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J. VAN DER SLOT

A NOTE ON PERFECT IRREDUCIBLE MAPPINGS

2e boerhaavestraat 49 amsterdam

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by

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<u>Until explicitly stated all spaces considered here are assumed to be regular</u>.

<u>INTRODUCTION</u>. Let X be a space and $\mathcal{U}$ an open base for X which is closed for the taking of finite intersections. Then we can consider the collection $X'_{\mathcal{U}}$ consisting of all maximal centered systems of members of $\mathcal{U}$. By defining $U^* = \{\mu \in X'_{\mathcal{U}} \mid U \in \mu\}$ we get a Hausdorff topology on $X'_{\mathcal{U}}$ and a natural irreducible continuous map i of a dense subspace $X_{\mathcal{U}}$ (consisting of those $\mu \in X'_{\mathcal{U}}$ for which $\cap\{\overline{U} \mid U \in \mu\} \neq \emptyset$) onto X, sending each $U^* = U' \cap X_{\mathcal{U}}$ onto $\overline{U}$. We shall derive necessary and sufficient conditions on $\mathcal{U}$ in order that the induces map of $X_{\mathcal{U}}$ onto X is perfect.

Moreover, let f be a perfect and irreducible map of a space X onto a space Y and $\mathcal{U}$, $\mathcal{V}$ open bases for X and Y respectively, closed for the taking of finite intersections and such that $\overline{\mathcal{V}} = \{f(\overline{U}) \mid U \in \mathcal{U}\}$. (It is well known that if $\mathcal{U}$ is closed for finite unions then the collection $\{Y \setminus f(X\setminus U) \mid U \in \mathcal{U}\}$ is such a base). We will show that there is a natural homeomorphism of $X'_{\mathcal{U}}$ onto $Y'_{\mathcal{V}}$ sending each U^* onto V^* if $f(\overline{U}) = \overline{V}$, and which maps $X_{\mathcal{U}}$ onto $Y_{\mathcal{V}}$.

In the sequal $\mathcal{U}$ is a base for the space X which is <u>closed for finite intersections</u>. By greek letters we denote maximal centered families of elements of $\mathcal{U}$. We set $X'_{\mathcal{U}} = \{\mu \mid \mu$ maximal centered system of elements of $\mathcal{U}\}$ and for $U \in \mathcal{U}$ $U' = \{\mu \in X'_{\mathcal{U}} \mid U \in \mu\}$. Furthermore, $X_{\mathcal{U}} = \{\mu \in X'_{\mathcal{U}} \mid \cap\{\overline{U} \mid U \in \mu\} \neq \emptyset\}$ and $U^* = U' \cap X_{\mathcal{U}}$.

<u>PROPOSITION 1</u>. a) <u>The collection</u> $\mathcal{U}'$ <u>for</u> $U \in \mathcal{U}$ <u>is a base for a (Hausdorff)topology on</u> $X'_{\mathcal{U}}$. <u>Moreover, for each</u> $U_1, \ldots, U_n \in \mathcal{U}$ <u>we have</u> $(U_1 \cap \ldots \cap U_n)' = U'_1 \cap \ldots \cap U'_n$. <u>Each centered system of members of</u> $\mathcal{U}'$

has non-empty intersection.

b) Each U' is open and closed i.e. $X'_{\mathfrak{U}}$ is zerodimensional.

c) The natural mapping i which assigns to each $\mu \in X_{\mathfrak{U}}$ the point $i(\mu) = \cap\{\overline{U} \mid U \in \mu\}$ of X is continuous, irreducible and sends each U^* onto $\overline{U}$.

