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AFDELING ZUIVERE WISKUNDE ZN 36/71 ~~ZW 1/71~~ JANUARI

T.A. CHAPMAN

ON SOME APPLICATIONS OF INFINITE-DIMENSIONAL MANIFOLDS TO THE THEORY OF SHAPE

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Contents

On Some Applications of Infinite-Dimensional Manifolds
To The Theory of Shape

T.A. Chapman[1]

1. Introduction.

In this paper we apply some recent results concerning the point-set topology of infinite-dimensional manifolds to the concept of "shape", as introduced by Borsuk in [5].

Let the Hilbert cube I^∞ be represented by $\Pi_{i=1}^{\infty} I_i$, where each I_i is the closed interval $[-1,1]$, and let s denote $\Pi_{i=1}^{\infty} I_i^0$, where each I_i^0 is the open interval $(-1,1)$. We let $\mathcal{S}$ denote the category whose objects are compacta in s and whose morphisms are fundamental equivalence classes of fundamental sequences (in I^∞) between these compacta. (This constitutes a subcategory of the <u>fundamental category</u> introduced in [5].) We let $\mathcal{P}$ denote the category whose objects are subsets of I^∞, with complements in I^∞ which are compacta in s, and whose morphisms are weak proper homotopy classes of proper maps (see Section 2 for a more precise definition).

The first result we establish enables us to translate problems concerning the shape of compacta to problems concerning contractible open subsets of I^∞.

<u>Theorem 1</u>. <u>There is a category isomorphism</u> T <u>from</u> $\mathcal{P}$ <u>onto</u> $\mathcal{S}$ <u>such that</u> $T(X) = I^\infty \setminus X$, <u>for each object</u> X <u>in</u> $\mathcal{P}$.

We also show that the shape of a compactum in s depends on (and determines) the homeomorphism type of its complement in I^∞.

<u>Theorem 2</u>. <u>If</u> X <u>and</u> Y <u>are compacta in</u> s, <u>then</u> X <u>and</u> Y <u>have the same shape</u> (i.e. $Sh(X) = Sh(Y)$) <u>iff</u> $I^\infty \setminus X$ <u>and</u> $I^\infty \setminus Y$ <u>are homeomorphic</u> ($\cong$).

This result enables us to identify the fundamental absolute retracts (abbreviated FAR) in s, as introduced in [6].

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1. Supported in part by NSF grant GP 14429.

Theorem 3. *If* $X \subset s$ *is a compactum, then* X *is a* FAR *iff* $I^\infty \setminus X \cong I^\infty$ {*point*}.

We remark that as a corollary of Theorem 3 we show that each FAR in s is the intersection of a decreasing sequence of Hilbert cubes, which gives another proof of the main result in [10]. In a separate paper we will apply these results to obtain some solutions to some concrete problems concerning FAR's [9].

2. General preliminaries.

Concerning the fundamental category S we will use the results and notation from [5] and [6].

Concerning the proper category P we define a map (i.e. a continuous function) $f: X \to Y$ to be proper iff for each compactum $B \subset Y$ there exists a compactum $A \subset X$ such that $f(X\setminus A) \cap B = \emptyset$. (This is just a reformulation of the usual notion of a proper map.) Then maps $f, g: X \to Y$ are said to be weakly properly homotopic iff for each compactum $B \subset Y$ there exists a compactum $A \subset X$ and a homotopy $F = \{F_t\}: X \times I \to Y$ (where $I = [0,1]$) such that $F_0 = f$, $F_1 = g$, and $F((X\setminus A) \times I) \cap B = \emptyset$. (If, in fact, there exists a proper map $F: X \times I \to Y$ which satisfies $F_0 = f$ and $F_1 = g$, then we say that f and g are properly homotopic.) We write $f \sim g$ to indicate that f and g are weakly properly homotopic.

If $f: X \to Y$ and $g: Y \to X$ are proper maps such that $f \circ g \sim id_Y$ (the identity on Y), then we say that X weakly properly homotopically dominates Y. If, additionally, $g \circ f \sim id_X$, then we say that X and Y have the same weak proper homotopy type. If $f: X \to Y$ is a proper map, then we use $\{f\}$ to denote the class of proper maps of X into Y which are weakly properly homotopic to f.

It is easy to see that $\sim$ is an equivalence relation on the class of proper maps from a space X to a space Y. It is also easy to see that if $f, f': X \to Y$ and $g, g': X \to Y$ are proper maps such that $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$. This verifies that the composition of the equivalence classes $\{f\}$ and $\{g\}$ can be well defined by $\{g \circ f\}$. Thus we can define a category P whose objects are subsets of I^∞, with complements in I^∞ which are compacta in s, and whose morphisms are weak proper homotopy equivalence classes of proper maps.

3. Infinite-dimensional preliminaries.

We will need the following definition, as introduced by Anderson in [1]. A closed set K in a space X is said to be a Z-set in X iff for each non-null, homotopically trivial open set U in X, $U\setminus K$ is non-null and homotopically trivial. From [1] we find that compacta in s are Z-sets in s and I^∞ and compacta in $I^\infty \setminus s$ are Z-sets in I^∞. More generally it is easy to see that if K is a Z-set in a space X and U is open in X, then $U \cap K$ is a Z-set in U.

