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A NOTE ON THE MOVABILITY OF CHAINS

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A note on the movability of chains

Prof. Dwinger posed the following question:
Is it true that, given a chain C, if for each a C there is an automorphism ϕ_a of C (i.e. a 1-1 monotonic map of C onto itself) such that $\phi_a(a) \neq a$ then there is an automorphism ϕ of C such that $\forall a \in C \quad \phi(a) \neq a$?
For various reasons the answer is negative, and some counterexamples will be given here.

1. Definitions

A chain is a totally ordered set.
If C is a chain and $p \in C$ then $R(p) = \{q \mid \phi \in Aut(C) \text{ and } \phi(p) \neq p \rightarrow \rightarrow \phi(q) \neq q\}$ where Aut(C) is the set of automorphisms of C, that is, R(p) is the set of points which must move as soon as p is moved. (In particular, if p is not movable $R(p) = C$).
If ϕ is an automorphism of C and $\phi(p) \neq p$ then ϕ^p is the authomorphism defined as follows:

$$\phi^p(x) = \begin{cases} \phi(x) & \text{if for some } n \in Z \quad \phi^{-n}(p) < x < \phi^n(p) \\ x & \text{otherwise} \end{cases}$$

and $I(\phi;p) = \{q \in C \mid \phi^p(q) \neq q\}$.
A chain C is called homogeneous if $\forall a \in C \; \forall b \in C \; \exists \phi_{ab} \in Aut(C) : \phi_{ab}(a) = b$ and bihomogeneous if $\forall a,b,p,q \quad \exists \phi_{ab,pq} \in Aut(C) : \phi_{ab,pq}(a) = p$ and $\phi_{ab,pq}(b) = q$.

2. Propositions

<u>Prop</u> 1. If $\phi(p) \neq p$ then $R(p) \subset I(\phi;p)$

<u>Prop</u> 2. $p \in R(q) \rightarrow R(p) \subset R(q)$

<u>Prop</u> 3. If $p \notin R(q)$ and $q \notin R(p)$ then $R(p) \cap R(q) = \emptyset$.

<u>Prop</u> 4. If C is homogeneous then either $\forall a \in C \; R(p) = \{p\}$ or C is the sum of a number of chains C_α, all isomorphic, homogeneous and without first or last point such that $p \in C_\alpha \rightarrow R(p) = C_\alpha$.

Proof 1. and 2. are immediate from the definitions.

3. Let ϕ_1 be an automorphism of C such that $\phi_1(p) \neq p$ and $\phi_1(q) = q$. We may suppose $p < \phi_1(p) < q$ (otherwise if $\phi_1(p) < p$ then $\phi_1^{-1}(p) > p$). ϕ_1 maps R(q) onto itself, therefore $\phi_1(p) \notin R(q)$, so there exists a $\phi_2 \in Aut(C)$ such that $\phi_2\phi_1(p) = \phi_1(p)$ and $\phi_2(q) \neq q$. We may suppose $p < \phi_1(p) < \phi_2(q) < q$. Now $\phi_2\phi_1(p) > p$ and $\phi_2\phi_1(q) < q$ so $I(\phi_2\phi_1;p) \cap I(\phi_2\phi_1;q) = \emptyset$ and because of prop 1. the proposition is proved.

4. Suppose $p < q \in R(p)$. Let $\phi(p) = q$, $\phi \in Aut(C)$, then $\phi^2(p) \in R(q) \subset R(p)$ and by induction $\phi^n(p) \in R(\phi^{n-1}(p)) \subset \ldots \subset R(p) \subset I(\phi;p)$. Since $\{\phi^n(p)\}_n$ is cofinal with $I(\phi;p)$ it follows that for each n $R(\phi^n(p))$ is cofinal in $I(\phi;p)$, and hence R(q) is cofinal with R(p).

   (i) Suppose moreover $p \notin R(q)$.
   Let $\phi_1 \in Aut(C)$ such that $\phi_1(q) > q$ and $\phi_1(p) = p$.
   If $\phi_1(q) < \phi(q)$ then $\phi_1(q) \in R(q)$ so by the above reasoning $\{\phi_1^n(q)\}_{n \in N}$ is cofinal with R(q) and therefore for some $n \in N$: $\phi_1^n(q) \geq \phi(q)$. So we may assume $\phi_1(q) \geq \phi(q)$ but in this case $\phi^{-1}\phi_1(p) < p$ and $\phi^{-1}\phi_1(q) \geq q$ which would imply $q \notin R(p)$, a contradiction.