PROOF. It is obvious that for each $U_1, \ldots, U_n \in \mathfrak{U}$ we have $(U_1 \cap \ldots \cap U_n)' = U'_1 \cap \ldots \cap U'_n$ because $\mathfrak{U}$ is closed for finite intersections. Thus $\mathfrak{U}'$ is a base for a topology on $X'_{\mathfrak{U}}$. Now, let $\mathfrak{U}'_1 = \{U' \mid U \in \mathfrak{U}_1 \subset \mathfrak{U}\}$ be a center-system of elements of $\mathfrak{U}'$. One easily verifies that $\mathfrak{U}_1$ is a centered family of members of $\mathfrak{U}$; hence $\mathfrak{U}_1$ is contained in some $\mu \in X'_{\mathfrak{U}}$. It follows $\mu \in \cap\mathfrak{U}'_1$.

b) The fact that each centered family of members of $\mathfrak{U}'$ has non empty intersections in $X'_{\mathfrak{U}}$ implies that each U' is open and closed in $X'_{\mathfrak{U}}$.

c) We shall first prove that $i(U^*) = \overline{U}$ for each $U \in \mathfrak{U}$. If $p \in i(U^*)$, then clearly $p \in \overline{U}$. Conversely, if $p \in \overline{U}$ then the neighbourhood system consisting of all $U \in \mathfrak{U}$ containing p, together with U is contained in some maximal centered system μ of $X_{\mathfrak{U}}$. Hence $i(\mu) = p$. To prove the continuity of i, let $i(\mu) = p \in X$. Let U be a member of $\mathfrak{U}$ containing p and $V \in \mathfrak{U}$ be such that $p \in V \subset \overline{V} \subset U$. Clearly $\mu \in V^*$ and $i(V^*) = \overline{V} \subset U$. To prove that i is irreducible, let S be closed in X. If $S \neq X_{\mathfrak{U}}$ there is $U \in \mathfrak{U}$ such that $U^* \cap S = \emptyset$; $U \neq \emptyset$. Let $p \in U$, then $i^{-1}(p) \subset U^*$; hence $p \notin i(S)$.

We shall recall one more proposition which we shall use later. With the notation of proposition 1 we have

PROPOSITION 2. If $V \in \mathfrak{U}$ and $\mathfrak{U}_1$ is a subcollection of $\mathfrak{U}$ such that $V \subset \cup \overline{\mathfrak{U}_1}$, then $V' \subset \overline{\cup \mathfrak{U}'_1}$. If $\mathfrak{U}_1$ is finite, then $V' \subset \cup \mathfrak{U}'_1$.

PROOF. Let $\mu \in V'$ and suppose, on the contrary, that $\mu \notin \overline{\cup \mathfrak{U}'_1}$. Hence there exists $W \in \mu$ such that $W' \cap (\cup\mathfrak{U}'_1) = \emptyset$, i.e. $W \cap U = \emptyset$ and also $W \cap \overline{U} = \emptyset$ for each $U \in \mathfrak{U}_1$. It follows $W \cap (\cup\overline{\mathfrak{U}_1}) = \emptyset$. Since $V \subset \cup \overline{\mathfrak{U}_1}$ we have $V \cap W = \emptyset$ which is impossible.

COROLLARY. If $\mathfrak{U}$ is the collection of all open subsets of X, then the closure in $X'_{\mathfrak{U}}$ of each open set of $X'_{\mathfrak{U}}$ is open. Indeed, if O is open in $X'_{\mathfrak{U}}$ then $O = \cup\mathfrak{U}'_1$ for some subcollection $\mathfrak{U}_1$ of $\mathfrak{U}$; hence $\overline{O} = \overline{\cup\mathfrak{U}'_1} \supset (\cup\mathfrak{U}_1)' \supset \cup\mathfrak{U}'_1 = O$. Because $(\cup\mathfrak{U}_1)'$ is closed the statement follows. Thus we conclude that in the case that $\mathcal{U}$ is the collection of all open subsets of X, then $X'_{\mathfrak{U}}$ (and also $X_{\mathfrak{U}}$) is extremely disconnected.

DEFINITION. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be collections of subsets of a space X. We shall write $\mathfrak{U}_1 * \mathfrak{U}_2 = \emptyset$ in case that for each $U_1 \in \mathfrak{U}_1$ there is $U_2 \in \mathfrak{U}_2$ such that $U_1 \cap U_2 = \emptyset$ and conversely with $\mathfrak{U}_1$ and $\mathfrak{U}_2$ interchanged.