We will need the notion of a Q-manifold, which is a separable metric space which has an open cover by sets homeomorphic to open subsets of I^∞. In [2] it is shown that if X is a Q-manifold, then $X \times I^\infty \cong X$. Thus for each Q-manifold X we have $X \cong X \times [0,1]$. The following results on Q-manifolds are established in [8].

Lemma 3.1. If X is any Q-manifold, then there is a locally-compact polyhedron P such that $X \times [0,1) \cong P \times I^\infty$

Lemma 3.2. If X is a Q-manifold, P is a locally-compact polyhedron, and $\phi : P \to X$ is a closed embedding such that $\phi(P)$ is a Z-set in X, then there exists a closed embedding $h : P \times I^\infty \to X$ such that $h(x,(0,0,\ldots)) = \phi(x)$, for all $x \in P$, and $Bd(h(P\times I^\infty)) = h(P\times W^+)$.

(For the representation $I^\infty = \Pi_{i=1}^\infty I_i$ as given in Section 1 we use the notation $W^+ = \{(x_i) \in I^\infty \mid x_1 = 1\}$ and $W^- = \{(x_i) \in I^\infty \mid x_1 = -1\}$. We also use Bd for the topological boundary operator.)

Let X and Y be spaces and let $\mathcal{U}$ be an open cover of Y. Then functions $f,g : X \to Y$ are said to be $\mathcal{U}$-close provided that for each $x \in X$ there exists a $U \in \mathcal{U}$ such that $f(x), g(x) \in U$. A function $F: X \times I \to Y$ is said to be limited by $\mathcal{U}$ provided that for each $x \in X$ there exists a $U \in \mathcal{U}$ such that $F(\{x\}\times I) \subset U$.

If X is a metric space and $K \subset X$ is closed, then from [3] there exists an open cover $\mathcal{U}$ of $X\setminus K$ such that if $h: X\setminus K \to X\setminus K$ is any homeomorphism which is $\mathcal{U}$-close to $id_{X\setminus K}$, then h can be extended to a homeomorphism $\tilde{h}: X \to X$ which satisfies $\tilde{h}|K = id_K$. Such a cover $X\setminus K$ will be called normal (with respect to K).

We will need the following mapping replacement result which appears in [4].

Lemma 3.3. Let X be a Q-manifold, $\mathcal{U}$ be an open cover of X, A be a closed subset of a locally-compact separable metric space Y, and let f: Y → X be a proper map such that f|A is a homeomorphism of A onto a Z-set in X. Then there exists an embedding g: Y → X such that g(Y) is a Z-set, g|A = f|A, and g is $\mathcal{U}$-close to f.

We will also need a version of this result for Q-manifolds which are [0,1)-stable. The proof is given in [4].

Lemma 3.4. Let X be a Q-manifold which satisfies $X \cong X \times [0,1)$, A be a closed subset of a locally-compact separable metric space Y, and let f: Y → X be a map such that f|A is a homeomorphism of A onto a Z-set in X. Then there exists an embedding g: Y → X such that g(Y) is a Z-set in X, g|A = f|A, and g ≃ f (i.e. g is homotopic to f).

(Note that if X is any Q-manifold, then

$$(X\times[0,1)) \times [0,1) \cong (X\times[0,1]) \times [0,1) \cong X \times [0,1)).$$

The following homeomorphism extension theorem will be useful [4].

Lemma 3.5. Let X be a Q-manifold, $\mathcal{U}$ be an open cover of X, A be a locally-compact separable metric space, and let f,g: A → X be closed embeddings such that f(A) and g(A) are Z-sets in X and such that there exists a proper homotopy F: A × I → X which is limited by $\mathcal{U}$ and which satisfies $F_0 = f$, $F_1 = g$. Then there exists a homeomorphism h: X → X which satisfies h ∘ f = g and which is $St^4(\mathcal{U})$-close to id_X.

We now combine these results to prove the following lemma which will be needed in Section 5.

Lemma 3.6. Let X and Y be Q-manifolds such that $X \cong X \times [0,1)$ and let f: X → Y be any continuous function. Then there exists an open embedding g: X → Y which satisfies g ≃ f.

Proof. Let h: Y → Y × [0,1] be any homeomorphism. It is clear that h ∘ f is homotopic to a continuous function f': X → Y × [0,1). Let $Y' = h^{-1}(Y\times[0,1))$ (which is an open subset of Y) and define $f'' = h^{-1} \circ f'$, which is a continuous function of X into Y' which is homotopic to f. Note also that $Y' \cong Y' \times [0,1)$.

We know that $X \cong P \times I^\infty$, for some locally-compact polyhedron P. Thus without loss of generality assume that $X = P \times I^\infty$. Using Lemma 3.4 there exists an embedding $\phi: P \times \{(0,0,\ldots)\} \to Y'$ such that $\phi(P \times \{(0,0\ldots)\}$ is a Z-set and $\phi \simeq f''|P \times \{(0,0,\ldots)\}$. Using Lemma 3.2 there exists an open embedding $g: P \times (I^\infty \setminus W^+) \to Y'$ such that $g(x,(0,0,\ldots)) = \phi(x,(0,0,\ldots))$, for all $x \in P$. Let $r: P \times (I^\infty \setminus W^+) \to P \times \{(0,0,\ldots)\}$ be the retraction which satisfies $r(x,t) = (x,(0,0,\ldots))$, for all $(x,t) \in P \times (I^\infty \setminus W^+)$. Then we observe that $r \simeq id_{P\times(I^\infty\setminus W^+)}$. We thus have

$$g = g\circ id \simeq g\circ r = \phi\circ r \simeq (f''|P\times\{(0,0,\ldots)\})\circ r = f''\circ r \simeq f''\circ id = f''.$$

We will also need the following result.

