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SHAPES OF FINITE-DIMENSIONAL COMPACTA

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Shapes Of Finite-Dimensional Compacta

by

T.A. Chapman [1]

1. Introduction

In [4] Borsuk defined the notion of shape for compact metric spaces (compacta), a concept which enables one to compare global properties of compacta, while virtually ignoring their local properties. Roughly speaking shape is an equivalence relation on compacta which generalizes the notion of homotopy type, and we write Sh(X) = Sh(Y) to indicate that X and Y have the same shape. As an example the arc and the pseudo-arc are compacta having radically different local properties, but it follows from [4] that they have the same shape. In the same vein it also follows from [4] that the circle and the so-called "Warsaw circle" have the same shape, where by the "Warsaw circle" we mean the compactum in E^2 obtained as the union of the compactum

$$A = (\{0\} \times [-1,1]) \cup \{(x, \sin \frac{1}{x}) \mid 0 < x \leq \frac{2}{\pi}\}$$

and an arc which has endpoints $(0,0)$, $(\frac{2}{\pi},1)$ and which intersects A only in these points. In Section 5 we will give a precise definition of shape for compacta, but for the time being this intuitive description given above will suffice. However it will be helpful to cite some results which compare this concept with homotopy type.

It is shown in [4] that if X and Y are compacta which have the same homotopy type, then Sh(X) = Sh(Y). Above we have given examples of compacta which have the same shape, but obviously different homotopy type. These examples clearly emphasize the fact that if two compacta have the same shape, but "insufficient local structure", then they do not necessarily have the same homotopy type. However in [4] it is shown that if X and Y are compacta which are ANR's (metric), then X and Y have the

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same homotopy type iff Sh(X) = Sh(Y). Thus with "sufficient local structure", the two concepts coincide.

The results of this paper deal only with shapes of finite-dimensional compacta. Indeed in Theorem 1 below we give a characterization of shapes of finite-dimensional compacta in terms of embeddings in "large dimensional" Euclidean spaces. In an earlier paper [8] the author has obtained a characterization of shapes of compacta (with no dimensional restriction) in terms of embeddings in the Hilbert cube. In a sense the results obtained here are motivated by [8], and to some extent the general structure of the proof of Theorem 1 is a modification of argument used in [8], but the present paper does not involve any infinite-dimensional topology. For the sake of completeness we give a short summary of the infinite-dimensional characterization of [8] at the end of the Introduction.

In order to state our characterization we will first need a definition. Let X be a compact subset of some n-dimensional Euclidean space E^n. Then we say that X is stably embedded in E^n provided that X lies in some Euclidean subspace of E^n having codimension at least $2 \cdot \dim X + 1$. It is easy to see that if $X \subset E^n$ is a stably embedded compactum and E^n is identified with an Euclidean subspace of E^m, for any $m \geq n$, then X is also stably embedded in E^m. We also remark that if X is a compactum and $X', X'' \subset E^n$ are stably embedded copies of X, then there exists a homeomorphism of E^n onto itself which takes X' onto X" (see Lemma 3.1). Thus all stably embedded copies of a given compactum in E^n are equivalent (via a space homeomorphism).

Theorem 1. For any integer $m > 0$ there exists an integer $n_1 > 0$ such that the following property is satisfied: If $X, Y \subset E^n$ are stably embedded compacta such that $\dim X, \dim Y \leq m$ and $n \geq n_1$, then

$$\mathrm{Sh}(X) = \mathrm{Sh}(Y) \text{ iff } E^n \setminus X \text{ and } E^n \setminus Y \text{ are homeomorphic.}$$

We remark that in the "only if" part of Theorem 1, the homeomorphism can be constructed to be the identity off of any topological n-cell in E^n containing $X \cup Y$ in its interior. Also the integer n_1 can be chosen to be any integer greater than or equal to $10m + 17$, but we remark that this number is the result of several calculations and in all probability is not the lowest possible one.

In an analogous fashion we can give an appropriate definition of stably embedded for the n-sphere S^n and we can easily obtain the following result directly from Theorem 1.

Corollary. *For any integer* $m > 0$ *there exists an integer* $n_1 > 0$ *such that the following property is satisfied: If* $X, Y \subset S^n$ *are stably embedded compacta such that* $\dim X, \dim Y \leq m$ *and* $n \leq n_1$, *then*

$$Sh(X) = Sh(Y) \text{ iff } S^n \backslash X \text{ and } S^n \backslash Y \text{ are homeomorphic.}$$

We remark that no prior knowledge of shape theory is required for reading this paper; in fact we require nothing other than the definition of "$Sh(X) = Sh(Y)$", which we give in Section 5. For other prerequisites we will need some elementary facts concerning the piecewise-linear topology of E^n plus an isotopy extension theorem from [11]. We also use a characterization of dimension in terms of mappings onto polyhedra in E^n (see [13], page 111) and a result of Klee [12] concerning extensions of homeomorphisms between certain compacta in E^n to homeomorphisms of E^n onto itself.

As for techniques we remark that the "only if" part of Theorem 1 is the most difficult part of the proof. Roughly the idea is to construct sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ of homeomorphisms of E^n onto itself such that the sequences $\{f_i \circ f_{i-1} \circ \ldots \circ f_1\}_{i=1}^{\infty}$ and $\{g_i \circ g_{i-1} \circ \ldots \circ g_1\}_{i=1}^{\infty}$ of left products converge pointwise (on $E^n \backslash X$ and $E^n \backslash Y$, respectively) to define embeddings $f: E^n \backslash X \to E^n$ and $g: E^n \backslash Y \to E^n$ such that $f(E^n \backslash X) = g(E^n \backslash Y)$. Then the desired homeomorphism of $E^n \backslash X$ onto $E^n \backslash Y$ is just $g^{-1} \circ f$.

We now make some comments concerning the infinite-dimensional characterization obtained in [8]. We represent the Hilbert cube Q by $\Pi_{i=1}^{\infty} I_i$, where each I_i is the closed interval $[0,1]$, and the pseudo-interior of Q is $s = \Pi_{i=1}^{\infty} I_i^{o}$, where each I_i^{o} is the open interval $(0,1)$. The characterization obtained in [8] is as follows:

Theorem 2. *Let* X, Y ⊂ s *be compacta. Then* Sh(X) = Sh(Y) *iff* Q\X *and* Q\Y *are homeomorphic.*

The condition "X, Y ⊂ s" in Theorem 2 is crucial and in general cannot be replaced by the weaker condition "X, Y ⊂ Q". Also it follows from [2] that if X, Y ⊂ s are any two compacta, then s\X and s\Y are homeomorphic to s. However the characterization is generally applicable to compacta, since any compactum can be embedded in s. We remark that the proof of Theorem 2 given in [8] is non-elementary and uses some recent developments in the theory of infinite-dimensional manifolds modeled on Q (see [7] for a summary). The proof we give here of Theorem 1 is a bit more complicated since there are some infinite-dimensional techniques used in the proof of Theorem 2 which have no finite-dimensional analogues.