DEFINITION. Let $\mathfrak{U}$ be a base for a space X. $\mathfrak{U}$ is called semi-complemented provided that given $\mathfrak{U}_1 \subset \mathfrak{U}$ and p is a boundary point of each $\overline{U}_1 \cup \ldots \cup \overline{U}_n$ $(U_i \in \mathfrak{U}_1)$ then these exists a subcollection $\mathfrak{U}_2 \subset \mathfrak{U}$ such that $\mathfrak{U}_1 * \mathfrak{U}_2 = \emptyset$ and p is a boundary point of each $V_1 \cap \ldots \cap V_n$ $(V_i \in \mathfrak{U}_2)$.

If $\mathfrak{U}$ is a complemented base for X (i.e. $U \in \mathfrak{U}$ implies $X \setminus \overline{U} \in \mathfrak{U}$) then $\mathfrak{U}$ is semicomplemented. It is also easy to prove that if $\mathfrak{U}$ is a semiring (i.e. $U \in \mathfrak{U}, V \in \mathfrak{U} \Rightarrow U \setminus \overline{V} \in \mathfrak{U}$) then $\mathfrak{U}$ is also semicomplemented. If each $U \in \mathfrak{U}$ is open and closed then $\mathfrak{U}$ is semicomplemented.

DEFINITION. A mapping f of a space X onto a space Y is called perfect provided that it is continuous, closed (the images of closed sets are closed) and the preimage of points of Y are compact. f is called irreducible provided that $f(S) \neq Y$ for each proper closed subset S of X.

Hereafter we will show that under very general hypotheses on a base $\mathfrak{U}$ (namely $\mathfrak{U}$ be semicomplemented) the induced mapping $i: X_{\mathfrak{U}} \to X$ defined on page 1 is perfect and irreducible.

First we mention a few properties of such mappings.

PROPOSITION 4. Let f be an irreducible continuous map of a space X onto a space Y. If 0 is open in X, then $\overline{f(0)} = \overline{Y \setminus f(X\setminus 0)}$.

PROOF. It suffices to show that $\overline{f(0)} \subset \overline{Y \setminus f(X\setminus 0)}$. It is evident that $f[X \setminus 0 \cup \overline{f^{-1}(Y\setminus f(X\setminus 0))}] = Y$, and since f is an irreducible map, it follows that $(X\setminus 0) \cup \overline{f^{-1}(Y\setminus f(X\setminus 0)} = X$, i.e., $0 \subset \overline{f^{-1}(Y\setminus f(X\setminus 0)}$. Thus $\overline{f(0)} \subset \overline{Y \setminus f(X\setminus 0)}$.

PROPOSITION 5. Let f be a perfect mapping of X onto Y. If $\mathfrak{U}$ is a base for X which is closed under the taking of finite unions, then the collection $\{Y \setminus f(X\setminus U) \mid U \in \mathfrak{U}\}$ is an open base for Y.

PROOF. This is well known (see e.g. [2] or [5]).

PROPOSITION 6. Let f be a perfect irreducible map of X onto Y; $\mathfrak{U}$ a base for X consisting of open and closed subsets and $\mathfrak{V}$ a base of Y such that $\overline{\mathfrak{V}} = \{f(U) \mid U \in \mathcal{U}\}$. Then $\mathfrak{V}$ is semicomplemented.

PROOF. Let $y \in Y$ and y be a boundary point of each $\overline{V}_1 \cup \ldots \cup \overline{V}_n$ where $V_1, \ldots, V_n$ run through a subcollection $\mathfrak{V}_1$ of $\mathfrak{V}$. For $V \in \mathfrak{V}$ let $U(V) \in \mathcal{U}$ be such that $f(U(V)) = \overline{V}$. We propose that the collection $f^{-1}(Y) \cap \{X \setminus U(V) \mid V \in \mathfrak{V}_1\}$ is a centered system. Indeed, if $V_1, \ldots, V_n \in \mathfrak{V}_1$ then

$$y \in \overline{Y \setminus \cup\{f(U(V_i))|i=1,2,\ldots,n\}} = \overline{Y \setminus f(\cup\{U(V_i)|i=1,2,\ldots,n\})} =$$

$$= f(X\setminus\cup\{U(V_i)|i=1,2,\ldots,n\}.$$ It follows $f^{-1}(y) \cap \cap\{X \setminus U(V_i)|i=1,\ldots,n\} \neq \emptyset$.