Lemma 3.7. Let X be a Q-manifold and let $K \subset X$ be a Z-set. Then there exists an open set $U \subset X$ such that $K \subset U$ and $U \cong U \times [0,1)$.
Proof. From [7] it follows that there exists a homeomorphism $h: X \to X \times [0,1]$ such that $h(K) \subset X \times \{\frac{1}{2}\}$. Then $U = h^{-1}(X\times[0,1))$ fulfills our requirements.

A subset K of a space X is said to be bicollared provided that there exists an open embedding $h: K \times (-1,1) \to X$ such that $h(x,0) = x$, for all $x \in K$. We will need the following result, which appears in [11].

Lemma 3.8. Let $f: I^\infty \to I^\infty$ be an embedding such that $f(I^\infty)$ is bicollared. Then $I^\infty \setminus f(I^\infty) = A \cup B$, where A and B are disjoint sets such that $Cl(A) \cap Cl(B) = f(I^\infty)$ and $Cl(A) \cong Cl(B) \cong I^\infty$, where Cl denotes closure.
(Note that $f(I^\infty)$ is a Z-set in each of $Cl(A)$ and $Cl(B)$).

4. Proof of Theorem 1. We will need the following result in the proof of Theorem 1.

Lemma 4.1. *If $X \subset I^\infty$ is a Z-set, then there exists a homotopy $F: I^\infty \times I \to I^\infty$ which satisfies the following properties.*

(1) $F_0 = \mathrm{id}$,

(2) *for each open neighborhood* U *of* X *there exists a* $t_1 \in (0,1)$ *such that* $F_t | I^\infty \setminus U = \mathrm{id}$, *for* $0 \leq t \leq t_1$,

(3) $F_t(I^\infty) \cap X = \emptyset$, *for all* $t \in (0,1]$.

Proof. Using Lemma 3.5 we can assume that $X \subset W^+$. Then the construction of F is straightforward.

We will use the notation $\mathcal{F}(X)$ to denote the class of homotopies $F: I^\infty \times I \to I^\infty$ as described in Lemma 4.1.

We now construct an isomorphism T from $\mathcal{P}$ onto $\mathcal{S}$. As indicated in the statement of Theorem 1 we let $T(X) = I^\infty \setminus X$, for each X in $\mathcal{P}$.

We now show how T assigns morphisms.

Let $\{f\}: X \to Y$ be a morphism in $\mathcal{P}$, choose any $F \in \mathcal{F}(I^\infty \setminus X)$, and for each integer $k > 0$ let $f_k = f \circ F_{1/k}$. We show that $\underline{f} = \{f_k, I^\infty \setminus X, I^\infty \setminus Y\}$ is a fundamental sequence. To see this let $V \subset I^\infty$ be an open neighborhood of $I^\infty \setminus Y$ and use the fact that f is proper to choose an open neighborhood $U \subset I^\infty$ of $I^\infty \setminus X$ which satisfies $f(U \cap X) \subset V$. Now choose $t_1 \in (0,1)$ such that $F_t | I^\infty \setminus U = \mathrm{id}$, for $0 \leq t \leq t_1$. If k,l are positive integers such that $1/k,\ 1/l \leq t_1$, then $f_k|U = f \circ F_{1/k}|U \simeq f \circ F_{1/l}|U$ (in V) $= f_l|U$, as we wanted. Thus $\underline{f}$ is a fundamental sequence.

To see that $\underline{f}$ is uniquely defined in terms of F choose $F' \in \mathcal{F}(I^\infty \setminus X)$ and let $\underline{f}' = \{f \circ F'_{1/k}, I^\infty \setminus X, I^\infty \setminus Y\}$ be similarly defined. We show that $\underline{f} \simeq \underline{f}'$. Let $V \subset I^\infty$ be an open neighborhood of $I^\infty \setminus Y$ and choose $U \subset I^\infty$ an open neighborhood of $I^\infty \setminus X$ satisfying $f(U \cap X) \subset V$. Choose $t_1 \in (0,1)$ such that $F_t | I^\infty \setminus U = \mathrm{id}$ and $F'_t | I^\infty \setminus U = \mathrm{id}$, for $0 \leq t \leq t_1$. If k is a positive integer satisfying $1/k \leq t_1$ we clearly have $F_{1/k}|U \simeq F'_{1/k}|U$ (in U), with the image of the homotopy possibly intersecting $I^\infty \setminus X$. If this is the case we cannot use f to transfer this homotopy to one joining $f \circ F_{1/k}|U$ to $f \circ F'_{1/k}|U$.

To remedy this let $G: U \times I \to U$ be a homotopy which satisfies $G_0 = F_{1/k}|U$, $G_1 = F'_{1/k}|U$, and let $H: U \times I \to U$ be defined by $H_t = F_{t(1-t)} \circ G_t$. We note that $H_0 = F_{1/k}|U$, $H_1 = F'_{1/k}|U$, and for $0 < t < 1$ we have $H_t(U) = F_{t(1-t)}(G_t(U)) \subset F_{t(1-t)}(U) \subset U \cap X$. Thus $f \circ H_t$ defines a homotopy which joins $f \circ F'_{1/k}|U$. This means that $\underline{f} \simeq \underline{f}'$.