The author is grateful to Morton Brown for suggesting something on the order of Theorem 1, in the sense that he felt "shape" for finite-dimensional compacta should mean "homeomorphic complements" (in an appropriate setting in Euclidean space). The author also wishes to thank R.D. Anderson for making some valuable comments on the manuscript.

2. Definitions and notation

For any topological space X and any set $A \subset X$ we let $Bd_X(A)$, $Int_X(A)$, and $Cl_X(A)$ denote, respectively, the topological boundary, interior, and closure of A in X. When no ambiguity results we will suppress the subscipt X. If Y is another space and $f: X \to Y$ is a function, then $f|A$ will denote the restriction of f to A.

All homeomorphisms will be onto and we use the notation $X \cong Y$ to indicate that spaces X and Y are homeomorphic. The identity homeomorphism of X onto itself will be denoted by id_X and by a map we will mean a continuous function. If (Y,d) is a metric space and $f, g: X \to Y$ are maps, then we use

$$d(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\} \quad \text{(if it exists)}$$

for the distance between f and g. In the sequel we will indiscriminately use d to denote the metric of any space under consideration.

For products $X \times Y$ we use $p_X: X \times Y \to X$ to denote projection. In Euclidean space E^n and any integer $m \leq n$ we use $p_m: E^n \to E^m$ to denote projection onto the first m coordinates, i.e.

$$p_m(x_1,x_2,\dots,x_n) = (x_1,x_2,\dots,x_m),$$

for all $(x_1,x_2,\dots,x_n) \in E^n$.

We use I to denote the unit interval $[0,1]$ and by a homotopy we mean a continuous function $F: X \times I \to Y$. The levels of F are the maps $F_t: X \to Y$, defined by $F_t(x) = F(x,t)$, for all $(x,t) \in X \times I$. For a map $G: X \times I \to Y \times I$ we will also use the notation $G_t: X \to Y$ for the map defined by $G_t(x) = p_Y \circ G(x,t)$, for all $(x,t) \in X \times I$. If $B \subset Y$ and $f, g: X \to Y$ are maps satisfying $f(X), g(X) \subset B$, then we use the notation $f \simeq g$ (in B) to mean that there exists a homotopy $F: X \times I \to Y$ such that $F_0 = f$, $F_1 = g$, and $F(X\times I) \subset B$.

In Euclidean space E^n and any $\epsilon > 0$ we let

$$B^n_\epsilon = \{x \in E^n \mid ||x|| \leq \epsilon\},$$

$$\partial B^n_\epsilon = \{x \in E^n \mid ||x|| = \epsilon\}.$$

For any integer $m \leq n$ we will use $E^m \times 0 \subset E^n$ to indicate the Euclidean subspace of E^n defined by

$$E^m \times 0 = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_{m+1} = x_{m+2} = \ldots = x_n = 0\}.$$

By a <u>polyhedron</u> we will mean a (locally-finite) union of linear cells contained in some Euclidean space E^n and by a <u>topological polyhedron</u> we will mean any space homeomorphic to a polyhedron. Generally we will use notation and results from [10] concerning elementary piecewise linear (PL) topology, including such concepts as PL maps, derived and regular neighborhoods, etc.

3. Extending homeomorphisms

The main results of this section are Propositions 3.3 and 3.5, where we establish two different types of homeomorphism extension theorems. In Proposition 3.3 we will be concerned with extending homeomorphisms between certain compacta in E^n and in Proposition 3.5 we will be concerned with extending PL homeomorphisms between compact polyhedra in E^n. The following lemma, which is essentially due to Klee [12], is needed in the proof of Proposition 3.3.

Lemma 3.1. (Klee [12]) *Let $X, Y \subset E^n$ be stably embedded compacta and let $h: X \to Y$ be a homeomorphism. Then h can be extended to a homeomorphism of E^n onto itself.*

We will also need the following result concerning the replacement of mappings by embeddings.

Lemma 3.2. *Let X be a compactum satisfying $\dim X \leq n$, $A \subset X$ be closed, and for any integer $m > 0$ let $f: X \to E^m \times 0 \subset E^{m+2n+1}$ be a map such that $f|A$ is an embedding. Then for any $\epsilon > 0$ there exists an embedding $g: X \to E^{m+2n+1}$ such that $g|A = f|A$ and $d(f,g) < \epsilon$.*

Proof. If $X|A$ is the quotient space in which A has been identified with a point, then we have $\dim(X|A) \leq n$ (since $\dim X = \operatorname{Ind} X$). Let $\alpha: X|A \to \operatorname{Int}(B_\epsilon^{2n+1})$ be an embedding such that $\alpha(a) = 0$, where $a \in X|A$ is the point to which A has been identified. Let $p: X \to X|A$ be the canonical projection and define $g: X \to E^{m+2n+1}$ by

$$g(x) = (p_m \circ f(x), \alpha \circ p(x)),$$

for all $x \in X$. It is clear that g fulfills our requirements.

Proposition 3.3. *For any integer $n > 0$ let $U \subset E^n \times 0 \subset E^{5n+7}$ be an open subset of $E^n \times 0$, $X \subset U$ be compact, and let $F: X \times I \to U$ be a homotopy such that $F_0 = id_X$ and $F_1: X \to U$ is an embedding. Then F_1 can be extended to a homeomorphism*

$$h\colon Cl_{E^n}(p_n(U)) \times B_1^{4n+7} \to Cl_{E^n}(p_n(U)) \times B_1^{4n+7}$$

which satisfies

$$h|Bd(Cl_{E^n}(p_n(U)) \times B_1^{4n+7}) = id.$$

Proof. Let $G\colon X \times I \to p_n(U) \times E^{4n+7}$ be the function defined by

$$G_t(x) = \begin{cases} F_{2t-1}(x), & \text{for } 0 \leq t \leq \frac{1}{2} \\ (p_n \circ F_1(x), t-\frac{1}{2}, 0, 0, \ldots, 0), & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then G is a map and $G|(X\times 0) \cup (X\times 1)$ is an embedding. Since $\dim(X\times I) \leq n+1$ we can use Lemma 3.2 to choose an embedding

$$H\colon X \times I \to p_n(U) \times B_1^{2n+4} \times 0 \subset p_n(U) \times B_1^{4n+7}$$

such that $H|(X\times 0) \cup (X\times 1) = G|(X\times 0) \cup (X\times 1)$.

Now let $\alpha\colon H(X\times I) \to E^{5n+7}$ be defined by

$$\alpha \circ H(x,t) = (p_n(x),t,0,0,\ldots,0),$$

for all $(x,t) \in X \times I$. Since $H(X\times I) \subset E^{3n+4} \times 0 \subset E^{5n+7}$ and $\alpha \circ H(X\times I) \subset E^{3n+4} \times 0$ we can use Lemma 3.1 to extend α to a homeomorphism $f_1\colon E^{5n+7} \to E^{5n+7}$. Note that $W = f_1(Cl_{E^n}(p_n(U)) \times B_1^{4n+7})$ is a neighborhood of $p_n(X) \times [0,1] \times 0$ in E^{5n+7}. Using motions only in the (n+1)-st coordinate direction we can clearly obtain a homeomorphism $f_2\colon E^{5n+7} \to E^{5n+7}$ such that

$$f_2(p_n(x),0,0,\ldots,0) = (p_n(x),1,0,0,\ldots,0)$$

and

$$f_2|E^{5n+7}\backslash W = id,$$

for all $x \in X$.

