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UNIVERSAL TOPOLOGICAL TRANSFORMATION GROUPS

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Universal Topological Transformation Groups

J. de Vries

1. Introduction

A topological transformation group (ttg) is a triple (G,X,π) with G a topological group, X a topological space and $\pi : G \times X \to X$ a continuous function, satisfying the conditions

(i) $\pi(e,x) = x$ for every $x \in X$ (e denotes the identity of G)

(ii) $\pi(t_1,\pi(t_2,x)) = \pi(t_1t_2,x)$ for every t_1, $t_2 \in G$ and $x \in X$.

In this context, G is called the phase group, X is called the phase space, and π is called the action of the group G on the space X. Let (G,X,π) be a ttg. Define $\pi^t : X \to X$ and $\pi_x : G \to X$ by $\pi^t(x) = \pi_x(t) = \pi(t,x)$ $(t \in G, x \in X)$. It is clear that $t \mapsto \pi^t$ is a homomorphism of the group G onto a subgroup of the group of all autohomeomorphisms of X, and that, for every $x \in X$, π_x is a continuous function on G into X. A subset B of X is called invariant under the action π of G on X provided $\pi(t,x) \in B$ for every $t \in G$ and $x \in B$. If it is clear from the context what action on X is meant, then we simply call B an invariant subset of X. For any invariant subset B of X a ttg can be defined with B as a phase space by means of the action $(t,x) \to \pi(t,x) : G \times B \to B$. To simplify the notation we denote this action also by π (instead of $\pi|_{G\times B}$). The ttg (G,B,π) is called a subsystem of (G,X,π).

Let $\underline{C}$ be a class of topological transformation groups. A ttg(G,X,π) is said to be universal with respect to $\underline{C}$ provided every member of $\underline{C}$ may be considered as a subsystem of (G,X,π). To be precise, let (H,Y,ρ) and (H,Z,σ) be ttgs. An embedding of (H,Y,ρ) en (H,Z,σ) is a topological embedding $f: Y \to Z$ such that

$$f(\rho(t,y)) = \sigma(t,f(y))$$

for every $t \in H$ and $y \in Y$. So a topological embedding $f: Y \to Z$ is an embedding of the topological transformation groups if and only if, for every $t \in H$, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\rho^t} & Y \\ \downarrow f & & \downarrow f \\ Z & \xrightarrow{\sigma^t} & Z \end{array}$$

Now a ttg (G,X,π) is said to be <u>universal</u> with respect to a class $\underset{\sim}{C}$ of ttgs if and only if every member of $\underset{\sim}{C}$ has G as a phase group and is embeddable in (G,X,π). Observe, that we do <u>not</u> require (G,X,π) to belong to $\underset{\sim}{C}$. When looking for universal systems one desires to have $\underset{\sim}{C}$ large and (G,X,π) easily describable. The classical result concerning this problem with $G = \mathbb{R}$ is due to Bebutov, which in a generalization by Kakutani [12] reads as follows:

Let $C(\mathbb{R})$ be the space of continuous, real valued bounded functions on $\mathbb{R}$, endowed with the compact-open topology, and let $\pi : \mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R})$ be the translation operator, defined by $\pi(t,f)(s) = f(s+t)$ $(f \in C(\mathbb{R}), t \in \mathbb{R}$ and $s \in \mathbb{R})$. Then the ttg $(\mathbb{R},C(\mathbb{R}),\pi)$ is universal for the class of all ttgs $(\mathbb{R},Y,\rho)$ with Y a compact metrizable space, and ρ such that the set of invariant points of $(\mathbb{R},Y,\rho)$ is homeomorphic to a subset of $\mathbb{R}$.

This result is generalized by Hájek [8], who exhibited, for every $n \in \mathbb{N}$, a ttg which is universal for the class of all ttgs $(\mathbb{R},Y,\rho)$ with Y a <u>locally</u> compact separable metrizable space, and ρ such that the set of invariant points is homeomorphic with a closed subset of $\mathbb{R}^n$.

Finally, Carlson[4] exhibits a ttg which is universal with respect to the class of all ttgs $(\mathbb{R},Y,\rho)$ with Y a seperable metrizable space. The phase space of this universal ttg is a subspace of $C(\mathbb{R}^2)$, endowed with the topology of uniform convergence, and the action is a "weighted" translation.

In these examples only actions of the topological group $\mathbb{R}$ are considered. For topological groups other than $\mathbb{R}$ there is the following chain of results.

Extending results of J. de Groot and others, in Ch.3 of [2] many examples of universal topological transformation (semi) groups are given (mainly with discrete phase groups). The case of actions of a certain class of locally compact groups is handled in Ch.4 of [2] and in [3].