This gives a means of assigning to each proper map $f: X \to Y$ (where $I^\infty \setminus Y$ and $I^\infty \setminus X$ are compacta in s) a fundamental sequence $\underline{f}$ from $I^\infty \setminus X$ to $I^\infty \setminus Y$. In order to see that this assignment depends only on the weak proper homotopy class of f assume that $g: X \to Y$ is proper and $f \sim g$. We wish to show that if $F \in \mathcal{F}(I^\infty \setminus X)$, $\underline{f} = \{f \circ F_{1/k}, I^\infty \setminus X, I^\infty \setminus Y\}$, $\underline{g} = \{g \circ F_{1/k}, I^\infty \setminus X, I^\infty \setminus Y\}$, then $\underline{f} \simeq \underline{g}$. To see this let $V \subset I^\infty$ be an open neighborhood of $I^\infty \setminus Y$ and choose a compact set $A \subset X$ and a homotopy $G: X \times I \to Y$ such that $G_0 = f$, $G_1 = g$, and $G((U \cap X) \times I) \subset V$, where $U = I^\infty \setminus A$. Let $t_1 \in (0,1)$ be chosen so that $F_t | I^\infty \setminus U = id$, for $0 \leq t \leq t_1$. Then for each positive integer k satisfying $1/k \leq t_1$ we find that $G_t \circ F_{1/k}|U$ gives a homotopy (in V) which joins $f \circ F_{1/k}|U$ to $g \circ F_{1/k}|U$ (in V), as we needed.

Thus to each morphism $\{f\} : X \to Y$ in $\mathcal{P}$ we have shown how to assign a unique morphism $[\underline{f}] : I^\infty \setminus X \to I^\infty \setminus Y$ in $\mathcal{S}$, and we write $T(\{f\}) = [\underline{f}]$. We now demonstrate that T is a functor and it is an isomorphism from $\mathcal{P}$ onto $\mathcal{S}$. To show that $T(id) = id$ choose an object X in $\mathcal{P}$ and $F \in \mathcal{F}(I^\infty \setminus X)$, and let $\underline{f} = \{F_{1/k}, I^\infty \setminus X, I^\infty \setminus X\}$. We must show that $\underline{f} \simeq \underline{i}$, the identity fundamental sequence on $I^\infty \setminus X$. Choose an open set U containing $I^\infty \setminus X$ and $t_1 \in (0,1)$ such that $F_t | I^\infty \setminus U = id$, for $0 \leq t \leq t_1$. Clearly $F_{1/k}|U \simeq id_{I^\infty}|U$ (in U), for all positive integers k satisfying $1/k \leq t_1$.

To show that T preserves compositions choose morphisms $\{f\}: X \to Y$ and $\{g\}: Y \to Z$ in $\mathcal{P}$ and choose $F \in \mathcal{F}(I^\infty \setminus X)$, $G \in \mathcal{F}(I^\infty \setminus Y)$. We must show that $\{g \circ f \circ F_{1/k}, I^\infty \setminus X, I^\infty \setminus Z\} \simeq \{g \circ G_{1/k} \circ f \circ F_{1/k}, I^\infty \setminus X, I^\infty \setminus Z\}$.

Choose open neighborhoods $U \subset I^\infty$ of $I^\infty \backslash X$, $V \subset I^\infty$ of $I^\infty \backslash Y$, and $W \subset I^\infty$ of $I^\infty \backslash Z$ such that $f(U \cap X) \subset V$ and $g(V \cap Y) \subset W$. Also choose $t_1 \in (0,1)$ such that $F_t | I^\infty \backslash U = \text{id}$ and $G_t | I^\infty \backslash V = \text{id}$, for $0 \leq t \leq t_1$. Then for each positive k satisfying $1/k \leq t_1$ we have $g \circ G_{1/k} \circ f \circ F_{1/k} | U \simeq g \circ f \circ F_{1/k} | U$ (in W).

To show that T is an isomorphism we show first that if $\{f\}: X \to Y$ and $\{g\}: X \to Y$ are morphisms in $\mathcal{P}$ such that $T(\{f\}) = T(\{g\})$, then $\{f\} = \{g\}$. Choose $F \in \mathcal{F}(I^\infty \backslash X)$ and note that $\{f \circ F_{1/k}, I^\infty \backslash X, I^\infty \backslash Y\} \simeq \{g \circ F_{1/k}, I^\infty \backslash X, I^\infty \backslash Y\}$. Choose $B \subset Y$ a compact set and put $V = I^\infty \backslash B$. Then there exists an open neighborhood $U \subset I^\infty$ of $I^\infty \backslash X$ and an integer $n_1 > 0$ such that $k \geq n_1$ implies that $f \circ F_{1/k} | U \simeq g \circ F_{1/k} | U$ (in V). We note first that $f | U \cap X \simeq f \circ F_{1/k} | U \cap X$ (in V), for each $k \leq n_1$. Similarly we have $g | U \cap X \simeq g \circ F_{1/k} | U \cap X$, hence $f | U \cap X \simeq g | U \cap X$ (in V).