Let $f_3: E^{5n+7} \to E^{5n+7}$ be a homeomorphism such that

$$f_3(F_1(x)) = (p_n \circ F_1(x), \tfrac{1}{2}, 0, 0, \ldots, 0)$$

and

$$f_3 | E^{5n+7} \setminus (Cl_{E^n}(p_n(U)) \times B_1^{4n+7}) = id,$$

for all $x \in X$.
Then we can define $h: E^{5n+7} \to E^{5n+7}$ by

$$h = f_3^{-1} \circ f_1^{-1} \circ f_2 \circ f_1$$

to fulfill our requirements.

We now establish Proposition 3.5, which is a PL version of Proposition 3.3. Modulo some simple observation on dimension we shall see that Proposition 3.5 is just a spacial case of Corollary 1.4 of [11]. In Lemma 3.4 below we state a version of Corollary 1.4 of [11] which we will use.

Lemma 3.4. (Hudson [11]). _Let_ X _be a compact topological polyhedron,_ M _be an open manifold such that_ $\dim M - \dim X \geq 3$, _and let_ $f, g: X \to M$ _be_ PL _embeddings for which there exists a_ PL _embedding_ $F: X \times I \to M \times I$ _satisfying_ $F_0 = f$ _and_ $F_1 = g$. (_Under these circumstances we say that_ f _and_ g _are concordant and_ F _is a concordance._) _Then there exists a_ PL _homeomorphism_ $h: M \to M$ _such that_ $h|f(X) = g \circ f^{-1}$.

Proposition 3.5. _Let_ X _be a compact topological polyhedron such that_ $\dim X \leq n$ _and let_ $f, g: X \to E^{2n+2}$ _be_ PL _embeddings. Then there exists a_ PL _homeomorphism_ $h: E^{2n+2} \to E^{2n+2}$ _such that_ $h|f(X) = g \circ f^{-1}$.

Proof. For $n = 0$ the result is trivial and we therefore assume $n \geq 1$, in which case $\dim E^{2n+2} - \dim X \geq 3$. In order to apply Lemma 3.4 all we need do is establish the existence of an appropriate concordance. This can easily be accomplished in the following three steps.

1. Using the contractibility of E^{2n+2} we can construct a map $F: X \times I \to E^{2n+2} \times I$ such that $F_0 = f$ and $F_1 = g$.

2. Using Lemma 4.2 of [10] (which is concerned with approximating maps by PL maps) we can replace F by a PL map $G: X \times I \to E^{2n+2} \times I$ such that $G_0 = f$ and $G_1 = g$.
3. If we note that $2.\dim(X \times I) + 1 \leq \dim(E^{2n+2} \times I)$, then we can use standard procedures on general positioning to modify G to obtain a PL embedding $H: X \times I \to E^{2n+2} \times I$ which satisfies $H_0 = f$ and $H_1 = g$.

Then H is our desired concordance.

4. Embedding compacta in E^n

If X is any compactum satisfying dim X $\leq$ n, then it is well-known that X can be embedded in E^{2n+1}. In Proposition 4.4 below we prove that X can be embedded into E^{2n+1} in a "nice" way which will be useful in the sequel. This "niceness" condition is described in the following definition.

Definition 4.1. *Let $X \subset E^n$ be a compactum which satisfies* dim X $\leq$ m, *for some* m > 0. *Then we say that* X *is in standard position iff there exist sequences* $\{P_i\}_{i=1}^{\infty}$ *and* $\{N_i\}_{i=1}^{\infty}$ *such that the following properties are satisfied.*

(1) *Each* P_i *is a compact polyhedron in* E^n *satisfying* dim $P_i \leq m$,

(2) *Each* N_i *is a regular neighborhood of* P_i *in* E^n,

(3) *Each* $N_{i+1} \subset \text{Int}(N_i)$, *and*

(4) $X = \cap_{i=1}^{\infty} N_i$.

We remark that this condition does not necessarily imply tameness. For example if $X \subset E^3$ is the wild arc of Artin-Fox (as described on page 177 of [9]), then it is easily verified that X is in standard position. On the other hand if $X \subset E^n$ ($n \geq 4$) is any arc such that $E^n \backslash X$ is not simply connected (such arcs exist from [3]), then it can be verified that X is not in standard position. We omit the details because these observations are not needed in the sequel. One obvious fact which will be needed in the sequel is the following: If $X \subset E^n$ is a compactum in standard position, then $X \times 0 \subset E^{n+m}$ is in standard position, for all $m \geq 0$.

In proposition 4.4 below we show that every compactum of dimension less than or equal to n can be embedded into E^{2n+1} in standard position. The following characterization of dimension will be needed in the proof of Proposition 4.4.

Lemma 4.2. ([13], page 111). *A compactum* $X \subset E^n$ *satisfies* dim X $\leq$ m *iff for each* $\epsilon > 0$ *there exists a polyhedron* $P \subset E^n$ *satisfying* dim P $\leq$ m *and a map* $f: X \to P$ *such that* $f(X) = P$ *and* $d(f, id) < \epsilon$.

We will also need a convergence procedure for sequences of embeddings of compacta into complete metric spaces. Various forms of this type of convergence procedure are known and have been used occasionally (for example see Lemma 2.1 of [1]). It is for this reason that we state the result with no proof. For notation let (Y,d) be a metric space and let $X \subset Y$ be a compactum. Then for any embedding $f\colon X \to Y$ and any $\delta > 0$ let

$$\epsilon(f,\delta) = \mathrm{glb}\{d(f(x),f(y)) \mid x,\ y \in X \text{ and } d(x,y) \geq \delta\},$$

which is clearly a positive number.

Lemma 4.3. *Let (Y,d) be a complete metric space and let $X \subset Y$ be a compactum. Moreover let*

$$X \xrightarrow{f_1} Y,\ f_1(X) \xrightarrow{f_2} Y,\ f_2 \circ f_1(X) \xrightarrow{f_3} Y,\ \ldots$$

be a sequence of embeddings such that

$$d(f_i,\mathrm{id}) < \min(3^{-i},(3^{-i})\cdot \epsilon(f_{i-1}\circ\ldots\circ f_1,2^{-i})),$$

for all $i > 1$. Then the sequence $\{f_i \circ f_{i-1} \circ \ldots \circ f_1\}_{i=1}^{\infty}$ converges pointwise to an embedding of X into Y.