The compactness of $f^{-1}(y)$ yields the existence of a point $q \in \cap\{X \setminus U(V) \mid V \in \mathfrak{V}_1\} \cap f^{-1}(y)$. For each $V \in \mathfrak{V}_1$ let W(V) be an element of $\mathfrak{U}$ such that $q \in W(V) \subset X \setminus U(V)$. And $V' \in \mathfrak{V}$ be such that $\overline{V'} = f(W(V))$. We will show that $\mathfrak{V}_2 = \{V' \mid V \in \mathfrak{V}_1\}$ satisfies the desired conditions. Obviously $V \cap V' = \emptyset$ since $Y \setminus f(X\setminus U(V)) \cap Y \setminus f(X\setminus W(V)) = \emptyset$ so $\mathfrak{V}_2 * \mathfrak{V}_1 = \emptyset$. We will show that y is a boundary point of each $\overline{V'_1 \cap \ldots \cap V'_n}$. Indeed, $y \in \underline{f(\cap\{W(V_i) | i = 1, \ldots, n\})} = \overline{\cap\{Y \setminus f(X \setminus W(V_i) | i = 1, \ldots, n)\}} = \underline{\cap \{V'_i \mid i = 1, \ldots, n\}}$. We also have $\cap \{Y \setminus f(X\setminus W(V_i)) | i = 1, \ldots, n\} \cap \cup \{Y \setminus f(X\setminus U(V_i)) | i = 1, \ldots, n\} = \emptyset$. So $y \notin \cap \{Y \setminus f(X\setminus W(V_i)) | i=1,\ldots,n\}$ i.e. $y \notin \cap \{V'_i | i = 1, \ldots, n\}$. This completes the proof of the proposition.

THEOREM 1. Let $\mathcal{U}$ be a base for a space X and let $\mathcal{U}$ be closed under the taking of finite intersections. Let i be the natural continuous map of $X_{\mathcal{U}}$ onto X. Then i is perfect if and only if $\mathcal{U}$ is semicomplemented.

PROOF. The "only if" part has already been proved in the foregoing proposition. To prove the "if" part we shall first show that $i^{-1}(p)$ is compact for each p of X. Let $\{X_{\mathcal{U}} \setminus U^* | U \in \mathcal{U}_1\} \cap i^{-1}(p)$ be a centered system of members of $X_{\mathcal{U}} \setminus \mathcal{U}^*$. We may suppose that $X_{\mathcal{U}} \setminus U^* \not\supset i^{-1}(p)$ for each $U \in \mathcal{U}_1$. Then p is a boundary point of each $\cup \{\overline{U}_i \mid i = 1, \ldots, n\}$ $(U_i \in \mathcal{U}_1)$. Indeed, $i^{-1}(p) \cap U_i^* \neq \emptyset$ for all i, so $p \in \cup \{\overline{U}_i | i = 1, \ldots, n\}$. We also have $p \notin \text{int} \cup \{\overline{U}_i | i = 1, \ldots, n\}$, because otherwise there is $V \in \mathcal{U}$ containing p such that $V \subset \cup \{\overline{U}_i | i = 1, \ldots, n\}$. Hence $V^* \subset \cup \{U_i^* | i = 1, \ldots, n\}$ (prop. 2) which is impossible since $i^{-1}(p) \subset V^*$. Because $\mathcal{U}$ is semicomplemented there exists $\mathcal{U}_2 \subset \mathcal{U}$ such that $\mathcal{U}_1 * \mathcal{U}_2 = \emptyset$ and such that p is a boundary point of each $\cap \{V_i | i = 1, \ldots, n\} (V_i \in \mathcal{U}_2)$. Let $\mathcal{U}(p) = \{U \in \mathcal{U} | p \in U\}$, then $\mathcal{U}(p) \cup \mathcal{U}_2$ is centered and is contained in some $\mu \in X_{\mathcal{U}}$. We propose $\mu \in \cap \{X_{\mathcal{U}} \setminus U^* | U \in \mathcal{U}_1\} \cap i^{-1}(p)$. $\mu \in i^{-1}(p)$ is obvious, and since for each $U \in \mathcal{U}_1$ there is $V \in \mathcal{U}_2$ such that $V \cap U = \emptyset$ μ cannot belong to some U^* for $U \in \mathcal{U}_1$. Thus we have proved that $i^{-1}(p)$ is compact for each $p \in X$.