Choose an open neighborhood $U' \subset I^\infty$ of $I^\infty \backslash X$ such that $\mathrm{Cl}(U') \subset U$ and use the above remarks to obtain a homotopy $G: (\mathrm{Cl}(U') \cap X) \times I \to V$ which satisfies $G_0 = f | \mathrm{Cl}(U') \cap X$ and $G_1 = g | \mathrm{Cl}(U') \cap X$. Let $A = (\mathrm{Cl}(U') \cap X) \times I \cup ((X \backslash \mathrm{Cl}(U')) \times \{0,1\}$, which is a closed subset of $X \times I$, and let $\alpha: A \to I^\infty$ be defined by $\alpha | (\mathrm{Cl}(U') \cap X) \times I = G$, $\alpha(x,0) = f(x)$, and $\alpha(x,1) = g(x)$, for all $x \in X \backslash \mathrm{Cl}(U')$. Extend α to a continuous function $\beta: X \times I \to I^\infty$. Then for $t \in I$ let $\gamma_t = F_{t(1-t)} \circ \beta_t$. We see that $\gamma: X \times I \to Y$ is a continuous function which satisfies $\gamma_0 = f$, $\gamma_1 = g$, and $\gamma(\mathrm{Cl}(U') \times I) \subset V$.

This implies that $f \sim g$.

Now choose a morphism $[\underline{f}]: X \to Y$ in $\mathcal{S}$. We must show that there exists a morphism $\{f\}: I^\infty \backslash X \to I^\infty \backslash Y$ in $\mathcal{P}$ such that $T(\{f\}) = [\underline{f}]$.

Using techniques like those used above we can choose a representative $\underline{f} = \{f_k, X, Y\}$ from the class $[\underline{f}]$ such that $f_k(I^\infty) \cap Y = \emptyset$, for all $k > 0$. Choose a sequence $\{U_k\}_{k=1}^\infty$ of open sets in I^∞ such that $X = \bigcap_{i=1}^\infty U_i$ and $U_i \supset \mathrm{Cl}(U_{i+1})$, for all $i > 0$. Also choose a sequence $\{V_i\}_{i=1}^\infty$ of open subsets of I^∞ such that $Y = \bigcap_{i=1}^\infty V_i$. We can pick a sequence $\{n_i\}_{i=1}^\infty$ of positive integers such that $n_1 < n_2 < \dots$ and for each $i \geq 0$ and $k, l \geq n_i$, we have $f_k | \mathrm{Cl}(U_{n_i}) \simeq f_l | \mathrm{Cl}(U_{n_i})$ (in V_i).

Let $\phi_i : I^\infty \to [0,1]$ be a continuous function such that $\phi_i(x) = 0$, for $x \in I^\infty \backslash U_{n_i}$, and $\phi_i(x) = 1$, for $x \in Cl(U_{n_{i+1}})$. Let $F^i: Cl(U_{n_i}) \times I \to V_i$ be a homotopy such that $F_0^i = f_{n_i}|Cl(U_{n_i})$ and $F_1^i = f_{n_{i+1}}|Cl(U_{n_i})$. Using tricks similar to those already employed we can additionally require that $F^i(Cl(U_{n_i}) \times I) \cap Y = \emptyset$, for all $i > 0$. Then define $f: I^\infty \backslash X \to I^\infty \backslash Y$ by $f(x) = f_{n_1}(x)$, for $x \in I^\infty \backslash U_{n_1}$, and $f(x) = F^i_{\phi_i(x)}$, for $x \in Cl(U_{n_i}) \backslash U_{n_{i+1}}$. It then follows that f is a proper map. It remains to be shown that $T(\{f\}) = [\underline{f}]$.

To see this choose $F \in F(X)$ and note that $T(\{f\}) = [\{f \circ F_{1/k}, X,Y\}]$. Thus we must show that $\underline{f} \simeq \{f \circ F_{1/k}, X, Y\}$. If V is an open neighborhood of Y, then we can choose $i > 0$ such that $k, l \geq n_i$ implies that $f_k|U_{n_i} \simeq f_l|U_{n_i}$ (in V) and such that $0 \leq t \leq 1/n_i$ implies that $F_t|I^\infty \backslash U_{n_i} = id$. If we can show that $k \geq n_i$ implies that $f_k|U_{n_i} \simeq f \circ F_{1/k}|U_{n_i}$ (in V), then we will be done. For such a fixed $k \geq n_i$ we have $F_{1/k}(U_{n_i}) \subset I^\infty \backslash U_{n_j}$, for some $j > i$. We can then use a finite induction to conclude that $f|F_{1/k}(U_{n_i}) \simeq f_{n_i}|F_{1/k}(U_{n_i})$ (in V). Hence $f \circ F_{1/k}|U_{n_i} \simeq f_k \circ F_{1/k}|U_{n_i}$ (in V) $\simeq f_k|U_{n_i}$ (in V), and we are done.

5. <u>Relative fundamental sequences</u>. We will need to define a relative notion of a fundamental sequence. Let A and B be subsets of a space X. Then a <u>relative fundamental sequence</u> $\underline{f}$ from A to B in X consists of an open set G containing A and a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions, $f_k\colon G \to X$, such that the following properties are satisfied.

(1) $f_k \simeq id_G$, for all $k \geq 1$,

(2) for each open neighborhood V of B there exists an open neighborhood $U \subset G$ of A and an integer $n_1 > 0$ such that if $k, l \geq n_1$ are integers, then $f_k|U \simeq f_l|U$ (in V).

