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COUNTING FAMILIES OF MUTUALLY INTERSECTING SETS
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COUNTING FAMILIES OF MUTUALLY INTERSECTING SETS

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**Introduction**

A family of sets is called **linked** if every two sets have a non-empty intersection. As application of a recent results of Kleitman estimating the number of antichains on an n-point-set, we derive asymptotic formula for the $2^{\log n}$ of the following numbers:

- the number $\lambda(n)$ of maximal linked families of subsets of \{1,2,...,n\}
- the number $\Lambda(n)$ of all linked families of subsets of \{1,2,...,n\}

In his survey [1] p. 79, P. Erdös asked for an asymptotic formula for $\Lambda(n)$. Our concern came forth from an investigation [4], on maximal linked families of closed sets in topological spaces.

**Notation**

- $S_n = \{1,2,\ldots,n\}$
- $P_n = P(S_n)$ = powerset of $S_n$
- $M \subseteq P_n$ is **linked** if $\forall S,S' \in M \land S \cap S' \neq \emptyset$
- $M \subseteq P_n$ is an **antichain** if $\forall S,S' \in M \land S \neq S'$
- an **mls** is a maximal linked (sub)system of $P_n$
- $L_n = \{McP_n \mid M \text{ is an mls}\}$
- $A_n = \{McP_n \mid M \text{ is a non-empty antichain}\}$
- $I_n = \{McP_n \mid nM \neq \emptyset\}$
- $\lambda(n) = |L_n|$
- $\Lambda(n) = |\{McP_n \mid M \text{ is linked}\}|$
- $\alpha(n) = |A_n|$
- $i(n) = |I_n|$

For arbitrary $M \subseteq P_n$ we define

$$M_{\text{MIN}} = \text{MIN}(M) = \{S \in M \mid \forall T \in S \; T \subseteq M \Rightarrow T = S\}$$

Finally for two function $f,g : \mathbb{N} \rightarrow \mathbb{R}$ we write
\[ f \sim g \]

iff \[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1. \]

The first lemma is trivial.

**Lemma 1**

(a) A linked family \( M \subset P_n \) is an mls iff \( M \) contains \( S_n \) and moreover (precisely) one set of each pair of complementary proper subsets of \( S_n \).

(b) Each linked family is contained in (at least one) mls.

(c) Two mls's \( M, M' \subset P_n \) are different iff \( \exists S \in M \exists S' \in M' \) \( S \cap S' = \emptyset \).

**Lemma 2.**

\[ \binom{n-1}{\lceil n/2 \rceil - 1} \leq \lambda(n). \]

**Proof.**

We give slightly different proofs for even and for odd \( n \).

Let \( n = 2k \). Let \( \{A_i, S_n \setminus A_i \mid 1 \leq i \leq \binom{2k}{k-1} = \binom{2k-1}{k-1} \} \) be the family of all unordered pairs of complementary \( k \)-point-sets in \( S_n \). If we choose one \( k \)-point-set from each pair then we obtain a linked system. \( \binom{2k-1}{k-1} \)

Thus we obtain \( 2^{k-1} \) different linked families, with the properties that for two such families, say \( A \) and \( A' \), \( \exists A \in A \exists A' \in A' \) \( A \cap A' = \emptyset \). By \( \binom{2k-1}{k-1} \)

It follows that there are at least \( 2^{k-1} \) different mls's.

Let \( n = 2k-1 \). Consider the family \( \{\{A_i, S_n \setminus A_i \mid 1 \leq i \leq \binom{2k-2}{k-2} \} \) of all pairs of complementary sets \( A_i, S_n \setminus A_i \) satisfying \( 1 \in A_i \) and \( A_i \) has \( k-1 \) \( \binom{2k-2}{k-2} \) points. The same reasoning as above leads to \( \lambda(n) \leq 2^{k-2} \).