Using a result of A.B. Paalman-de Miranda [14], the main theorem of [3] may be stated in the following way.
Let G be a locally compact, σ-compact topological group and let κ be a cardinal number. Then there is a real Hilbert space K and a group isomorphism $L : G \to GL(K)$ with the following universal property: for every ttg (G,X,π) with X a metrizable space of weight $\leq \kappa$, there is a topological embedding $f : X \to K$ such that

$$\forall t \in G : L(t) \circ f = f \circ \pi^t.$$

Here K is the space of all square (Haar-) integrable functions on G into a fixed real Hilbert space of dimension κ (equivalently: of weight κ) and, for every $t \in G$, $L(t)$ is a "weighted" translation operator. It is well-known that this Hilbert space K is topologically isomorphic with a direct sum of κ copies of $L^2(G)$, the ordinary space of all real valued square-integrable functions on G. In [15] this representation of K is used to show that $L : G \to GL(K)$ is a topological isomorphism, provided $GL(K)$ is endowed with the strong operator topology. In addition, if $\rho : G \times K \to K$ is defined by $\rho(t,\xi) = L(t)\ (\xi)\ (\xi\in K, t\in G)$, then ρ is shown to be continuous. Consequently, (G,K,ρ) is a ttg, and the result of Baayen and de Groot mentioned above states, in fact, that (G,K,ρ) is universal with respect to the class of all ttgs (G,X,π) with X is a metrizable space of weight $w(X) \leq \kappa$.
In order to obtain universal ttgs with less untransparant phase spaces the question may be raised for which classes of ttgs there is a universal ttg with $L^2(G)$ as a phase space. In Theorem 3.5 of [15] it is proved that for every σ-compact, locally compact group G, such that the dimension λ of $L^2(G)$ is not finite)*, there is a ttg $(G,L^2(G),\bar{\rho})$ which is universal for the class of all ttgs (G,X,π) with X a metrizable space of weight $w(X) \leq \lambda$. However, this theorem is rather unsatisfactory, since it is merely an existence theorem, and nothing can be said about the action $\bar{\rho}$ in this universal ttg.

In this paper we describe universal ttgs with respect to actions of arbitrary infinite locally compact groups G. Extending methods of

)* i.e. G is not finite

Carlson [4] we construct in Section 4 a ttg F_G with phase space $C(G\times G)$, the space of all real valued bounded continuous functions on $G \times G$ endowed with the compact-open topology, on which G acts by means of translations. This ttg F_G is universal for the class of all ttgs (G,X,ρ) with X a completely regular space of weight $w(X) \leq L(G)$, where $L(G)$ denotes the Lindelöf degree of G. Since $L(G)$ is infinite for the groups G under consideration, all actions of these groups on separable metrizable spaces X are included in the class for which F_G is universal.

As an application we find that the concept of boundedness, defined by Carlson in [4] and adapted to our needs in Section 3 below, is exactly what is needed for a ttg with locally compact phase group H to be embeddable in a ttg with a compact phase space. If H has the additional property of being σ-compact and infinite, then this compact phase space may be chosen to be metrizable if we consider bounded actions of H on separable metrizable spaces. This generalizes results of J. de Groot and R.H. McDowell [7].

Using the results of Section 4 we are able to exhibit in Section 5 a ttg H_G with phase space $L^2(G\times G)$ which is universal for the class of all ttgs (G,X,ρ) with G an infinite, σ-compact locally compact group, X a separable metrizable space and ρ a bounded action.

The ttg F_G is more or less a generalization of the Bebutov system and the system of Carlson (cf.[4], [8] and [12]), and the ttg H_G is related to the universal system of Baayen and de Groot (cf.[2], [3] and [15]). H_G may be considered as rather nice, since it has a Hilbert space as a phase space, and G acts on it by means of invertible bounded linear operators, which are, however, rather difficult to handle (cf. Definition 5.1). The ttg F_G has only a complete locally convex topological vector space (a Frechet space if G is σ-compact) as a phase space, but it is universal with respect to a class of ttgs which is, in general, more extensive than that of H_G.

2. Preliminaries and conventions

As usual, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, integers and positive integers respectively. We shall use $\mathbb{R}$ and $\mathbb{Z}$ mainly as examples of locally compact, σ-compact groups.
The symbol "$\subset$" is always used as a strict inclusion. The formula "$A \subset B$ or $A = B$" is abbreviated "$A \subseteq B$". The cardinality of a set A is denoted $|A|$; in particular, $\aleph_0 := |\mathbb{N}|$.
Every topological space under consideration will be a Hausdorff space. The interior (resp. closure) of a set A in a topological space will be denoted by int(A) or intA (resp. $\bar{A}$ or A^-). Whenever X is a topological space, C(X) will denote the space of all bounded continuous real valued functions on X, endowed with the compact-open topology. Recall that a base for the neighbourhood system of an element $f \in C(X)$ is formed by all sets

$$U_f(C,\varepsilon) := \{g \mid g \in C(X) \ \& \ \forall x \in C : |g(x) - f(x)| < \varepsilon\}$$