   (ii) Therefore if $q \in R(p)$ then $p \in R(q)$ and the relation $\sim$ defined by $p \sim q$ iff $p \in R(q)$ is an equivalence relation. If we call the equivalence classes C_α, the proposition is proved.

Prop 5. If ϕ is an automorphism moving each point of C, then C is the sum of a number of chains each cofinal with ω_0 and coinitial with ω_0^*.

Proof The relation $\sim$ defined by $p \sim q$ iff $I(\phi;p) = I(\phi;q)$ is an equivalence relation, and the equivalence classes $I(\phi;s)$ are cofinal and coinitial with $\{\phi^n(s)\}_{n \in Z}$.

<u>Prop</u> 6. If C is bihomogeneous and the sum of a number of chains each cofinal with ω_0 and coinitial with ω_0^* then there is an automorphism ϕ of C moving each point of C.

<u>Proof</u> Let C be a bihomogeneous chain, and let $\{p_n\}_{n\in\mathbb{Z}}$ be cofinal and coinitial with C. ($p_n < p_{n+1}$ $\forall n\in\mathbb{Z}$)

For each $n\in Z$ let ϕ_n be an automorphism of C such that $\phi_n(p_n) = p_{n-1}$ and $\phi_n(p_{n+1}) = p_n$. Define an automorphism ϕ of C by $\phi(x) = \phi_n(x)$ if $p_n \leq x \leq p_{n+1}$, then ϕ moves each point of C.

In the general case, paste together automorphisms of the summands to get a ϕ moving all of C.

<u>3. Examples</u>

1. By prop 5. it is easy to construct examples of chains in which each point can be moved by an automorphism, but in which not all points can be moved by the same automorphism; for example, the long line $(\omega_0^* + \omega_1) \times [0,1)$ ordered lexicographically is connected and bihomogeneous, but since it has no gaps with right- or left-character ω_0 there cannot be an automorphism moving each of its points. Likewise $Z \times (\omega_0^* + \omega_1) \times [0,1)$, ordered lexicographically is a homogeneous example which is cofinal with ω_0 and coinitial with ω_0^*.

2. An example with $R(x) \neq \{x\}$ for all x:

   Let $C_0 = Z$

   $C_1 = (C_0 + A_1) \times Z$ ordered antilexicographically (i.e. according to last difference)

   $\vdots$

   $C_{n+1} = (C_n + A_{n+1}) \times Z$

   where $Z = \omega_0^* + \omega_0$ and $A_n = \omega_n + 1$; let a_n be the last element of A_n.

   Then C_n can be thought of as embedded in C_{n+1} by $x\in C_n \to (x,1)\in C_{n+1}$.

   In this way an inductive system is obtained with inductive limit (union) C.

   Let $x\in C$, then for some n $x\in C_n$; now ϕ_n defined by

$$\phi_n(z) = \begin{cases} (y,k+1) & \text{if } z = (y,k)\in C_n \\ z & \text{otherwise} \end{cases}$$

is an automorphism of C such that $\phi_n(x) \neq x$.
Now suppose ϕ is an automorphism of C such that for each $x\in C$ $\phi(x) \neq x$. Since (a_n,k) is the limit of an ascending sequence of type ω_n for each k,n but no other point in the interval $[(a_n,k),(a_n,k+1)) = I_{n,k}$ is, it follows that $\phi[I_{n,k}] \cap I_{n,k} = \emptyset$ (because either $\phi((a_n,k)) \geq (a_n,k+1)$ or $\phi((a_n,k+1)) \leq (a_n,k)$).
(In fact this says that $R(a_n) = C_n$).
Let $x_0\in C$. Then $\exists m : x_0\in C_m$ and $\phi(x_0)\in C_m$.
Now look to C_{m+1}. Here x_0 is called $(x_0,1)$ and $\phi(x_0)$ is called $(\phi(x_0),1)$. Since both $(x_0,1)$ and $(\phi(x_0),1)$ lie in $I_{m+1,0}$ it follows that $(\phi(x_0),1)\in I_{m+1,0} \cap \phi[I_{m+1,0}] = \emptyset$, a contradiction.

3. The countable case

If C is countable and moreover homogeneous then either there exists a pair of neighbours in C, and in that case C is a sum of copies of $Z = \omega_0^* + \omega_0$, or C is dense and therefore isomorphic to $\mathbb{Q}$.
However, a non-homogeneous example which is countable can be constructed by modifying the previous construction:

take $C_0 = \mathbb{Q}$ and $A_n = \{0,1,2,\ldots,n\}$.