Proposition 4.4. *Let $X \subset E^{2n+1}$ be a compactum such that $\dim X \leq n$. Then there exists an embedding $f\colon X \to E^{2n+1}$ such that $f(X)$ is in standard position.*

Proof. Let $Y = E^{2n+1}$ in Lemma 4.3 and let ϵ be the function defined in the paragraph preceeding Lemma 4.3. Using Lemma 4.2 there exists a polyhedron $P_1 \subset E^{2n+1}$ satisfying $\dim P_1 \leq n$ and a map $g_1\colon X \to P_1$ such that $g_1(X) = P_1$ and $d(g_1,\mathrm{id}) < \frac{1}{2}$. Choose a regular neighborhood N_1 of P_1 in E^{2n+1} such that there exists a retraction $r_1\colon N_1 \to P_1$ satisfying $d(r_1,\mathrm{id}) < \frac{1}{2}$. It is well-known that any continuous function of X into E^{2n+1} can be approximated by an embedding. Thus there exists an embedding $f_1\colon X \to \mathrm{Int}(N_1)$ such that $d(f_1,g_1) < \frac{1}{2}$. This implies that $d(f_1,\mathrm{id}) < 1$. Then put $\delta_1 = d(f_1(X), E^{2n+1} \setminus \mathrm{Int}(N_1))$ to complete the first step of the construction.

Now use Lemma 4.2 to obtain a polyhedron $P_2 \subset E^{2n+1}$ satisfying $\dim P_2 \leq n$ and a map $g_2\colon f_1(X) \to P_2$ such that $g_2 \circ f_1(X) = P_2$ and

$$d(g_2,id) < \min(\frac{\delta_1}{4},\frac{\epsilon_1}{2}),$$

where $\epsilon_1 = \min(3^{-2},(3^{-2})\cdot \epsilon\ (f_1,2^{-2}))$ (from Lemma 4.3). Since $d(g_2,id) < \frac{\delta_1}{4}$ we have $P_2 \subset \text{Int}(N_1)$. Thus we can choose a regular neighborhood N_2 of P_2 in E^{2n+1} such that $N_2 \subset \text{Int}(N_1)$ and for which there exists a retraction $r_2\colon N_2 \to P_2$ satisfying $d(r_2,id) < \frac{1}{2^2}$. Now let $f_2\colon f_1(X) \to \text{Int}(N_2)$ be an embedding satisfying

$$d(f_2,g_2) < \min(\frac{\delta_1}{4},\frac{\epsilon_1}{2}).$$

It then follows that $d(f_2,id) < \min(\frac{\delta_1}{2},\epsilon_1)$. It is also easy to see that $d(X,x) < 1$ and $d(f_1(X),y) < \frac{1}{2}$, for all $x \in N_1$ and $y \in N_2$. This completes the second step of the construction.

Inductively continuing this process we can clearly obtain a sequence $\{P_i\}_{i=1}^{\infty}$ of compact polyhedra in E^{2n+1} such that each P_i satisfies $\dim P_i \leq n$, a sequence $\{N_i\}_{i=1}^{\infty}$ of regular neighborhoods, with each N_i a regular neighborhood of P_i in E^{2n+1}, and a sequence of embeddings

$$X \xrightarrow{f_1} E^{2n+1},\ f_1(X) \xrightarrow{f_2} E^{2n+1},\ f_2\circ f_1(X) \xrightarrow{f_3} E^{2n+1},\ \ldots$$

such that the following properties are satisfied:

(1) $d(f_i,id) < \min(3^{-i},(3^{-i})\cdot \epsilon\ (f_{i-1}\circ\ldots\circ f_1,2^{-i}))$, for all $i > 1$,

(2) $f_i\circ f_{i-1}\circ\ldots\circ f_1(X) \subset \text{Int}(N_i)$ and $N_{i+1} \subset \text{Int}(N_i)$, for all $i > 0$,

(3) if $\delta_i = d(f_i\circ\ldots\circ f_1(X),E^{2n+1}\backslash\text{Int}(N_i))$, then $d(f_{i+j},id) < \frac{\delta_i}{2^j}$, for all i, $j > 0$,

(4) $d(f_{i-1}\circ\ldots\circ f_1(X),x) < \frac{1}{2^{i-1}}$, for all $i > 1$ and $x \in N_i$.

It follows from (1) and Lemma 4.3 that the sequence $\{f_i\circ\ldots\circ f_1\}_{i=1}^{\infty}$ converges to an embedding $f\colon X \to E^{2n+1}$. From (2) and (3) we have $f(X) \subset N_i$, for all $i > 0$. Thus all we have to do is show that $\cap_{i=1}^{\infty} N_i \subset f(X)$. To see this choose any $x \in \cap_{i=1}^{\infty} N_i$ and use (4) to conclude that $d(f_{i-1}\circ\ldots\circ f_1(X),x) < \frac{1}{2^{i-1}}$, for all $i > 1$. Since we have $d(f_i,id) < 3^{-i}$, for all $i > 1$, it is routine to verify that $x \in f(X)$.

5. Shape preliminaries

In this section we give a definition of shape for compacta and also describe some related apparatus which will be needed in the proof of Theorem 1. The approach we use here is essentially that of Borsuk's [4]. The central concept is the notion of a fundamental sequence. This is intuitively a generalization of the notion of a continuous function between compacta X and Y, where we consider a sequence of "approximate maps" of X towards Y, rather than one map of X into Y. We then define the notion of homotopy for fundamental sequence and then define the notion of shape, which we will see is just the natural analogue of homotopy equivalence for this new category.

Consider compacta X, Y contained in a space W and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of maps f_k: W → W such that the following condition is satisfied: for each neighborhood V ⊂ W of Y there exists a neighborhood U ⊂ W of X such that

$$f_k|U \simeq f_l|U \text{ (in V)},$$

for almost all integers k and l. Then we say that $\{f_k, X, Y\}$ is a fundamental sequence from X to Y (in W) and we write $\underline{f}$: X → Y, where $\underline{f} = \{f_k, X, Y\}$. Whenever no ambiguity results we will simply call $\underline{f}$ a fundamental sequence.

If X, Y, Z are compacta in W and $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, Y, Z\}$ are fundamental sequences, then it is easily seen that $\underline{g} \circ \underline{f} = \{g_k \circ f_k, X, Z\}$ is a fundamental sequence from X to Z. It is also clear that if X is any compactum in W, then we can define a fundamental sequence $\{f_k, X, X\}$ by setting $f_k = id_W$, for all k > 0. We call this the identity fundamental sequence from X to X (in W) and we denote it by $\underline{id}_X$.

Now consider compacta X, Y contained in a space W and let $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, X, Y\}$ be fundamental sequences. Then we say that $\underline{f}$ and $\underline{g}$ are homotopic (and we write $\underline{f} \simeq \underline{g}$) iff for each neighborhood V ⊂ W of Y there exists a neighborhood U ⊂ W of X such that

$$f_k|U \simeq g_k|U \text{ (in V)},$$

for almost all integers k.