with C a compact subset of X and $\varepsilon > 0$.
It is well-known that C(X) is a locally convex real vector space, complete in the corresponding additive uniformity. Observe, that C(X) is metrizable (i.e. a Frechet space) if and only if 0 has a countable neighbourhood base, if and only if $X = \bigcup\{D_n \mid n \in \mathbb{N}\}$ with each D_n compact such that, for every compact subset $C \subseteq X$ we have $C \subseteq D_n$ for some $n \subseteq \mathbb{N}$. Now a locally compact space X is σ-compact if and only if $X = \bigcup\{U_n \mid n \in \mathbb{N}\}$ with each U_n open, $\bar{U}_n$ compact and $\bar{U}_n \subseteq U_{n+1}$ (X is σ-compact, by definition, whenever X is a countable union of compact sets; to construct the sets U_n by induction, use the fact that in a locally compact space every compact set is contained in an open set with compact closure). In this case the sets $D_n = \bar{U}_n$ satisfy the condition, mentioned above. Consequently we have for any locally compact space X:

$$C(X) \text{ is metrizable} \iff X \text{ is } \sigma\text{-compact}.$$

A topological group is a group G endowed with a Hausdorff topology such that the mapping $(s,t) \mapsto st^{-1}: G \times G \to G$ is continuous. In the sequel this mapping will be denoted by p. A topological group endowed with a locally compact Hausdorff topology will simply be called a locally compact group. The identity element of a topological group G will be denoted by e_G; if there is no danger for confusion we simply shall write e instead of e_G. The neighbourhood system of the identity in a topological group will be denoted by $\mathcal{V}_e$. The <u>right</u> (<u>left</u>) <u>uniformity</u> in a topological group G is the uniformity which has as a base all sets of the form $\{(s,t) \mid (s,t) \in G \times G \,\&\, st^{-1} \in U\}$ ($\{(s,t) \mid (s,t) \in G \times G \,\&\, s^{-1}t \in U\}$, respectively) with $U \in \mathcal{V}_e$.
The right Haar measure in a locally compact group G will be denoted by μ_G or μ if it is clear from the context which group we consider. Instead of $\int_G f d\mu_G$ we mostly write $\int_G f(t)dt$. The space of (equivalence classes of) μ_G-square integrable real valued functions on G is denoted by $L^2(G)$. In the usual way this space is endowed with the structure of a Hilbert space. The norm of an element $f \in L^2(G)$ will be denoted by $||f||$; it is defined by

$$||f||^2 = \int_G |f|^2 d\mu_G.$$

Recall that the <u>weight</u> of a topological space X is defined by

$$w(X) := \min \{|\mathcal{B}| \mid \mathcal{B} \text{ is an open base for } X\}$$

and that the <u>Lindelöff degree</u> of X is defined by

$$L(X) := \min \{\kappa \mid \text{each open covering of } X \text{ has a subcovering of cardinality } \kappa\}.$$

It is well-known that $L(X) \le w(X)$ (for a systematic treatment of cardinal numbers reflecting topological properties of topological spaces, and the relations between them we refer to [11]). A family of subsets of X is said to be <u>locally finite</u> whenever each $x \in X$ has a neighbourhood which meets only finitely many members of that family. In [16] it is shown that in any topological space X a locally finite, disjoint family $\mathcal{W}$ of non-void open subsets of X has cardinality $|\mathcal{W}| \le L(X)$.

If G is a locally compact group, then G has a locally finite disjoint family of non-void open subsets of cardinality L(G) if and only if G is non-compact. For further reference we state this result in the following form (cf. [16]):

P R O P O S I T I O N 2.1. Let G be a non-compact locally compact group. For every cardinal number $\kappa \leq L(G)$ there is a locally finite disjoint family of non-void open subsets of G with cardinality κ.

To give an idea about the values of L(G) for various groups we mention the following facts: $L(G) < \aleph_0$ if and only if G is finite (hence discrete); for locally compact groups G, $L(G) \leq \aleph_0$ if and only if G is σ-compact and, finally, for metrizable groups G we have $L(G) = w(G)$ (Conversily, if G is σ-compact and $L(G) = w(G)$, then G is metrizable, since $w(G) \leq \aleph_0$).

Let G be a locally compact group. A weight function on G is an element $w \in L^2(G)$ with the following properties:

(i) $\forall t \in G : w(t) > 0$

(ii) $\forall (s,t) \in G \times G : w(st) \geq w(s)\, w(t)$

(iii) The function $t \mapsto w(t)^{-1} : G \to \mathbb{R}$ is bounded on every compact subset of G.

Changing the value of w at e it may always be supposed that $w(e) = 1$, but we shall not use this fact. In a modified form, weight functions were introduced by P.C. Baayen. In [2], [3] and [14] an element $w \in L^2(G)$ is called a proper weight function if $w(e) = 1$ and if w satisfies conditions (i) and (ii) above.
As to the existence of weight functions we have

P R O P O S I T I O N 2.2. A locally compact group G admits a weight function if and only if G is σ-compact.
P R O O F . Cf. [14] and the remarks preceding Theorem B in [15].

E X A M P L E S . 1°. If G is compact, a weight function w on G is given by $w(t) = 1$ $(t \in G)$.

2°. For $G = \mathbb{R}$ weight functions are given by $w(t) = 2^{-|t|}$ and by $w_\alpha(t) = (1+|t|)^{-\alpha}$, $\alpha > \frac{1}{2}$.

