A NOTE ON MAGIC GRAPHS
A.E. BROUWER
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2e boerhaavestraat 49 amsterdam
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A note on magic graphs

In [1] Stewart defined for a finite graph $G$ not containing isolated vertices the spaces $S(G)$ and $Z(G)$; $S(G)$ is the space of all real-valued functions $f$ defined on the set of edges $E(G)$ of $G$ with the property that $\sum \{f(e) \mid e \text{ is incident with } v\} = c_v(f)$ is independent of the vertex $v$ in $G$, and $Z(G)$ is the subspace of $S(G)$ consisting of the functions $f$ with $c(f) = c_v(f) = 0$.

He proved that if $G$ is connected then

1. $E - n + 1 \leq \dim S(G) \leq E - n + 2$
2. $\dim Z(G) \leq \dim S(G) \leq 1 + \dim Z(G)$

where $E$ is the number of edges and $n$ the number of vertices of $G$, but he was unable to determine the exact values of $\dim S(G)$ and $\dim Z(G)$.

In this note I shall prove:

**Theorem 1.** If $G$ is connected then $\dim S(G) = E - n + 2$ iff the vertices of $G$ can be coloured with blue and red in such a way that no two vertices of the same colour are adjacent and, moreover, the number of blue vertices equals the number of red vertices.

In other words: $\dim S(G) = E - n + 2$ if $G$ is bipartite in two sets of equal cardinality

and $\dim S(G) = E - n + 1$ otherwise.

**Theorem 2.** If $G$ is connected then

$$\dim Z(G) = \dim S(G) - 1 = E - n$$

if $G$ contains a circuit of odd length

and $\dim Z(G) = E - n + 1$ otherwise.

**Corollary.** Call a graph $G$ semimagic if $\dim S(G) > \dim Z(G)$; then we have:

$K_{n,m}$ is semimagic iff $n = m$; $K_n$ is semimagic for all $n \geq 2$.

**Theorem 3.** Let $G$ consist of the components $G_i \ (1 \leq i \leq \tau(G))$, where $\tau(G)$ is the number of components of $G$.

Then $\dim Z(G) = \sum \dim Z(G_i)$, and if $\delta_i = \dim S(G_i) - \dim Z(G_i)$

then $\dim S(G) = \sum \dim Z(G_i) + n\delta_i = \sum \dim S(G_i) - \sum \delta_i + n\delta_i$.

In particular if for all $i \ S(G_i) \neq Z(G_i)$ then

$$\dim S(G) = \sum \dim S(G_i) - \tau(G) + 1$$

and if $S(G_i) = Z(G_i)$ for
\( \tau'(G) > 0 \) components of \( G \) then
\[
\dim S(G) = \sum \dim S(G_i) - \tau(G) + \tau'(G).
\]
(This is obvious, but Stewart gives the incorrect result
\[
\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i) \text{ if for all } i \text{ dim } S(G_i) > 0.
\]
A counterexample to this is the graph
\[
\begin{array}{c}
\square \\
\rightarrow \\
\square
\end{array}
\]

Proof.

Let \( G \) be connected, \( \dim S(G) = E - n + \delta \), \( \dim Z(G) = E - n + \zeta \) and let \( f \in S(G) \).

The proof is with induction on \( E \), the number of edges of \( G \).

We distinguish several cases:

(A) \( G \) contains a circuit of even length \( C = (v_0, v_1, \ldots, v_{2k-1}) \).

Let \( G' \) be the graph obtained from \( G \) by deleting the edge \( v_0 v_{2k-1} \).

\( G' \) is connected, and \( G' \) is bipartite in equal parts if \( G \) is and
\( G' \) contains an odd circuit if \( G \) does.

Define \( f' \) by \( f'(v_i v_{i+1}) = f(v_i v_{i+1}) + (-1)^i f(v_0 v_{2k-1}) \) \( (i=0, \ldots, 2k-2) \)
and \( f'(e) = f(e) \) if \( e \notin C \).

Then \( f' \in S(G') \) and \( \sigma(f') = \sigma(f) \).