If X and Y are compacta contained in W, where W is now an AR (metric), then we say that X and Y have the same _shape_ (in W) (and we write Sh(X) = Sh(Y)) provided that there exist fundamental sequences $\underline{f}$: X → Y and $\underline{g}$: Y → X such that $\underline{g} \circ \underline{f} \simeq \underline{id}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{id}_Y$. The following result of Borsuk makes this definition unambiguous and generally applicable to compacta.

Proposition 5.1. (Borsuk [5]) _Let_ W, W' _be_ AR's _metric_, _let_ X, Y ⊂ W _be compacta_, _and let_ X', Y' ⊂ W' _be compacta such that_ X ≅ X' _and_ Y ≅ Y'. _Then_ Sh(X) = Sh(Y) (_in_ W) _iff_ Sh(X') = Sh(Y') (_in_ W').

Thus the shape of any compactum can be uniquely determined by embedding it into any AR (metric) and applying the above definition. Normally one uses separable infinite-dimensional Hilbert space l_2 or Q for the AR, since any compactum can be embedded into either of these spaces.

6. Relative shape

In this section we define a relative notion of shape which will be needed in the proof of Theorem 1. This apparatus was also employed in [8] to prove Theorem 2, as cited at the end of our Introduction.

Consider compacta X, Y contained in a space W and let $G \subset W$ be a neighborhood of X. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of maps $f_k\colon G \to W$ such that the following conditions are satisfied:

(1) $f_k \simeq id_G$, for all $k > 0$,

(2) for each neighborhood $V \subset W$ of Y there exists a neighborhood $U \subset G$ of X such that

$$f_k|U \simeq f_l|U \text{ (in } V),$$

for almost all integers k and l.

Then we define $\{f_k,X,Y,G\}$ to be a <u>relative fundamental sequence</u> (in W) and we write $\underline{f} = \{f_k,X,Y,G\}$ to indicate this. As in Section 5 we will simply call $\underline{f}$ a relative fundamental sequence whenever there is no ambiguity.

Now choose compacta X, Y, Z in a space W and relative fundamental sequences $\underline{f} = \{f_k,X,Y,G\}$, $\underline{g} = \{g_k,Y,Z,H\}$ (in W). It is clear that there exists a neighborhood $G_1 \subset G$ of X and an integer $n_1 > 0$ such that $\underline{g}\circ\underline{f} = \{g_k\circ f_k|G_1,X,Z,G_1\}$ $(k \geq n_1)$ is a relative fundamental sequence (in W). In order that $\underline{g}\circ\underline{f}$ be well defined we will identify relative fundamental sequences $\underline{f}' = \{f'_k,X,Y,G'\}$ and $\underline{f}'' = \{f''_k, X,Y,G''\}$ (in W) provided that there exists a neighborhood $G_0 \subset G' \cap G''$ of X such that $f'_k|G_0 = f''_k|G_0$, for almost all integers k.

If X, Y are compacta in W and $\underline{f} = \{f_k,X,Y\}$ is a fundamental sequence (in W), then $\{f_k|G,X,Y,G\}$ uniquely defines a relative fundamental sequence (in W), for any neighborhood G of X. We also see that if $X \subset W$ is a compactum and G is any neighborhood of X, then $\{id_G,X,X,G\}$ uniquely defines a relative fundamental sequence (in W). We denote this sequence by $\underline{id}_X$ (when no ambiguity results) and call it the <u>identity relative fundamental sequence</u> from X to X.

If X, Y are compacta in W and $\underline{f} = \{f_k,X,Y,G\}$, $\underline{g} = \{g_k,X,Y,H\}$ are relative fundamental sequences (in W), then we write $\underline{f} \simeq \underline{g}$ iff for each neighborhood $V \subset W$ of Y there exists a neighborhood $U \subset G \cap H$ of X such that

$$f_k|U \simeq g_k|U \text{ (in } V),$$

for almost all integers k.

Now let X, Y be compacta in W and assume that there exist relative fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, Y, X, H\}$ (in W) such that $\underline{g} \circ \underline{f} \simeq \underline{id}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{id}_Y$. Then we say that X and Y have the same <u>relative shape</u> (in W).

We emphasize the fact that the notion of relative shape depends on W and the positioning of X and Y in W. We will not need an analogue of Proposition 5.1 for relative shape.

7. The main lemma

In Lemma 7.1 below we establish what amounts to the inductive step in the proof of the "only if" part of Theorem 1. This is the only place that it becomes necessary to get deeply involved with the apparatus of Section 6.

Lemma 7.1. *For any integer* $n > 0$ *let* $W \subset E^{2n+2}$ *be an open set and let* $X, Y \subset W$ *be compacta such that* X *is in standard position*, $\dim X \leq n$, *and* X, Y *have the same relative shape* (*in* W). *If* $W_1 \subset W$ *is any neighborhood of* Y, *then there exists an embedding* $\psi: X \to W_1$ *such that* $\psi(X)$ *is in standard position*, $\psi \simeq id_X$ (*in* W), *and* $\psi(X)$, Y *have the same relative shape* (*in* W).

Proof. Since X is in standard position we can find sequences $\{P_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ which satisfy properties (1) - (4) of Definition 4.1. Choose neighborhoods $G \subset W$ of X, $H \subset W$ of Y, and fundamental sequences $\underline{f} = \{f_k, X, Y, G\}$ and $\underline{g} = \{g_k, Y, X, H\}$ (in W) such that $\underline{g} \circ \underline{f} \simeq \underline{id}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{id}_Y$ (in W).

Now choose an integer $n_1 > 0$ and an integer $i_1 > 0$ such that $N_{i_1} \subset G$ and

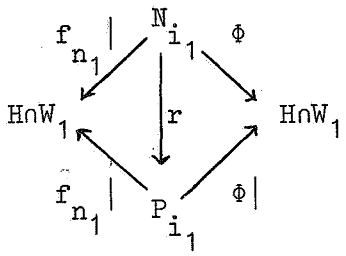
$$f_k | N_{i_1} \simeq f_l | N_{i_1} \quad (\text{in } H \cap W_1),$$

for all integers $k, l \geq n_1$. Since $f_{n_1} | P_{i_1} : P_{i_1} \to H \cap W_1$ is a map and $\dim P_{i_1} \leq n$, we can find a PL embedding $\phi: P_{i_1} \to E^{2n+2}$ which is as close to $f_{n_1} | P_{i_1}$ as we like. We can therefore choose ϕ close enough to $f_{n_1} | P_{i_1}$ so that $\phi(P_{i_1}) \subset H \cap W_1$ and $\phi \simeq f_{n_1} | P_{i_1}$ (in $H \cap W_1$) (for example we can use the straight-line homotopy joining ϕ to $f_{n_1} | P_{i_1}$). Using Proposition 3.5 we can extend ϕ to a PL homeomorphism $\Phi_1: E^{2n+2} \to E^{2n+2}$.

Since $\Phi_1^{-1}(H \cap W_1)$ is a neighborhood of P_{i_1}, we can find a derived neighborhood N of P_{i_1} such that $N \subset \Phi_1^{-1}(H \cap W_1)$. Using Theorem 2.11 of [10] there exists a PL homeomorphism $\alpha: N_{i_1} \to N$ such that $\alpha | P_{i_1} = id$.