If $X = I^{\infty}$ and $\underline{f} = \{f_k, A, B\}$ is a fundamental sequence, then it is clear that $\{f_k, A, B, G\}$ is a relative fundamental sequence, for each open neighborhood G of A. If A, B, C are subsets of X and $\{f_k, A, B, G\}$, $\{g_k, B, C, H\}$ are relative fundamental sequences, then there exists an integer $n_1 > 0$ and an open set G' satisfying $A \subset G' \subset G$ such that $\{g_k \circ f_k | G', A, C, G'\}_{k=n_1}^{\infty}$ is a relative fundamental sequence. We will agree to identify relative fundamental sequences $\{f_k, A, B, G\}$ and $\{g_k, A, B, H\}$ provided that there exists an open neighborhood $G' \subset G \cap H$ of A such that $f_k|G' = g_k|G'$, for all but finitely many values of k. Thus composition is well defined.

If $\underline{f} = \{f_k, A, B, G\}$ and $\underline{g} = \{g_k, A, B, H\}$ are relative fundamental sequences then we write $\underline{f} \simeq \underline{g}$ iff for each open neighborhood V of B there exists an open neighborhood $U \subset G \cap H$ of A and an integer $n_1 > 0$ such that $f_k|U \simeq g_k|U$ (in V), for all integers $k \geq n_1$. In analogy with [5] we say that A <u>relatively fundamentally dominates</u> B (in X) iff there exist relative fundamental sequences $\underline{f} = \{f_k, A, B, G\}$ and $\underline{g} = \{g_k, B, A, H\}$ such that $\underline{f} \circ \underline{g} \simeq \underline{i}_B$, i.e. for each open neighborhood V of B there exists an open neighborhood $U \subset V \cap H$ of B and an integer $n_1 > 0$ such that $k \geq n_1$ implies that U is in the domain of $f_k \circ g_k$ and $f_k \circ g_k|U \simeq id_U$ (in V). In like manner we can also define what is meant by <u>relative fundamental equivalence</u>.

We now establish a result which plays a key role in the inductive step in the proof of Theorem 2. We do it in two steps.

Lemma 5.1. *Let X be a Q-manifold and let A, B be compact Z-sets in X such that A relatively fundamentally dominates B in X. If W is an open subset of X containing B, then there exists an embedding $\emptyset$: $A \to W$ such that $\emptyset(A)$ is a Z-set, $\emptyset \simeq id_A$, and $\emptyset(A)$ relatively fundamentally dominates B in W.*

Proof. Choose relative fundamental sequences $\underline{f} = \{f_k, A, B, G\}$ and $\underline{g} = \{g_k, B, A, H\}$ such that $\underline{f} \circ \underline{g} \simeq \underline{i}_B$. Choose an integer $n_1 > 0$ and an open set U such that $A \subset U \subset G$, $f_k(U) \subset H \cap W$, and $f_k|U \simeq f_l|U$ (in $H \cap W$), for all k, $l \geq n_1$. Using Lemma 3.7 we may assume that $U \cong U \times [0,1)$. Now apply Lemma 3.6 to get an open embedding Φ: $U \to W$ such that $\Phi \simeq f_{n_1}|U$ (in W). We can find an open neighborhood $V \subset H \cap W$ of B and an integer $n_2 \geq n_1$ such that $g_k(V) \subset U$, for all $k \geq n_2$, $g_k|V \simeq g_l|V$ (in U), for all k, $l \geq n_2$, and $f_k \circ g_k|V \simeq id_V$ (in $H \cap W$), for all $k \geq n_2$. Now let $\phi = \Phi|A$, $G' = \Phi(U)$, $H' = V$, $f'_k = f_k \circ \Phi^{-1}$, and $g'_k = \Phi \circ g_k|V$, for all $k \geq n_2$.

To see that $\underline{f}' = \{f'_k, \phi(A), B, G'\}$ is a relative fundamental sequence in W first note that for each $k \geq n_2$ we have $f'_k = f_k \circ \Phi^{-1} \simeq f_{n_1} \circ \Phi^{-1}$ (in W) $\simeq \Phi \circ \Phi^{-1}$ (in W) $= id_{G'}$. Now let $V' \subset W$ be an open neighborhood of B and choose an open neighborhood $U' \subset U$ of A and an integer $n_3 \geq n_2$ such that $f_k|U' \simeq f_l|U'$ (in V'), for all k, $l \geq n_3$. Then $\Phi(U')$ is an open set in W containing $\phi(A)$ such that $f'_k|\Phi(U') \simeq f'_l|\Phi(U')$ (in V'), for all k, $l \geq n_3$.

To see that $\underline{g}' = \{g'_k, B, \phi(A), H'\}$ is a relative fundamental sequence in W we have $g'_k = \Phi \circ (g_k|V) \simeq f_k \circ (g_k|V)$ (in W) $\simeq id_V$ (in W), for all $k \geq n_2$. Now let U' be an open set in W containing $\phi(A)$ and choose an integer $n_3 \geq n_2$ and an open set $V' \subset V$ containing B such that $g_k(V') \subset \Phi^{-1}(U' \cap \Phi(U))$, for all $k \geq n_3$, and $g_k|V' \simeq g_l|V'$ (in $\Phi^{-1}(U' \cap \Phi(U))$), for all k, $l \geq n_3$. Then it follows that $g'_k|V' \simeq g'_l|V'$ (in U'), for all k, $l \geq n_3$.

