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AFDELING ZUIVERE WISKUNDE ZN 41/72 MARCH

A.E. BROUWER and A. VERBEEK

COUNTING FAMILIES OF MUTUALLY INTERSECTING SETS

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Introduction

A family of sets is called linked if every two sets have a non-empty intersection. As application of a recent results of Kleitman estimating the number of antichains on an n-point-set, we derive asymptotic formula for the $^2\log$ of the following numbers:

the number $\lambda(n)$ of maximal linked families of subsets of $\{1,2,\ldots,n\}$

the number $\Lambda(n)$ of all linked families of subsets of $\{1,2,\ldots,n\}$

In his survey [1] p. 79, P. Erdös asked for an asymptotic formula for $\Lambda(n)$. Our concern came forth from an investigation [4], on maximal linked families of closed sets in topological spaces.

Notation

$S_n = \{1,2,\ldots,n\}$

$P_n = P(S_n)$ = powerset of S_n

$M \subset P_n$ is linked if $\forall S,S' \in M \quad S \cap S' \neq \emptyset$

$M \subset P_n$ is an antichain if $\forall S,S' \in M \quad S \not\subset S'$

an mls is a maximal linked (sub)system (of P_n)

$L_n = \{M \subset P_n \mid M \text{ is an mls}\}$

$A_n = \{M \subset P_n \mid M \text{ is a non-empty antichain}\}$

$I_n = \{M \subset P_n \mid \cap M \neq \emptyset\}$

$\lambda(n) = |L_n|$

$\Lambda(n) = |\{M \subset P_n \mid M \text{ is linked}\}|$

$\alpha(n) = |A_n|$

$i(n) = |I_n|$

For arbitrary $M \subset P_n$ we define

$$M_{MIN} = MIN(M) = \{S \in M \mid \forall T \subset S \quad T \in M \Rightarrow T = S\}$$

Finally for two function f,g: $N \to R$ we write

$$f \sim g$$

$$\text{iff} \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 .$$

The first lemma is trivial.

Lemma 1

(a) A linked family $M \subset P_n$ is an mls iff M contains S_n and moreover (precisely) one set of each pair of complementary proper subsets of S_n.

(b) Each linked family is contained in (at least one) mls.

(c) Two mls's $M, M' \subset P_n$ are different iff $\exists S \in M \exists S' \in M'$ $S \cap S' = \emptyset$.

Lemma 2.

$$2^{\binom{n-1}{[n/2]-1}} \leq \lambda(n).$$

Proof.

We give slightly different proofs for even and for odd n.
Let $n = 2k$. Let $\{\{A_i, S_n \backslash A_i\} \mid 1 \leq i \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}\}$ be the family of all unordered pairs of complementary $k = n/2$-point-sets in S_n. If we choose one k-point-set from each pair then we obtain a linked system. Thus we obtain $2^{\binom{2k-1}{k-1}}$ different linked families, with the properties that for two such families, say A and A', $\exists A \in A\ \exists A' \in A'$ $A \cap A' = \emptyset$. By 1b + 1c it follows that there are at least $2^{\binom{2k-1}{k-1}}$ different mls's.
Let $n = 2k-1$. Consider the family $\{\{A_i, S_n \backslash A_i\} \mid 1 \leq i \leq \binom{2k-2}{k-2}\}$ of all pairs of complementary sets A_i, $S_n \backslash A_i$ satisfying $1 \in A_i$ and A_i has $k - 1$ points. The same reasoning as above leads to $2^{\binom{2k-2}{k-2}} \leq \lambda(n)$.

Lemma 3.

$$\lambda(n) \leq \alpha(n-1).$$

Proof.

Define $f\colon L_n \to A_{n-1}$ by

$$f(M) = \{S \mid S \in \mathrm{MIN}\{T \mid n \notin T \in M\}\}.$$

By 1a the family $M' = \{T \mid n \notin T \in M\}$ uniquely determines M (viz. $M = M' \cup \{S \mid n \in S \subset S_n \text{ and } S_n \setminus S \notin M'\}$), and hence also MIN M' uniquely determines M, as $M' = \{T \subset S_{n-1} \mid \exists\, S \in \mathrm{MIN}\, M' : S \subset T\}$. Finally MIN M' obviously is an antichain in P_{n-1}.