3°. For $G = \mathbb{Z}$ a weight function is given by $w(n) = 2^{-n}$.

4°. For $i = 1, \ldots, n$, let G_i be a σ-compact locally compact group with weight function w_i. Then a weight function on $G_1 \times \ldots \times G_n$ is given by $w(s_1,\ldots,s_n) = w_1(s_1) \ldots w_n(s_n)$.

For other examples we refer to [3].

We need weight functions to construct ttgs of the following form:

P R O P O S I T I O N 2.3. *Let H be a locally compact, σ-compact topological group and let w be a fixed weight function on H. Define, for every $t \in H$ and $f \in L^2(H)$ the function $\sigma^t(f)$ on H by*

$$\sigma^t(f)\ (s) = \frac{v(s)}{v(st)}\ f(st) \qquad (s \in H).$$

Then $\sigma : (t,f) \mapsto \sigma^t(f)$ maps $H \times L^2(H)$ continuously onto $L^2(H)$, and $(H, L^2(H), \sigma)$ is a topological transformation group.

P R O O F . Cf. [15], Corollary 2.1.

In Section 5 we show that a ttg derived from a system of this kind is universal for a certain class of ttgs. The disadvantage of this system is, of course, the occurrence of the weight function in its definition (cf. Definition 5.1). It should be noticed that, in the ttg of Proposition 2.3, H acts on the Hilbert space $L^2(H)$ by means of invertible bounded linear operators, since each σ^t is linear, invertible and bounded such that $v(t) \le ||\sigma^t|| \le v(t)^{-1}$ $(t \in H)$.

Our next ttg is a generalization of the Bebutov universal system:

P R O P O S I T I O N 2.4. *Let H be a locally compact group and let, for every $t \in H$ and $f \in C(H)$, the function $\rho^t(f)$ on H be defined by*

$$\rho^t(f)\ (s) = f(st) \qquad (s \in H).$$

Then $\rho : (t,f) \mapsto \rho^t(f)$ maps $H \times C(H)$ continuously onto $C(H)$, and $(H, C(H), \rho)$ is a ttg.

P R O O F . This is well-known (cf. [6], Definition 1.63). For completeness we insert a proof of the continuity of ρ.
Let $(t_0,f_0) \in H \times C(H)$ and consider the neighbourhood $U_{\rho(t_0,f_0)}(C,\varepsilon)$ of $\rho(t_0,f_0)$ in $C(H)$ ($\varepsilon>0$, $C\subseteq H$ compact). Fix a compact neighbourhood V_0 of e in H. Then Ct_0V_0 is compact, hence f_0 is uniformly continuous on Ct_0V_0 with respect to the restriction of the left uniformity of H to Ct_0V_0. Consequently, there is a symmetrical neighbourhood V_1 of e, $V_1 \subseteq V_0$, such that

$$(2.1) \qquad |f_0(st) - f_0(st_0)| < \frac{\varepsilon}{2}$$

for every $s \in C$ and every $t \in t_0V_1^{-1} = t_0V_1$.
Now let $f \in U_{f_0}(Ct_0V_0,\frac{\varepsilon}{2})$. Then, for every $s \in C$ and $t \in t_0V_1$, we have $st \in Ct_0V_0$, so that

$$|f(st) - f_0(st_0)| \le |f(st) - f_0(st)| + |f_0(st) - f_0(st_0)| < \varepsilon.$$

Thus $\rho(t,f) \in U_{\rho(t_0,f_0)}(C,\varepsilon)$ for every (t,f) in the neighbourhood $t_0V_1 \times U_{f_0}(Ct_0V_0,\frac{\varepsilon}{2})$ of (t_0,f_0).

3. The concept of boundedness

L E M M A 3.1 *Let (G,X,π) be a ttg with X a uniform space with uniformity $\mathcal{U}$. The following conditions are equivalent:*

(i) $\forall \alpha \in \mathcal{U}, \exists U \in \mathcal{V}_e : (t,x) \in U \times X \Rightarrow (\pi^t(x),x) \in \alpha.$

(ii) *The family $\{\pi_x \mid x \in X\}$ of functions from G into X is equicontinuous at e.*

(iii) *The family $\{\pi_x \mid x \in X\}$ of functions from G into X is equi-uniformly continuous on G, provided G is endowed with its right uniformity.*

P R O O F . (i) $\Longleftrightarrow$ (ii). Obvious from the definition of equicontinuity.
(i) $\Longrightarrow$ (iii). Let $\alpha \in \mathcal{U}$. There is a $U \in \mathcal{V}_e$ such that, for every $t \in U$, we have $(\pi^t(x),x) \in \alpha$ for all $x \in X$. In particular, if we replace x by $\pi^s x$ ($s \in G$), then we obtain $(\pi^{ts}(x), \pi^s(x)) \in \alpha$ for every $x \in X$, $s \in G$ and $t \in U$. This means precisely that $(\pi_x(u),\pi_x(s)) \in \alpha$ for every $x \in X$, provided $(u,s) \in G \times G$ satisfies $us^{-1} \in U$.
(iii) $\Longrightarrow$ (ii). Trivial.