To see that $\underline{f}' \circ \underline{g}' \simeq \underline{i}_B$ choose an open neighborhood $V' \subset W$ of B. Now choose an open neighborhood $V'' \subset V' \cap V$ of B and an integer $n_3 \geq n_2$ such that $f_k \circ g_k|V'' \simeq id_{V''}$ (in V'), for all $k \geq n_3$. Then it easily follows that $f'_k \circ g'_k|V'' \simeq id_{V''}$ (in V'), for all $k \geq n_3$.

Thus $\phi(A)$ relatively fundamentally dominates B in W. Finally we note that $\phi = \Phi|A \simeq f_{n_1}|A \simeq id_A$.

Using a similar argument we can establish the following result.

Lemma 5.2. _Let_ X _be a Q-manifold and let_ A, B _be compact Z-sets in_ X _such that_ A _and_ B _are relatively fundamentally equivalent in_ X. _If_ W _is an open subset of_ X _containing_ B, _then there exists an embedding_ $\phi\colon A \to W$ _such that_ $\phi(A)$ _is a Z-set_, $\phi \simeq id_A$, _and_ $\phi(A)$ _is relatively fundamentally equivalent to_ B (_in_ W).

6. Proof of Theorem 2. We note that if $I^\infty \backslash X \cong I^\infty \backslash Y$, then $I^\infty \backslash X$ has the same weak proper homotopy type as $I^\infty \backslash Y$, and we can thus use Theorem 1 to conclude that $Sh(X) = Sh(Y)$.

On the other hand assume that $Sh(X) = Sh(Y)$, where X and Y are compacta in s. We will inductively construct sequences $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ of open subsets of I^∞ and a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms of I^∞ onto itself such that the following properties are satisfied.

(1) $X = \bigcap_{i=1}^\infty U_i$ and $U_{i+1} \subset U_i$, for all $i > 0$,

(2) $Y = \bigcap_{i=1}^\infty V_i$ and $V_{i+1} \subset V_i$, for all $i > 0$,

(3) $h_{2i-1} \circ \ldots \circ h_1(X) \subset V_i$, for all $i > 0$,

(4) $h_j | I^\infty \backslash V_i = id$, for all $j > 2i-1$,

(5) $h_{2i} \circ \ldots \circ h_1(U_i) \supset Y$, for all $i > 0$,

(6) $h_j | I^\infty \backslash h_{2i} \circ \ldots \circ h_1(U_i) = id$, for all $j > 2i$.

Before proceeding with the construction of these sequences we will show how to use them to construct our desired homeomorphism of $I^\infty \backslash X$ onto $I^\infty \backslash Y$.

For each $x \in I^\infty \backslash X$ we have $x \notin U_i$, for some $i > 0$. Thus $h_{2i} \circ \ldots \circ h_1(x) \notin h_{2i} \circ \ldots \circ h_1(U_i)$ and we therefore have $h_j \circ \ldots \circ h_1(x) = h_{2i} \circ \ldots \circ h_1(x)$, for all $j > 2i$. This means that $h(x) = \lim_{j \to \infty} h_j \circ \ldots \circ h_1(x)$ is defined, for all $x \in I^\infty \backslash X$. It follows from (5) above that $h(x) \in I^\infty \backslash Y$. Thus we have defined a function from $I^\infty \backslash X$ into $I^\infty \backslash Y$, and the verification that it is indeed an onto homeomorphism is routine.

We now turn to the construction of the necessary sequences. We start by choosing $\{U_i'\}_{i=1}^\infty$ and $\{V_i'\}_{i=1}^\infty$ to be decreasing sequences of open subsets of I^∞ such that $X = \bigcap_{i=1}^\infty U_i'$ and $Y = \bigcap_{i=1}^\infty V_i'$. We will construct $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ as subsequences of $\{U_i'\}_{i=1}^\infty$ and $\{V_i'\}_{i=1}^\infty$, respectively.

For the first step choose $V_1 = V_1'$ and use Lemma 5.2 to get an embedding $\phi_1 \colon X \to V_1$ such that $\phi_1(X)$ is a Z-set, $\phi_1 \simeq id_X$, and $\phi_1(X)$ is relatively fundamentally equivalent to Y (in V_1). Then extend ϕ_1 to a homeomorphism $h_1 \colon I^\infty \to I^\infty$.

For the second step choose an integer $i_1 > 0$ large enough so that $U'_{i_1} \subset h_1^{-1}(V_1)$ and put $U_1 = U'_{i_1}$. Once more using Lemma 5.2 let $\phi_2\colon Y \to h_1(U_1)$ be an embedding so that $\phi_2 \simeq id_Y$ (in V_1), $\phi_2(Y)$ is a Z-set, and $\phi_2(Y)$ is relatively fundamentally equivalent to $h_1(X)$ in $h_1(U_1)$. Since $\phi_2 \simeq id_Y$ in V_1 we can extend ϕ_2 to a homeomorphism $\tilde{\phi}_2\colon V_1 \to V_1$ which in turn can be extended to a homeomorphism $\tilde{\phi}'_2\colon I^\infty \to I^\infty$ which satisfies $\tilde{\phi}'_2 | I^\infty \setminus V_1 = id$. The construction of $\tilde{\phi}_2$ requires an application of Lemma 3.5., where $\tilde{\phi}_2$ is limited by an open cover of V_1 which is normal with respect to $I^\infty \setminus V_1$. Then we put $h_2 = (\tilde{\phi}'_2)^{-1}$ for the second step of our construction. As this is essentially the inductive step we are done.

