$\bigcap \tilde{V}i^{-} = \{p\}$
page 31, l. 2	$\bigcap Vi = \emptyset$	$\bigcap \overline{V}_i^M = \emptyset$
page 31, l. 4	$\bigcap Vi = \emptyset$	$\bigcap \overline{V}_i^M = \bigcap B_i$
page 33, l. 17	$\ldots \subset O_\alpha^{1+1} \subset \overline{O}^{1+1}$	$\ldots \subset O_\alpha^{1+1} \subset \overline{O}_\alpha^{1+1}$