We shall now prove that i is a closed mapping. Let S be closed in X and $p \in \overline{f(S)}$. Let us suppose that $p \notin f(S)$. Thus $i^{-1}(p) \cap S = \emptyset$. We have just proved that $i^{-1}(p)$ is compact, so there are U_i, $i = 1, \ldots, n \in \mathcal{U}$ such that $i^{-1}(p) \subset \cup \{U_i^* | i = 1, \ldots, n\}$ and $U_i^* \cap S = \emptyset$ for all i. We shall first prove that p is a boundary point of $\cup \{\overline{U}_i | i = 1, \ldots, n\}$. It is clear that $p \in \cup \{\overline{U}_i | i = 1, \ldots, n\}$. Let us suppose that $p \in \text{int} \cup \{\overline{U}_i | i = 1, \ldots, n\}$. Hence there exists $V \in \mathcal{U}$ such that

$p \in V \subset \cup \{\overline{U}_i | i = 1, \ldots, n\}$. Thus $i^{-1}(p) \subset V^* \subset \cup \{U_i^* | i = 1, \ldots, n\}$ (prop 2). Since $\mu \notin V^*$ implies that there is $W \in \mathcal{U}$ such that $W \cap V = \emptyset$; hence $i(\mu) \in \overline{W} \subset \overline{X \setminus \overline{V}}$, it follows that $\overline{f(S)} \subset \overline{X \setminus \overline{V}}$. However, $p \notin \overline{X \setminus \overline{V}}$, contradicting $p \in \overline{f(S)}$. We conclude that p is a boundary point of $\cup \{\overline{U}_i | i = 1, \ldots, n\}$. Since $\mathcal{U}$ is semicomplemented there are $V_1, \ldots, V_n \in \mathcal{U}$ such that $V_i \cap U_i = \emptyset$ $(i=1,\ldots,n)$ and $p \in \overline{\cap \{V_i | i = 1, \ldots, n\}}$. Let μ be a member of $i^{-1}(p)$ that contains the collection $\{V_i | i = 1, \ldots, n\}$; then $\mu \in U_l^*$ for some l $(1 \le l \le n)$ i.e. $U_l \in \mu$. However $U_l \cap V_l = \emptyset$ gives a contradiction. This completes the proof of the theorem.

EXAMPLE 1. Consider the real numbers $\mathbb{R}$ with the usual order topology. Consider two bases $\mathcal{U}_1$ and $\mathcal{U}_2$ for $\mathbb{R}$.

1° $\mathcal{U}_1 = \{(a,b) \mid a, b \text{ are rational}\}$

2° $\mathcal{U}_2 = \{(a,b) \mid a \text{ is rational; } b \text{ is irrational}\}$.

Both $\mathcal{U}_1$ and $\mathcal{U}_2$ are closed for finite intersections. However, $\mathcal{U}_2$ is not semicomplemented, since for each $U_1, U_2 \in \mathcal{U}_2$, $U_1 \cap U_2 = \emptyset$ implies $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. The mapping i is one to one; i is not perfect because it would then be a homeomorfism, which is impossible since $\mathbb{R}$ is not zero-dimensional.