Then $\Phi = \Phi_1 \circ \alpha : N_{i_1} \to H \cap W_1$ is a PL embedding which satisfies $\Phi|P_{i_1} \simeq f|P_{i_1}$ (in $H \cap W_1$). Also it follows that $\Phi(X)$ is in standard position, since Φ is PL.

Since N_{i_1} is a regular neighborhood of P_{i_1} there exists a retraction $r: N_{i_1} \to P_{i_1}$ such that $r \simeq id$ (in N_{i_1}). Thus the two smaller triangles in the following diagram homotopy commute (where we use $|$ for the appropriate restriction):



That is, $\Phi \simeq (\Phi|P_{i_1}) \circ r$ (in $H \cap W_1$) and $f_{n_1}|N_{i_1} \simeq (f_{n_1}|P_{i_1}) \circ r$ (in $H \cap W_1$). We can now use this to prove that $f_{n_1}|N_{i_1} \simeq \Phi$ (in $H \cap W_1$) as follows:

$$f_{n_1}|N_{i_1} \simeq f_{n_1} \circ r \text{ (in } H \cap W_1) \simeq \Phi \circ r \text{ (in } H \cap W_1) \simeq \Phi \text{ (in } H \cap W_1).$$

This will be needed below.

Now let $\psi: X \to W_1$ be defined by $\psi = \Phi|X$. We have already seen that $\psi(X)$ is in standard position and also

$$\psi = \Phi|X \simeq f_{n_1}|X \text{ (in } H \cap W_1) \simeq id_X \text{ (in } W),$$

with the last homotopy following since it was assumed that $f_k \simeq id_G$ (in W), for all $k > 0$. Thus all that remains to be done is prove that $\psi(X)$, Y have the same relative shape (in W_1).

In order to do this choose an integer $n_2 \geq n_1$ and a neighborhood $H_1' \subset H \cap W_1$ of Y such that $g_k|H_1' \simeq g_l|H_1'$ (in N_{i_1}), for all integers $k, l \geq n_2$. Using the fact that $\underline{f} \circ \underline{g} \simeq \underline{id}_Y$ (in W) we can find an integer $n_3 \geq n_2$ and a neighborhood $H' \subset H_1'$ of Y such that $f_k \circ g_k|H' \simeq id_{H'}$ (in H_1'), for all $k \geq n_3$. Put $G' = \Phi(N_{i_1})$ and for all $k \geq n_3$ define

$f'_k : G' \to W_1$ by $f'_k = f_k \circ \Phi^{-1}$ and define $g'_k : H' \to W_1$ by $g'_k = \Phi \circ g_k | H'$. For all $k \geq n_3$ put $\underline{f}' = \{f'_k, \psi(X), Y, G'\}$ and $\underline{g}' = \{g'_k, Y, \psi(X), H'\}$. We will prove that $\underline{f}'$, $\underline{g}'$ are relative fundamental sequences (in W_1) which satisfy $\underline{g}' \circ \underline{f}' \simeq \underline{id}_{\psi(X)}$ (in W_1) and $\underline{f}' \circ \underline{g}' \simeq \underline{id}_Y$ (in W_1).

To see that $\underline{f}'$ is a relative fundamental sequence (in W_1) we first note that

$$f'_k = f_k \circ \Phi^{-1} \simeq \Phi \circ \Phi^{-1} \text{ (in } W_1) = id_{G'},$$

since $k \geq n_3$. Now choose a neighborhood $V \subset W_1$ of Y. Since $\underline{f}$ is a relative fundamental sequence (in W) there exists a neighborhood $U \subset N_{i_1}$ of X and an integer $n_4 \geq n_3$ such that $f_k | U \simeq f_l | U$ (in V), for all $k, l \geq n_4$. This obviously implies that $f'_k | \Phi(U) \simeq f'_l | \Phi(U)$ (in V), for all $k, l \geq n_4$. Thus $\underline{f}'$ is a relative fundamental sequence (in W_1).

To see that $\underline{g}'$ is a relative fundamental sequence (in W_1) we note that

$$g'_k = \Phi \circ g_k | H' \simeq f_{n_1} \circ g_k | H' \text{ (in } W_1) \simeq f_k \circ g_k | H' \text{ (in } W_1) \simeq id_{H'} \text{ (in } W_1),$$

for all $k \geq n_3$. Now choose a neighborhood $U \subset G'$ of $\psi(X)$. Then $\Phi^{-1}(U)$ is a neighborhood of X and there exists a neighborhood $V \subset H'$ and an integer $n_4 \geq n_3$ such that $g_k | V \simeq g_l | V$ (in $\Phi^{-1}(U)$), for all $k, l \geq n_4$. It is then clear that

$$g'_k | V = \Phi \circ g_k | V \simeq \Phi \circ g_l | V \text{ (in } U) = g'_l | V,$$

for all $k, l \geq n_4$. Thus $\underline{g}'$ is a relative fundamental sequence (in W_1).

To see that $\underline{f}' \circ \underline{g}' \simeq \underline{id}_Y$ (in W_1) choose a neighborhood $V \subset H'$ of Y. Since $\underline{f} \circ \underline{g} \simeq \underline{id}_Y$ (in W) we can find a neighborhood $V' \subset V$ of Y and an integer $n_4 \geq n_3$ such that $f_k \circ g_k | V' \simeq id_{V'}$ (in V), for all $k \geq n_4$. Then we have

$$f'_k \circ g'_k | V' = (f_k \circ \Phi^{-1}) \circ (\Phi \circ g_k) | V' = f_k \circ g_k | V' \simeq id_{V'} \text{ (in } V),$$

for all $k \geq n_4$. This implies that $\underline{f}' \circ \underline{g}' \simeq \underline{id}_Y$ (in W_1).

To see that $\underline{g}'\circ\underline{f}' \simeq \underline{id}_{\psi(X)}$ (in W_1), choose a neighborhood $U \subset G'$ of $\psi(X)$. Then $\Phi^{-1}(U)$ is a neighborhood of X and there exists an integer $n_4 \geq n_3$ and a neighborhood $U' \subset \Phi^{-1}(U)$ of X such that $g_k\circ f_k|U' \simeq id_{U'}$ (in $\Phi^{-1}(U)$), for all $k \geq n_4$. Clearly $\Phi(U') \subset U$ is a neighborhood of $\psi(X)$ and also

$$g_k'\circ f_k'|\Phi(U') = \Phi\circ g_k\circ f_k\circ\Phi^{-1}|\Phi(U') \simeq \Phi\circ id_{U'}\circ\Phi^{-1}|\Phi(U') \text{ (in } U) = id_{\Phi(U')},$$

for all $k \geq n_4$. Thus $\underline{g}'\circ\underline{f}' \simeq \underline{id}_{\psi(X)}$ (in W_1) and we are done.