Conversely if \( f' \) on \( G' \) is given, then \( f \) on \( G \) can be constructed by
\[
\begin{cases}
    f(v_0 v_{2k-1}) = x \\
    f(v_i v_{i+1}) = f'(v_i v_{i+1}) - (-1)^i x \\
    f(e) = f'(e) & \text{if } e \notin C.
\end{cases}
\]

Since \( x \) is arbitrary this proves \( \dim S(G') = \dim S(G) - 1 \) and
\( \dim Z(G') = \dim Z(G) - 1 \). Since \( n' = n \) and \( E' = E - 1 \) it follows
that \( \delta' = \delta \) and \( \zeta' = \zeta \), so the theorems are valid for \( G \) if and only
if they are valid for \( G' \).

(B) \( G \) does not contain a circuit of even length, but contains two circuits of odd length:
\( C_1 = (v_0, v_1, \ldots, v_{2k}) \) and \( C_2 = (w_0, w_1, \ldots, w_{2l}) \).
Since $G$ does not contain a circuit of even length, $C_1$ and $C_2$ do not have common edges.

Since $G$ is connected, $C_1$ and $C_2$ are connected by a way
\[ W = (u_0, u_1, \ldots, u_s) \]
where
\[ W \cap C_1 = \{u_0\} \text{ and } W \cap C_2 = \{u_s\} \]
and possibly $s = 0$.

Let $u_0 = v_0$, $u_s = w_0$, $w_{2k+1} := w_0$.

Let $G'$ be the graph obtained from $G$ by deleting the edge $v_0 v_{2k}$.

As above an $f'$ can be defined by
\[
\begin{align*}
    f'(e) &= f(e) \quad \text{for } e \notin C_1 \cup C_2 \cup W \\
    f'(v_i v_{i+1}) &= f(v_i v_{i+1}) - (-1)^i f(v_0 v_{2k}) \\
    f'(u_j u_{j+1}) &= f(u_j u_{j+1}) + 2(-1)^j f(v_0 v_{2k}) \\
    f'(w_j w_{j+1}) &= f(w_j w_{j+1}) + (-1)^j (-1)^s f(v_0 v_{2k})
\end{align*}
\]

and again it follows that $\delta' = \delta$ and $\zeta' = \zeta$.

(C) $G$ contains one circuit of odd length: $C = (v_0, v_1, \ldots, v_{2k})$. Define
\[ v_{2k+i} := v_{i-1}. \]

Here $E = n$, so we have to prove $\delta = 1$ and $\zeta = 0$.

Fix a $\sigma \in S(G)$; then an $f \in S(G)$ with $\sigma(f) = \sigma$ can be defined in one and only one way:

Each $v_i$ is the root of a (possibly empty) tree on which $f$ is completely determined.

To satisfy the conditions $\sigma_{v_i}(f) = \sigma$ we get $2k+1$ equations
\[
f(v_{i-1} v_i) + f(v_i v_{i+1}) = a_i \quad (1 \leq i \leq 2k+1) \]
with the unique solution
\[
f(v_{i-1} v_i) = \frac{1}{2} a_{i-1} + \frac{1}{2} \sum_{j=0}^{2k-1} (-1)^j a_{i+j} \quad (1 \leq i \leq 2k+1) \]
(where $a_{i+2k+1} = a_i$).

This proves both theorems for graphs which contain a circuit of odd length.

(D) $G$ contains no circuit, i.e. is a tree.

Fix a root $v_0$ of $G$ and a $\sigma \in S$, then there is a unique $f$ such that
$\sigma_{v_0}(f) = \sigma$ for $v \neq v_0$.

Now if $\sigma_{v_0}(f) = \sigma$ then $\dim S(G) = 1$ else $\dim S(G) = 0$, and in either case $\dim Z(G) = 0$.

Since $E = n-1$ and $\dim Z(G) = 0$, we have $\zeta = 1$, which proves theorem 2.
G is connected and does not contain a circuit of odd length, hence 
G is bipartite in a unique way: $G = G_1 \cup G_2$. Now if 
dim S(G) = 1 and $\sigma(f) \neq 0$ then $|G_1| = |G_2|$ since 
$\sigma \cdot |G_1| = \sum_{e} f(e) = \sigma \cdot |G_2|$. 
Conversely, if $|G_1| = |G_2|$ and $\sigma_v(f) = \sigma$ for $v \neq v_0$ then 
$\sigma_{v_0}(f) = \sum_{v} f(e) - \sigma \cdot (|G_1|-1) = \sigma$. 
Therefore if $|G_1| = |G_2|$ then $\delta = 2$ else $\delta = 1$. This proves everything.

Reference