Lemma 4. KLEITMAN [2]

$$^2\log \alpha(n) \sim \binom{n}{[n/2]}.$$

Lemma 5.

$$\binom{n-1}{[n/2]-1} \sim \binom{n-1}{[\frac{n-1}{2}]} \sim \frac{2^n}{\sqrt{2\pi n}} \sim \frac{2^n}{\sqrt{2\pi(n-1)}} .$$

The last lemma is trivial. From 2, 3, 4 and 5 we immediately obtain our main result:

Theorem 6.

$$^2\log \lambda(n) \sim {}^2\log \alpha(n-1) \sim \frac{2^n}{\sqrt{2\pi n}} .$$

From this result it is easy to deduce an asymptotic formula for $^2\log \Lambda(n)$. First we observe that $\Lambda(n) \geq i(n)$. The wellknown expression for $i(n)$, see below, can be obtained by first counting for all $k \in S_n$ all families A with $\{k\} \subset \cap A$. Then for $k \neq k'$ the families with $\{k,k'\} \subset \cap A$ were counted twice, so their number should be subtracted and so on.

Lemma 7.

(a) $$n.2^{2^{n-1}}(1-(n-1)/2.2^{2^{n-2}}) < \sum_{k=1}^{n} (-)^{k+1}\binom{n}{k}2^{2^{n-k}} = i(n) < \Lambda(n) < \lambda(n).2^{2^{n-1}} .$$

(b) $(\lambda(n)/2^{\binom{n}{[n/2]}}).2^{2^{n-1}} < \Lambda(n) < \lambda(n).2^{2^{n-1}}$

Proof.
Let $M \subset P_n$ be an arbitrary mls. Then, by 1a, M has 2^{n-1} members, and, by Sperner's lemma [3], M_{MIN} , being an antichain, has at most $\binom{n}{[n/2]}$ members. Thus M contains $2^{2^{n-1}}$ linked subfamilies, which proves the right-hand inequality of (a) and (b). To prove the left-hand side of (b), we observe that M is the only mls containing M_{MIN} . This means that no linked system N satisfies $M_{MIN} \subset N \subset M$ and $M'_{MIN} \subset N \subset M'$ for different mls's M and M'. As there are at least $2^{n-1} - \binom{n}{[n/2]}$ many sets in $M \backslash M_{MIN}$, the left-hand inequality follows.

From 6 and 7a we see that

Theorem 7.

$$^2\log i(n) \sim {}^2\log \Lambda(n) \sim 2^{n-1} .$$

In the numerical results (see page 5) $\lambda(6)$ (and $\lambda(1) - \lambda(5)$) were computed by means of the bijection

$$\phi: L_n \to \{M \subset P_n \mid M \text{ is a linked antichain}\}$$

defined as follows. Let $A = \{S_i \mid 1 \leq i \leq 2^{n-1}\}$ be a selection of subsets of S_n of at most n/2 points such that A contains precisely one of each pair of complementary subsets of S_n. Then for $M \in L_n$:

$$\phi(M) = \text{MIN}(M \cap A),$$

and

$$\phi^{-1}(N) = \{A \in P_n \mid \exists A' \in N \quad A' \subset A\} \cup$$
$$\cup \{A \in P_n \backslash A \mid \neg \exists A' \in N \quad A' \subset S_n \backslash A\}.$$

Moreover $\lambda(1) - \lambda(5)$ and $\alpha(1) - \alpha(4)$ were also computed by hand, and $\lambda(7)$, $\alpha(5)$ and $\alpha(6)$ have been obtained by means of a PDP-8 computer, but were evaluated only once.

Numerical results

n	$\lambda(n)$	$\alpha(n-1)$	$^2\log \lambda(n)$	$^2\log \alpha(n-1)$	$2^n/\sqrt{2\pi(n-1)}$	$2^n/\sqrt{2\pi n}$	$\binom{n-1}{[n/2]-1}$	$i(n)$	$\Lambda(n)$
0	1	–	0	–	–	∞	–	1	1
1	1	1	0	0	∞	.798	0	2	2
2	2	2	1	1	1.596	1.128	1	6	6
3	4	5	2	2.322	2.257	1.843	2	38	40
4	12	19	3.585	4.248	3.685	3.192	3	942	1.888
5	81	167	6.340	7.384	6.383	5.709	6	325.262	?
6	2.646	7.580	11.370	12.888	11.418	10.424	10	26.10^9	?
7	1.422.564	7.828.353	20.440	22.900	20.847	19.301	20	13.10^{19}	?
8	conjectured $7.10^{10}<\lambda(8)<4.10^{11}<\alpha(7)<4.10^{12}$		?	?	38.602	36.102	35	27.10^{38}	?
9	$10^{21}<\lambda(9)<5.10^{21}<\alpha(8)< \quad 10^{24}$		?	?	72.204	68.1	70	10.10^{77}	?

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