8. Proof of Theorem 1

Put $m = \max\{\dim X, \dim Y\}$, choose $n \geq 10m+17$, and let $r > 0$ be any number for which $X \cup Y \subset \mathrm{Int}(B_r^n)$. For the "if" part of the proof we will first establish the following weaker version and then use it for the more general case: If $B_r^n \setminus X \approx B_r^n \setminus Y$, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Using Lemma 3.1 there exist homeomorphisms $\alpha: B_r^n \to B_r^n$ and $\beta: B_r^n \to B_r^n$ such that $\alpha(X) \subset B_r^{2m+1} \times 0 \subset B_r^n$ and $\beta(X) \subset B_r^{2m+1} \times 0 \subset B_r^n$, where we use the notation of Section 2 for $B_r^{2m+1} \times 0 \subset B_r^n$. (Actually Lemma 3.1 does not explicitly give the homeomorphisms α and β, but it is clear that Klee's method yields this with no extra effort.) Define $X' = \alpha(X)$ and $Y' = \beta(Y)$. If $h: B_r^n \setminus X \to B_r^n \setminus Y$ is the homeomorphism of our hypothesis, then $h' = \beta \circ h \circ \alpha^{-1} : B_r^n \setminus X' \to B_r^n \setminus Y'$ is a homeomorphism.

Define a homotopy $F: (B_r^{2m+1} \times 0) \times I \to B_r^n$ by

$$F_t(x_1,\ldots,x_{2m+1},0,0,\ldots,0) = (x_1,\ldots,x_{2m+1},tr,0,0,\ldots,0),$$

for all $(x_1,\ldots,x_{2m+1},0,0,\ldots,0) \in B_r^{2m+1} \times 0$ and $t \in I$. Then $F_0 = \mathrm{id}$ and $F_t(B_r^{2m+1} \times 0) \cap (B_r^{2m+1} \times 0) = \emptyset$, for all $t \in (0,1]$. For each integer $k > 0$ define $f_k: B_r^n \to B_r^n$ and $g_k: B_r^n \to B_r^n$ by

$$f_k = h' \circ F_{1/k} \circ (p_{2m+1}, 0),$$

$$g_k = (h')^{-1} \circ F_{1/k} \circ (p_{2m+1}, 0),$$

where $(p_{2m+1}, 0) : B_r^n \to B_r^{2m+1} \times 0$ is projection. We will show that $\underline{f} = \{f_k, X', Y'\}$ and $\underline{g} = \{g_k, Y', X'\}$ are fundamental sequences which satisfy $\underline{g} \circ \underline{f} \simeq \underline{\mathrm{id}}_{X'}$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_{Y'}$ (in B_r^n).

To see that $\underline{f}$ is a fundamental sequence let $V \subset B_r^n$ be a neighborhood of Y'. Then $(h')^{-1}(V \setminus Y') \cup X'$ is a neighborhood of X' in B_r^n. Choose a closed neighborhood $U' \subset B_r^{2m+1} \times 0$ of X' and an $\epsilon > 0$ such that

$$U = p_{2m+1}(U') \times B_\epsilon^{n-(2m+1)} \subset (h')^{-1} (V \setminus Y') \cup X'.$$

Then we have $F_t(U') \subset (h')^{-1} (V \setminus Y') \cup X'$, for $0 \leq t \leq \frac{\epsilon}{r}$. For all integers k, $1 \geq \frac{r}{\epsilon}$ we have

$$f_k|U = h'\circ F_{1/k}\circ(p_{2m+1},0)|U \simeq h'\circ F_{1/l}\circ(p_{2m+1},0)|U \text{ (in } V) = f_l|U.$$

Thus $\underline{f}$ is a fundamental sequence. Similarly $\underline{g}$ is a fundamental sequence.

To show that $\underline{g}\circ\underline{f} \simeq \underline{id}_{X'}$ (in B^n_r) choose a neighborhood $U \subset B^n_r$ of X', and without loss of generality assume that $U = p_{2m+1}(U') \times B^{n-(2m+1)}_\epsilon$, for some neighborhood $U' \subset B^{2m+1}_r \times 0$ of X' and some $\epsilon > 0$. Then there exists a neighborhood $V \subset B^n_r$ of Y' and an integer $n_1 > 0$ such that $g_k|V \simeq g_l|V$ (in U), for all k, $l \geq n_1$. We can also choose $V = p_{2m+1}(V') \times B^{n-(2m+1)}_\delta$, where $V' \subset B^{2m+1}_r \times 0$ is a neighborhood of Y' and $\delta > 0$. Using the fact that h' is a homeomorphism we can additionally require that $(h')^{-1}(V\backslash Y') \subset U$. Since $\underline{f}$ is a fundamental sequence we can find a neighborhood $U_1 \subset U$ of X' and an integer $n_2 \geq n_1$ such that $f_k|U_1 \simeq f_l|U_1$ (in V), for all k, $l \geq n_2$. Then for $k \geq n_2$ we have $h'\circ F_{1/k}\circ(p_{2m+1},0)(U_1) \subset V$. We can clearly choose an integer $n_3 \geq n_2$ such that

$$F_{1/k}\circ(p_{2m+1},0)|V\backslash Y' \simeq id_{V\backslash Y'} \text{ (in } V\backslash Y'),$$

for all $k \geq n_3$. Thus for all k sufficiently large we have

$$\begin{aligned} g_k\circ f_k|U_1 &= (h')^{-1}\circ F_{1/k}\circ(p_{2m+1},0)\circ h'\circ F_{1/k}\circ(p_{2m+1},0)|U_1 \\ &\simeq (h')^{-1}\circ h'\circ F_{1/k}\circ(p_{2m+1},0)|U_1 \text{ (in } (h')^{-1}(V\backslash Y')) \\ &= F_{1/k}\circ(p_{2m+1},0)|U_1 \simeq id_{U_1} \text{ (in } U). \end{aligned}$$

This proves that $\underline{g}\circ\underline{f} \simeq \underline{id}_{X'}$. Similarly one can prove that $\underline{f}\circ\underline{g} \simeq \underline{id}_{Y'}$. Using Proposition 5.1 we then have Sh(X) = Sh(Y).