D E F I N I T I O N 3.1. Let (G,X,π) be a ttg with X a completely regular topological space (that is, X is uniformizable). (G,X,π) is said to be *bounded with respect to the uniformity* $\mathcal{U}$ provided $\mathcal{U}$ is compatible with the topology of X, and one of the conditions of Lemma 3.1 holds. The ttg (G,X,π) is said to be *bounded* if it is bounded with respect to some uniformity $\mathcal{U}$. If X is metrizable and if the uniformity $\mathcal{U}$ may be chosen to be a metrical uniformity (i.e. a uniformity with a countable base), then (G,X,π) is called *metrically bounded.*

R E M A R K . This definition is adapted from Carlson [4]. His concept of boundedness is what we call metric boundedness. It should be noticed that for ttgs with metrizable phase space boundedness may not imply metric boundedness, even not in the case that the phase group is an infinite, σ-compact locally compact group. However, we shall prove in Section 4 that in that case (i.e. an infinite, σ-compact locally compact group) both concepts coincide if the phase space is a *separable* metrizable space. We now exhibit an example of a ttg with a non-compact,

σ-compact locally compact phase group and a (non-separable) metrizable phase space, which is bounded but not metrically bounded.

E X A M P L E . Let $\{(H_\rho, Y_\rho, \rho)\}_{\rho\in A}$ be an uncountable collection of topological transformation groups with the following properties;

(a) For every $\rho \in A$, Y_ρ is a compact metrizable space, say with metric d_ρ and metrical uniformity $\mathcal{U}_\rho$.

(b) For every $\rho \in A$ and $(x,y) \in Y_\rho \times Y_\rho$, an element $t \in H_\rho$ exists such that $\rho(t,x) = y$.

(c) For every $\rho \in A$, H_ρ is a σ-compact, locally compact group. In addition, a finite number, but at least one, of the groups H_ρ is non-compact.

(Such collections do exist. Indeed, let $Y_\rho = \mathbb{T} := \{\lambda \mid \lambda \in \mathbb{C} \;\&\; |\lambda| = 1\}$ for every $\rho \in A$. Now fix $\sigma \in A$, let $H_\sigma = \mathbb{R}$ and let σ^t be the rotation of $Y_\sigma = \mathbb{T}$ over $2\pi t$ radians ($t \in H_\sigma = \mathbb{R}$). For all other $\rho \in A$, let $H_\rho = \mathbb{T}$ and $\rho(t,y) = ty$ (product in $\mathbb{T}$) whenever $(t,y) \in H_\rho \times Y_\rho$).
First we observe, that for every $\rho \in A$ the ttg (H_ρ, Y_ρ, ρ) is (metrically) bounded. This follows from the compactness of the phase space Y_ρ (cf. Proposition 3.1 below). However, we cannot replace the compactness of the spaces Y_ρ in our hypothesis by boundedness of the ttgs (H_ρ, Y_ρ, ρ) since we shall need the compactness of Y_ρ to ensure that, for every $\rho \in A$, $\mathcal{U}_\rho$ is the <u>unique</u> uniformity compatible with the topology of Y_ρ. We will use this fact without further reference.

Now let X be the topological direct sum (disjoint union) of the spaces Y_ρ, and let $i_\rho : Y_\rho \to X$ be the canonical injection ($\rho\in A$). Then X is a non-compact metrizable space. Next, let $G := \Pi\{H_\rho \mid \rho \in A\}$, the cartesian product group, endowed with the usual product topology. Then G is a non-compact, σ-compact, locally compact group. Finally, define $\pi : G \times X \to X$ in such a way that, for all $\sigma \in A$,

$$\pi(t, i_\sigma(x)) = i_\sigma[\sigma(t_\sigma, x)]$$

whenever $x \in Y_\sigma$ and $t = (t_\rho)_{\rho\in A} \in G$. Then π is continuous, and it is easy to see that (G, X, π) is a ttg.

Moreover, (G,X,π) is bounded. Indeed, for $\rho \in A$ and $n \in \mathbb{N}$, let

$$U(\rho,n) := \{(x,y) \mid (x,y) \in Y_\rho \times Y_\rho \ \& \ d_\rho(x,y) < \frac{1}{n}\},$$

and, for every finite subset B of A, let

$$V(B,n) := \bigcup_{\rho\in B} (i_\rho \times i_\rho)\,[U(\rho,n)] \cup \bigcup_{\rho\in A\setminus B} i_\rho(Y_\rho) \times i_\rho(Y_\rho)$$

Then $\mathcal{B} := \{V(B,n) \mid n \in \mathbb{N} \ \& \ B \subseteq A \text{ finite}\}$ is a base for a uniformity $\mathcal{U}$ in X which is compatible with the topology of X. Using the fact, that for any $\alpha \in \mathcal{U}$ we have $(i_\rho \times i_\rho)^{-1}(\alpha) \neq Y_\rho \times Y_\rho$ for at most finitely many $\rho \in A$ and that for those ρ's the ttg (H_ρ, Y_ρ, ρ) is bounded, it is easy to see that the family $\{\pi_x \mid x \in X\}$ of functions on G into X is equicontinuous at e, that is: (G,X,π) is bounded with respect to $\mathcal{U}$.