**Lemma 3.**

\[ \lambda(n) \leq a(n-1). \]
Proof.
Define \( f: L_n \rightarrow A_{n-1} \) by

\[
f(M) = \{ S \mid S \in \text{MIN}(T \mid n \notin T \in M) \}.
\]

By 1a the family \( M' = \{ T \mid n \notin T \in M \} \) uniquely determines \( M \) (viz. \( M = M' \cup \{ S \mid n \in S \in S_n \) and \( S_n \setminus S \notin M' \}) , and hence also \( \text{MIN} M' \) uniquely determines \( M \), as \( M' = \{ T \subseteq S_{n-1} \mid \exists S \in \text{MIN} M' : S \subseteq T \} \). Finally \( \text{MIN} M' \) obviously is an antichain in \( P_{n-1} \).

Lemma 4. KLEITMAN [2]

\[ 2^{\log a(n)} \sim (\binom{n}{\lfloor n/2 \rfloor}). \]

Lemma 5.

\[ 2^{1-\left[ \frac{n-1}{2} \right]} \sim \frac{2^n}{\sqrt{2\pi n}} \sim \frac{2^n}{v2\pi(n-1)}. \]

The last lemma is trivial. From 2, 3, 4 and 5 we immediately obtain our main result:

Theorem 6.

\[ 2^{\log \lambda(n)} \sim 2^{\log a(n-1)} \sim \frac{2^n}{\sqrt{2\pi n}}. \]

From this result it is easy to deduce an asymptotic formula for \( 2^{\log \lambda(n)} \). First we observe that \( \lambda(n) \geq i(n) \). The well-known expression for \( i(n) \), see below, can be obtained by first counting for all \( k \in S_n \) all families \( \lambda \) with \( \{ k \} \subset nA \). Then for \( k = k' \) the families with \( \{ k, k' \} \subset nA \) were counted twice, so their number should be subtracted and so on.

Lemma 7.

\[ (a) \ n.2^{n-1} \left( 1-(n-1)/2 \right)2^{n-2} < \sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}2^{n-k} = i(n) < \lambda(n) < \lambda(n).2^{n-1}. \]
(b) \( (\lambda(n)/2^{\binom{n}{[n/2]}}) \cdot 2^{2^n-1} < \Lambda(n) < \lambda(n) \cdot 2^{2^n-1} \)

Proof.
Let \( M \subseteq P_n \) be an arbitrary mls. Then, by 1a, \( M \) has \( 2^{n-1} \) members, and, by Sperner's lemma [3], \( M_{\text{MIN}} \), being an antichain, has at most \( \binom{n}{[n/2]} \) members. Thus \( M \) contains \( 2^{2^{n-1}} \) linked subfamilies, which proves the right-hand inequality of (a) and (b). To prove the left-hand side of (b), we observe that \( M \) is the only mls containing \( M_{\text{MIN}} \). This means that no linked system \( N \) satisfies \( M_{\text{MIN}} < N < M \) and \( M'_{\text{MIN}} < N < M' \) for different mls's \( M \) and \( M' \). As there are at least \( 2^{n-1} - \binom{n}{[n/2]} \) many sets in \( M \setminus M_{\text{MIN}} \), the left-hand inequality follows.

From 6 and 7a we see that

**Theorem 7.**

\[ 2 \log i(n) \sim 2 \log \Lambda(n) \sim 2^{n-1}. \]

In the numerical results (see page 5) \( \lambda(6) \) (and \( \lambda(1) - \lambda(5) \)) were computed by means of the bijection

\[ \phi: L_n \rightarrow \{ M \subseteq P_n \mid M \text{ is a linked antichain} \} \]

defined as follows. Let \( A = \{ S_i \mid 1 \leq i \leq 2^{n-1} \} \) be a selection of subsets of \( S_n \) of at most \( n/2 \) points such that \( A \) contains precisely one of each pair of complementary subsets of \( S_n \). Then for \( M \in L_n \):

\[ \phi(M) = \text{MIN}(M \cup A), \]

and

\[ \phi^{-1}(N) = \{ A \in P_n \mid \exists A' \in N \ A' \subseteq A \} \cup \{ A \in P_n \setminus A \mid \exists A' \in N \ A' \subseteq S_n \setminus A \}. \]

Moreover \( \lambda(1) - \lambda(5) \) and \( a(1) - a(4) \) were also computed by hand, and \( \lambda(7), a(5) \) and \( a(6) \) have been obtained by means of a PDP-8 computer, but were evaluated only once.
### Numerical results

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References