Now returning to the general case in the proof of the "if" part of Theorem 1 we have $E^n\backslash X \simeq E^n\backslash Y$ and we want to prove that Sh(X) = Sh(Y). Choose $r > 0$ such that $X \subset Int(B^n_r)$ and let $h: E^n\backslash X \to E^n\backslash Y$ be a homeomorphism. Since X is stably embedded, $Int(B^n_r)\backslash X$ must be connected. Since $h(\partial B^n_r)$ is a bicollared (n-1)-sphere in E^n we can use the Generalized Schoenflies Theorem of [6] to write

$$E^n\backslash h(\partial B^n_r) = A\cup B,$$

where A is the bounded component and B is the unbounded component. Since Y is stably embedded it follows that $A\setminus Y$ and $B\setminus Y$ are connected. Clearly there are exactly two possibilities that can occur: (1) $Y \cap B = \emptyset$ and (2) $Y \cap B \neq \emptyset$. If (1) occurs, then $Y \subset A$ and we have $h(\mathrm{Int}(B_r^n)\setminus X) = A\setminus Y$ or $h(E^n\setminus B_r^n) = A\setminus Y$. If $h(\mathrm{Int}(B_r^n)\setminus X) = A\setminus Y$, then we can use the fact that $\mathrm{Cl}_{E^n}(A) \simeq B_r^n$ (which follows from [6]) and the special case treated above, to conclude that $Sh(X) = Sh(Y)$. If $h(E^n\setminus B_r^n) = A\setminus Y$ (and therefore $h(\mathrm{Int}(B_r^n)\setminus X) = B$), then similar considerations can be used to get $Sh(X) = Sh(\{point\}) = Sh(Y)$. If (2) occurs, then it follows that $h(\mathrm{Int}(B_n^r)\setminus X) = B\setminus Y$ (since the possibility $h(E^n\setminus B_r^n) = B\setminus Y$ is absurd). The special case treated above implies that $Sh(X) = Sh((B\cap Y) \cup \{point\})$. Since $h(E^n\setminus B_r^n) = A\setminus Y$ the special case above implies that $Sh(A\cap Y) = Sh(\{point\})$. It follows easily from the definition that

$$Sh(Y) = Sh((B\cap Y)\cup(A\cap Y)) = Sh((B\cap Y)\cup\{point\}),$$

and therefore $Sh(X) = Sh(Y)$ (since $(B\cap Y) \cap (A\cap Y) = \emptyset$).

For the "only if" part of Theorem 1 we have stably embedded compacta $X, Y \subset E^n$ such that $Sh(X) = Sh(Y)$. We will construct a homeomorphism of $E^n\setminus X$ onto $E^n\setminus Y$ and it will be clear from the construction that the homeomorphism can be required to be the identity off of any n-cell in E^n containing $X \cup Y$ in its interior. Let m, n be chosen as before and also use Proposition 4.4 to get homeomorphisms $\alpha, \beta\colon E^n \to E^n$ such that $X' = \alpha(X) \subset E^{2m+1} \times 0 \subset E^n$, $Y' = \beta(Y) \subset E^{2m+1} \times 0$, and both X', Y' are in standard position in $E^{2m+1} \times 0$. Thus all we need do is prove that $E^n\setminus X' \simeq E^n\setminus Y'$.

We will inductively construct nested sequences $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ of open sets in E^n and sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ of homeomorphisms $f_i, g_i\colon E^n \to E^n$ such that the following properties are satisfied:

(1) $f_1(U_1) \subset V_1$ and $f_{i+1}\circ f_i\circ\cdots\circ f_1\,(U_{i+1}) \subset g_i\circ g_{i-1}\circ\cdots\circ g_1\,(V_{i+1})$, for all $i > 0$,

(2) $f_{i+1}|E^n\setminus f_i\circ\cdots\circ f_1\,(U_i) = \mathrm{id}$, for all $i > 0$,

(3) $g_i\circ\cdots\circ g_1\,(V_{i+1}) \subset f_i\circ\cdots\circ f_1\,(U_i)$, for all $i > 0$,

(4) $g_1|E^n\setminus V_1 = \mathrm{id}$ and $g_{i+1}|E^n\setminus g_i\circ\cdots\circ g_1\,(V_{i+1}) = \mathrm{id}$, for all $i > 0$,

(5) $X' = \bigcap_{i=1}^{\infty} U_i$ and $Y' = \bigcap_{i=1}^{\infty} V_i$.

We first show how to use these properties to prove that $E^n\backslash X' \simeq E^n\backslash Y'$. It follows from (2) and (5) above that $\{f_i \circ f_{i-1} \circ \dots \circ f_1\}_{i=1}^{\infty}$ converges on $E^n\backslash X'$ to define an embedding $f\colon E^n\backslash X' \to E^n$. It follows from (4) and (5) above that $\{g_i \circ \dots \circ g_1\}_{i=1}^{\infty}$ defines an embedding $g\colon E^n\backslash Y' \to E^n$. All we need do now is note that $f(E^n\backslash X') = g(E^n\backslash Y')$, a fact which can be easily verified using (1) and (3) above. Thus $g^{-1} \circ f$ defines a homeomorphism of $E^n\backslash X'$ onto $E^n\backslash Y'$. Now we must verify that it is possible to inductively construct these sequences. We will only do the first step in the induction, as this is essentially the inductive step.

Let $\{G_i\}_{i=1}^{\infty}$ and $\{H_i\}_{i=1}^{\infty}$ be nested sequences of open sets in E^n such that $X' = \cap_{i=1}^{\infty} G_i$ and $Y' = \cap_{i=1}^{\infty} H_i$. We are given $Sh(X') = Sh(Y')$, and therefore $X', Y' \subset E^{2m+2} \times 0 \subset E^n$ are compacta of dimension less than or equal to m, embedded in standard position, which have the same relative shape (in $E^{2m+2} \times 0$). Now choose $W_1 \subset E^{2n+2} \times 0$, an open neighborhood of Y', and $\epsilon_1 > 0$ such that

$$V_1 = p_{2m+2}(W_1) \times \operatorname{Int}(B_{\epsilon_1}^{n-(2m+2)}) \subset H_1.$$

Using Lemma 7.1 there exists an embedding $\psi_1\colon X' \to W_1$ such that $\psi_1(X')$ is in standard position and $\psi_1(X')$, Y' have the same relative shape (in W_1). Using Proposition 3.3 we can extend ψ_1 to a homeomorphism $f_1\colon E^n \to E^n$. Now choose $W_1' \subset E^{2n+2} \times 0$, an open neighborhood of $f_1(X')$, and $\epsilon_1' > 0$ such that

$$p_{2m+2}(W_1') \times \operatorname{Int}(B_{\epsilon_1'}^{n-(2m+2)}) \subset f_1(G_1) \cap V_1.$$

Then put

$$U_1 = f_1^{-1}(p_{2m+2}(W_1') \times \operatorname{Int}(B_{\epsilon_1'}^{n-(2m+2)}))$$

to complete the construction of U_1, V_1, and f_1.

We now construct g_1. Since $f_1(X')$, Y' are in standard position and have the same relative shape (in W_1), we can once more use Proposition 7.1 to get an embedding $\psi_2\colon Y' \to W_1'$ such that $\psi_2 \simeq id$ (in W_1), $\psi_2(Y')$ is in standard position, and $f_1(X')$, $\psi_2(Y')$ have the same relative shape (in W_1'). Using Proposition 3.3 we can extend ψ_2 to a homeomorphism $g_1\colon E^n \to E^n$ such that $g_1|E^n\setminus V_1 = \text{id}$. This completes the first step of the induction. We are now in a position to choose V_2 and construct f_2. This follows the same pattern as before. We simply remark that at each stage "auxiliary neighborhoods" W_i, $W_i' \subset E^{2m+2} \times 0$ will be needed, just as W_1, W_1' were used in the construction of f_1, g_1, U_1, V_1.

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