However, it is not difficult to show that $\mathcal{U}$ cannot have a countable base , because A is uncountable. Since X is metrizable, X has a uniformity for its topology with a countable base . Let $\mathcal{V}$ be any such a uniformity. Then for some $\beta \in \mathcal{V}$ we have $B_0 := \{\rho \mid \rho \in A \ \& \ (i_\rho \times i_\rho)^{-1}(\beta) \neq Y_\rho \times Y_\rho\}$ is infinite; otherwise $\mathcal{V}$ would equal the uniformity $\mathcal{U}$ described above, which has not a countable base. Now let V be a neighbourhood of e in G, say $V = \Pi\{V_\rho \mid \rho \in A\}$ with $V_\rho \neq H_\rho$ for only a finite number of ρ's in A. Since B_0 is an infinite subset of A, there is some $\sigma \in B_0$ such that $V_\sigma = H_\sigma$. Now take $t = (t_\rho)_{\rho\in A}$ such that $t_\rho \in V_\rho$ whenever $\rho \neq \sigma$ (for example, t_ρ is the identity of H_ρ) and take $t_\sigma \in V_\sigma$ such that $(\sigma(t_\sigma,y),y) \notin (i_\sigma \times i_\sigma)^{-1}(\beta)$ for some $y \in Y_\sigma$ (that this is possible follows from the hypothesis (b) about the ttgs (H_ρ,Y_ρ,ρ)). Then for $x := i_\sigma(y)$ we have $(\pi(t,x),x) = (i_\sigma(\sigma(t_\sigma,y)), i_\sigma(y)) \notin \beta$. Hence (G,X,π) is not bounded with respect to the uniformity $\mathcal{V}$. Since $\mathcal{V}$ was an arbitrary uniformity with countable base this proves that (G,X,π) cannot be metrically bounded.

P R O P O S I T I O N 3.1. <u>Let (G,X,π) be a ttg which is embeddable in a ttg (G,Y,ρ) with a compact phase space Y. Then (G,X,π) is bounded. If X is metrizable and Y can be chosen to be metrizable, then (G,X,π) is metrically bounded.</u>

P R O O F (Cf. Carlson [4]). An elementary compactness argument shows that the uniformity $\mathcal{U}$ of Y satisfies condition (i) of Lemma 3.1.

Hence (G,X,π) is bounded with respect to the relative uniformity of X in $(Y,\mathcal{U})$. The second statement of the Proposition is a trivial corollary of the first.

C O R O L L A R Y 3.1 (Carlson [4]). Let X be a locally compact topological space. Then any ttg (G,X,π) is bounded. If, in addition, X is separable and metrizable, then any ttg (G,X,π) is metrically bounded.

P R O O F . Let $Y := X \cup \{\infty\}$ be the one-point compactification of X. Notice that Y is metrizable whenever X is separable and metrizable. Define $\rho : G \times Y \to Y$ by the equations

$$\rho(t,x) = \begin{cases} \pi(t,x) & \text{whenever } x \in X \\ \infty & \text{for } x = \infty, \end{cases}$$

$(t \in G)$. Then ρ is continuous and (G,Y,ρ) is a ttg in which $(G,X,\pi)$is embeddable by the cannonical injection of X into Y. Now apply Proposition 3.1.

R E M A R K S . 1°. Other examples of bounded ttgs (G,X,π) are provided if we take discrete phase groups G. Such ttgs are bounded with respect to any uniformity for X. In particular, if X is metrizable, (G,X,π) is metrically bounded.

2°. In Section 4 we shall prove the converse of Proposition 3.1 for ttgs (G,X,π) with G a non-compact locally compact group and X a completely regular space of weight $w(X) \leq L(G)$, the Lindelöff degree of G.

We conclude this Section with a discussion of bounded subsystems of the ttg $(H,C(H),\rho)$ defined in Proposition 2.4 (H a locally compact group). A subset Ω of $C(H)$ is said to be uniformly bounded provided there is an $M > 0$ such that $\sup\{|f(t)| \mid t \in H\} \leq M$ for every $f \in \Omega$. In $C(H)$ we will consider only the additive uniformity $\mathcal{U}_a$, related with the compact-open topology in $C(H)$ and the structure of $C(H)$ as a topological vector space in the usual way. A base for this uniformity is formed by the sets

$$\{(f,g) \mid (f,g) \in C(H) \times C(H) \ \& \ \forall s \in K : |f(s) - g(s)| < \varepsilon\}$$

with $K \subseteq H$ compact and $\varepsilon > 0$.
For convenience the restriction of the uniformity $\mathcal{U}_a$ to a subset of $C(H)$ will also be called the additive uniformity (in that subset) and denoted by $\mathcal{U}_a$.