The base $\mathcal{U}_1$ for $\mathbb{R}$ is semicomplemented, hence $\mathbb{R}_{\mathcal{U}_1}$ is mapped perfectly onto $\mathbb{R}$.

EXAMPLE 2. Let X be a metric space. For i = 1, 2, ... there are locally finite open collections $\mathcal{U}_i$ of X, consisting of regularly open sets with the following properties:

a) the members of $\mathcal{U}_i$, i = 1, 2, ... are disjoint; $\overline{\mathcal{U}_i}$ covers X.

b) $\overline{\mathcal{U}}_{i+1}$ refines $\overline{\mathcal{U}}_i$.

c) $\operatorname{diam} \mathcal{U}_i < \frac{1}{i}$.

If we consider the base $\mathcal{U}$ for X consisting of all interiors of finite unions of members of $\overline{\mathcal{U}}_i$ for i = 1, 2, ..., then it is easy to see that

$\mathfrak{U}$ is closed for finite intersections and is semicomplemented. Thus X is mapped perfectly onto X and it is easy to see that $X_{\mathfrak{U}}$ is metrizable with covering dimension zero. Thus we have proved that each metrizable space is the image of a zerodimensional metrizable space under a perfect irreducible mapping (this is a well known result of Morita [5]).

THEOREM 2. *Let f be a perfect irreducible map of a space X onto a space Y. Let $\mathfrak{U}, \mathfrak{V}$ be bases for X and Y, respectively, closed for finite intersections and such that $\{f(\overline{U}) \mid U \in \mathfrak{U}\} = \overline{\mathfrak{V}}$. With the notation of proposition 1 there is a homeomorfism f^* of $X'_{\mathfrak{U}}$ onto $Y'_{\mathfrak{V}}$ which takes $X_{\mathfrak{U}}$ onto $Y_{\mathfrak{V}}$ and such that $f^*(U^*) = V^*$ for each $(U,V) \in (\mathfrak{U},\mathfrak{V})$ with the property $f(\overline{U}) = \overline{V}$.*

In our proof we make use of the following lemma

LEMMA. Let f, X, Y, $\mathfrak{U}$ and $\mathfrak{V}$ satisfy the above conditions. If U_i, $i = 1, \ldots, n$ and V_i, $i = 1, \ldots, n$ are finite subcollections of $\mathfrak{U}$ and $\mathfrak{V}$, respectively such that $f(\overline{U}_i) = \overline{V}_i$ then $\cap \{U_i \mid i = 1, \ldots, n\} = \emptyset$ is equivalent with $\cap \{V_i \mid i = 1, \ldots, n\} = \emptyset$.

PROOF. $\cap \{U_i \mid i = 1, \ldots, n\} = \emptyset$ is equivalent with $\cap \{Y \setminus f(X \setminus U_i) \mid i = 1, \ldots, n\} = \emptyset$, by the irreducibility of f. Because $\overline{Y \setminus f(X \setminus U_i)} = \overline{V}_i$ the statement follows.

Proof of the theorem: Let $\mu \in X'_{\mathfrak{U}}$. Then μ is a maximal centered system $\mathfrak{U}_1$ of members of $\mathfrak{U}$. Let $\mathfrak{V}_1 = \{V \in \mathfrak{V} \mid f(\overline{U}) = \overline{V}$ for some $U \in \mathfrak{U}_1\}$. One easily verifies (using the previous lemma) that $\mathfrak{V}_1$ is a maximal centered system of members of $\mathfrak{V}$, so $\mathfrak{V}_1$ defines an element $\nu = f^*(\mu)$ of $Y_{\mathfrak{V}}$. We will show that f^* satisfies all required conditions.