7. Proof of Theorem 3. Recall that an object in S is a FAR provided that it is the intersection of a decreasing sequence of AR's [6]. We use this to show that if X is a compactum in s satisfying $Sh(X) = Sh(\{point\})$, then X is a FAR. Using Theorem 2 there is a homeomorphism $h\colon I^\infty\backslash W^+ \to I^\infty\backslash X$. Then $I^\infty\backslash X = h[\bigcup_{i=1}^\infty([-1, 1-\frac{1}{i}] \times \Pi_{i=2}^\infty I_i)]$. We note that each $h(\{1-\frac{1}{i}\} \times \Pi_{i=2}^\infty I_i)$ is a bicollared copy of I^∞ in $I^\infty\backslash X$. Thus $I^\infty\backslash h(\{1-\frac{1}{i}\} \times \Pi_{i=2}^\infty I_i) = A_i \cup B_i$, where A_i and B_i are disjoint sets such that $Cl(A_i) \cap Cl(B_i) = h(\{1-\frac{1}{i}\} \times \Pi_{i=2}^\infty I_i)$ and $Cl(A_i) \cong Cl(B_i) \cong I^\infty$. Choose notation so that $Cl(A_i) = h([-1, 1-\frac{1}{i}] \times \Pi_{i=2}^\infty I_i)$ and thus we have $X = \bigcap_{i=1}^\infty Cl(B_i)$, a decreasing sequence of HIlbert cubes. Thus X is a FAR.

For the other implication assume that X is a FAR in S. Since we are interested only in $I^\infty\backslash X$ we can assume, by use of Lemma 3.5, that $X \subset W^+$. We will construct a homeomorphism of $I^\infty\backslash W^+$ onto $I^\infty\backslash X$. Choose a decreasing sequence $\{V_i\}_{i=1}^\infty$ of open subsets of I^∞ such that $X = \bigcap_{i=1}^\infty V_i$ and let $\underline{f} = \{f_k, W^+, X\}$ be a fundamental retraction of W^+ onto X. Then there exists an integer $n_1 > 0$ such that $f_{n_1}(W^+) \subset V_1$. Using Lemma 3.3 there exists an embedding $g_{n_1}\colon W^+ \to V_1$ such that $g_{n_1}|X = id$ and $g_{n_1}(W^+)$ is a Z-set. Then let $h_{n_1}\colon I^\infty \to I^\infty$ be an extension of g_{n_1} to a homeomorphism such that $h_{n_1}|W^- = id$. Since $h_{n_1}^{-1}(I^\infty\backslash V_1)$ is a compact set missing W^+ there exists ϵ_1, $0 < \epsilon_1 < 1$, such that $h_{n_1}^{-1}(I^\infty\backslash V_1) \subset [-1,\epsilon_1] \times \Pi_{i=2}^\infty I_i$, hence $I^\infty\backslash V_1 \subset h_{n_1}([-1,\epsilon_1] \times \Pi_{i=2}^\infty I_i^1)$. Now $V_2 \cap h_{n_1}([\epsilon_1,1] \times \Pi_{i=2}^\infty I_i)$ is an open subset of $h_{n_1}([\epsilon_1,1] \times \Pi_{i=2}^\infty I_i)$ containing X and we can use an argument similar to that above to produce an ϵ_2, $(1+\epsilon_1)/2 < \epsilon_2 < 1$, and a homeomorphism $\tilde{h}_{n_2}\colon h_{n_1}([\epsilon_1,1] \times \Pi_{i=2}^\infty I_i) \to h_{n_1}([\epsilon_1,1] \times \Pi_{i=2}^\infty I_i)$ which satisfies $\tilde{h}_{n_2}|(h_{n_1}(\{\epsilon_1\} \times \Pi_{i=2}^\infty I_i) \cup X) = id$ and $h_{n_1}([\epsilon_1,1] \times \Pi_{i=2}^\infty I_i) \setminus V_2 \subset \tilde{h}_{n_2} \circ h_{n_1}([\epsilon_1,\epsilon_2] \times \Pi_{i=2}^\infty I_i)$. Then $\tilde{h}_{n_2}$ extends to $h_{n_2}\colon h_{n_1}(I^\infty) \to I^\infty$ so that $h_{n_2}|h_{n_1}([-1,\epsilon_1] \times \Pi_{i=2}^\infty I_i) = id$. As this essentially the inductive step we can define a homeomorphism $h\colon I^\infty\backslash W^+ \to I^\infty\backslash X$ by putting $h(x) = \lim_{i\to\infty} h_{n_i} \circ \dots \circ h_{n_1}(x)$ for all $x \in I^\infty\backslash W^+$. The details are routine.

Corollary ([10]). *If* X *is a* FAR, *then* X *is the intersection of a decreasing sequence of Hilbert cubes*.

Proof. Assume X ⊂ s and then note that the assertion follows the first half of the proof of Theorem 3.

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