P R O P O S I T I O N 3.2 <u>Let</u> H <u>be a locally compact group and let</u> Ω <u>be a uniformly bounded invarient subset of</u> $C(H)$. <u>The following conditions are equivalent:</u>

(i) <u>The subsystem</u> (H,Ω,ρ) <u>of</u> $(H,C(H),\rho)$ <u>is bounded with respect to</u> $\mathcal{U}_a$.
(ii) Ω <u>is equicontinuous on</u> H.
(iii) <u>The ttg</u> (H,Ω,ρ) <u>is embeddable in a ttg</u> (H,Y,σ) <u>with compact phase space</u> Y.

<u>If one of these conditions holds and if, in addition</u> H <u>is</u> σ-<u>compact, then</u> Y <u>in</u> (iii) <u>may supposed to be metrizable</u>.

P R O O F . (i) $\Longrightarrow$ (ii). Boundedness of (H,Ω,ρ) with respect to $\mathcal{U}_a$ means that for every compact $K \subseteq H$ and every $\varepsilon > 0$ there is a $U \in \mathcal{V}_e$ such that, for all $(t,f) \in U \times \Omega$, we have

$$\forall s \in K : |f(st) - f(s)| < \varepsilon.$$

In particular, if we fix $s \in H$ and take $K = \{s\}$, then the corresponding $U \in \mathcal{V}_e$ satisfies the condition

$$|f(u) - f(s)| < \varepsilon$$

for every $f \in \Omega$ and every $u \in sU$. Consequently, Ω is equicontinuous at every point $s \in H$.
(ii) $\Longrightarrow$ (iii). Since Ω is uniformly bounded, equicontinuity of Ω implies that $\overline{\Omega}$ is compact (ASCOLI's theorem, cf. [13], p.233). Because each ρ^t is an autohomeomorphism of $C(H)$ which maps Ω onto itself, $\overline{\Omega}$ is an invariant subset of $C(H)$. Now $(H,\overline{\Omega},\rho)$ is a ttg with compact phase space in which (H,Ω,ρ) is embeddable in the obvious way. In addition, if H is σ-compact, $C(H)$ is metrizable, so that $\overline{\Omega}$ is metrizable as well.
(iii) $\Longrightarrow$ (i). Obvious from Proposition 3.1.

R E M A R K . The equivalence of (i) and (ii) can be established without any reference to ASCOLI's theorem. Bearing in mind the fact that an equicontinuous family of functions on a compact space is equiuniformly continuous, one may easily derive the boundedness of (G,Ω,ρ) from the equicontinuity of Ω along the lines of the proof of Proposition 2.4: formula (2.1) with $t_0 = e$ and f_0 replaced by all $f \in \Omega$ simultaneously is exactly what is needed. Notice that we are allowed to replace f_0 by any $f \in \Omega$ if Ω is equiuniformly continuous on CV_0.

4. The universal topological transformation group $\mathcal{F}_G$.

We begin with a describtion of the system $\mathcal{F}_G$. To this end we need the following observation.
Let (T,X,π) be a ttg, let S be a topological group, and let $\phi : S \to T$ be a continuous homomorphism. Then the function $\tilde{\pi} : S \times X \to X$, defined by $\tilde{\pi}(s,x) = \pi(\phi(s),x)$ $(s\in S, x\in X)$, is continuous, and $(S,X,\tilde{\pi})$ is a ttg (cf.[6],1.32, where $(S,X,\tilde{\pi})$ is called the (S,ϕ) restriction of (T,X,π)).
Now we consider a locally compact group G. Since $G \times G$ is locally compact as well, we may define a ttg $(G\times G, C(G\times G),\rho)$ following Proposition 2.4. If $\phi : G \to G \times G$ is defined by $\phi(s) = (s,s)$ $(s\in G)$, our universal system is, in fact, the (G,ϕ) restriction of $(G\times G, C(G\times G),\rho)$. We describe it explicitely in the following

D E F I N I T I O N 4.1. Let G be a locally compact group and let $\tau : G \times C(G\times G) \to C(G\times G)$ be defined by

$$\tau(t,f)\ (s_1,s_2) = f(s_1t,s_2t)$$

for every $t \in G$, $f \in C(G\times G)$ and $(s_1,s_2) \in G \times G$. The topological transformation group $(G,C(G\times G),\tau)$ which is defined in this way will be denoted by $\mathcal{F}_G$.

S T A N D I N G N O T A T I O N . Let X be a completely regular space of weight $w(X)$, and let I be an index set of cardinality $|I| = w(X)$. It is well known that there exists a set $\{\phi_i \mid i \in I\}$ of continuous functions on X into the interval $[0,1]$ such that, for every $x \in X$ and every neighbourhood U of x, there are a neighbourhood V of x, $V \subseteq U$, and an index $i \in I$ such that $\phi_i(y) > \frac{1}{2}$ whenever $y \in V$ and $\phi_i(y) = 0$ whenever $y \in X \setminus U$ (for finite discrete spaces X this is trivial; for the case that $w(X) \geq \aleph_0$, we refer to [5], the proof of Theorem 2.3.8). In particular, the collection $\{\phi_i \mid i \in I\}$ separates points and closed sets in X.