If $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ satisfy $f(\overline{U}) = \overline{V}$, then $\mu \in U'$ implies $U \in \mu$ and also $V \in f^*(\mu)$, i.e. $f^*(\mu) \in V'$. On the other hand $\mu \notin U'$ implies $U \notin \mu$; so there is $U_1 \in \mu$ such that $U \cap U_1 = \emptyset$. If $V_1 \in \mathfrak{V}$ satisfies $f(\overline{U}_1) = \overline{V}_1$, then we have by the previous lemma $V \cap V_1 = \emptyset$, i.e. $f^*(\mu) \notin V'$. Thus we have proved $f^*(\mu) \in V'$ if and only if $\mu \in U'$, hence f^* is continuous.

f^* is an onto-mapping: Indeed, if $\nu \in Y_{\mathcal{V}}$, and $\mathcal{U}_1 = \{U \in \mathcal{U} | f(\overline{U}) \in \overline{\nu}\}$, then $\mathcal{U}_1$ is a maximal centered system μ of members of $\mathcal{U}$, which is mapped onto ν by f^*.

f^* is one-to-one: If $\mu_1 \neq \mu_2 \in X'_{\mathcal{U}}$ then there are $U_1, U_2 \in \mathcal{U}$ such that $\mu_1 \in U'_1$, $\mu_2 \in U'_2$ and $U_1 \cap U_2 = \emptyset$. Let $V_1, V_2 \in \mathcal{V}$ satisfy $f(\overline{U}_1) = \overline{V}_1$ and $f(\overline{U}_2) = \overline{V}_2$. Then $V_1 \cap V_2 = \emptyset$ and $V'_1 \cap V'_2 = \emptyset$. Since $f^*(\mu_1) \in V'_1$ and $f^*(\mu_2) \in V'_2$ we have $f^*(\mu_1) \neq f^*(\mu_2)$.

The only which remains to show is that f^* maps $X_{\mathcal{U}}$ onto $Y_{\mathcal{V}}$. (Then we have also proved that $f^*(U^*) = V^*$ if $f(\overline{U}) = \overline{V}$ $(U \in \mathcal{U}, V \in \mathcal{V})$.) If $\mu \in X_{\mathcal{U}}$ then $\cap \{\overline{U} | U \in \mu\} \neq \emptyset$ and also $\cap \{f(\overline{U}) | U \in \mu\} \neq \emptyset$. Thus $f^*(\mu) \in Y_{\mathcal{V}}$. Conversely, if $\nu \in Y_{\mathcal{V}}$ then $\cap \{\overline{V} | V \in \nu\} \neq \emptyset$. Let $\mathcal{U}_1 = \{U \in \mathcal{U} | f(\overline{U}) \in \overline{\nu}\}$. As before, $\mathcal{U}_1$ is a maximal centered system of elements of $\mathcal{U}$ and we only need to show that $\cap \overline{\mathcal{U}_1} \neq \emptyset$. Let $p = \cap \{\overline{V} | V \in \nu\}$ then $\{\overline{U} \cap f^{-1}(p) | U \in \mathcal{U}_1\}$ is centered because for each $U_1, \ldots, U_n \in \mathcal{U}_1$ we have $p \in f(\overline{\cap \{U_i | i = 1, \ldots, n\}}) \subset f(\cap \{\overline{U}_i | i = 1, \ldots, n\})$. Compactness of $f^{-1}(p)$ yields indeed $\cap \overline{\mathcal{U}_1} \neq \emptyset$.

<u>N.B</u>. If each member of $\mathcal{U}$ is open and closed, then $X_{\mathcal{U}}$ is homeomorphic with X. In that case f^* establishes a homeomorphism of X onto $Y_{\mathcal{V}}$.

<u>REMARK</u>. Let X be a space and let $\mathcal{O}$ be the collection of all open subsets. In the literature $X_{\mathcal{O}}$ is called the <u>absolute</u> of X. $X_{\mathcal{O}}$ is extremely disconnected and is mapped perfectly onto X (see also [2], [4] and [6]). Two spaces which have homeomorphic absolutes are called <u>coabsolute</u>. If Y is a perfect irreducible image of X then X and Y are coabsolute. Furthermore, the property of being coabsolute is transitive, i.e. if X and Y are coabsolute; Y and Z are coabsolute, then X and Z are coabsolute.

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