Let G be a non-compact, locally compact group of Lindelöf degree $L(G) \geq w(X)$; Then G has a locally finite, disjoint family of non-empty open subsets, which has cardinality $w(X)$ (cf. Proposition 2.1).
Fix such a family and use I as an index set for it, say $\{C_i \mid i \in I\}$ (so we have, in fact, established a one-to-one mapping of the set $\{\phi_i \mid i \in I\}$ of functions mentioned above on this locally finite, disjoint family of non-empty open subsets of G).

Fix, for every $i \in I$, $t_i \in C_i$, and let $\psi_i : G \to [0,1]$ be a continuous function such that $\psi_i(t_i) = 1$ and $\psi_i(t) = 0$ for every $t \in G \setminus C_i$. In addition, let for every $i \in I$,

$$B_i := p^{-1}(C_i) = \{(s,t) \mid (s,t) \in G \times G \ \& \ st^{-1} \in C_i\}.$$

Observe that $\{B_i \mid i \in I\}$ is a locally finite, disjoint family of non-empty open subsets of $G \times G$ (p is continuous).

Finally, let $\rho : G \times X \to X$ be a continuous function. Then, for every $x \in X$, a function $\Psi(x) : G \times G \to [0,1]$ can be defined by

$$\Psi(x)\ (s,t) = \sum_{i \in I} \psi_i(st^{-1})\ \phi_i[\rho(st^{-1}s,x)]$$

$((s,t) \in G\times G)$. Indeed, $\psi_i(st^{-1}) = 0$ for $(s,t) \notin B_i$ and the sets B_i are pairwise disjoint.

In the case that G is an infinite compact topological group and $w(X) \leq L(G) = \aleph_0$ we may take $I = \mathbb{N}$ and for $\{C_i \mid i \in I\}$ we may take a disjoint sequence of non-empty open subsets of G(such a sequence exists whenever G is infinite). With ϕ_i and ψ_i defined as before we define $\Psi(x)$ in this case by

$$(*) \qquad \Psi(x)\ (s,t) = \sum_{i=1}^{\infty} 2^{-i}\ \psi_i(st^{-1})\ \phi_i[\rho(st^{-1}s,x)]$$

$(x \in X,\ (s,t) \in G\times G)$.

(In both cases, the functions $\Psi(x)$ replace the functions g_x of Carlson in Theorem 2 of [4]).

L E M M A 4.1. <u>For every</u> $x \in X$ <u>we have</u> $\Psi(x) \in C(G\times G)$.

P R O O F . Let $x \in X$. If $\Psi(x)$ is defined by $(*)$, observe that the right-hand member of $(*)$ is a series whose terms are continuous functions on $G \times G$ and which converges uniformly on $G \times G$. Consequently, $\Psi(x)$ is continuous on $G \times G$.

In the other case, observe that for every $j \in I$ the continuous function

$$\Psi_j(x) : (s,t) \mapsto \psi_j(st^{-1})\ \phi_j[\rho(st^{-1}s,x)]$$

vanishes outside B_j, while $\{B_i \mid i \in I\}$ forms a locally finite family. Consequently, the pointwise sum $\sum\{\Psi_i(x) \mid i \in I\}$ of these functions is continuous on $G \times G$, i.e. $\Psi(x) \in C(G\times G)$.

In the following lemma's and theorems proofs are given <u>only for the case that G is non-compact</u>, because the proofs for the case that $\Psi(x)$ is defined by (*) are almost literally the same.

L E M M A 4.2. <u>The mapping</u> $\Psi : x \mapsto \Psi(x) : X \to C(G\times G)$ <u>is continuous</u>.

P R O O F . Let $x \in X$ and consider the neighbourhood $U_{\Psi(x)}(K,\varepsilon)$ of $\Psi(x)$ in $C(G\times G)$ ($\varepsilon > 0$ and $K \subseteq G \times G$ compact). Because $\{B_i \mid i \in I\}$ is a locally finite family of subsets of $G \times G$, and because K is compact, there is a <u>finite</u> subset $I_0 \subseteq I$ such that

(4.1) $\quad K \cap B_i = \emptyset$ whenever $i \in I \setminus I_0$.)*

Now there is a neighbourhood U of x in X such that

(4.2) $\quad |\phi_i[\rho(st^{-1}s,x)] - \phi_i[\rho(st^{-1}s,y)]\ | < \varepsilon$

for every $y \in U$, $(s,t) \in K$ and $i \in I_0$. Indeed, for every $i \in I_0$, the

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)* In the case that Ψ is defined by (*), take I_0 such that $\sum_{i\in I\setminus I_0} 2^{-i} < \varepsilon$.

function $(u,y) \mapsto \phi_i[\rho(u,y)]$ is continuous on $G \times X$. Hence for every $u \in K^*: = \{st^{-1}s \mid (s,t) \in K\}$ there is a neighbourhood V_u of x in X and a neighbourhood W_u of u in G such that, for every $i \in I_0$, we have $|\phi_i[\rho(u,x)] - \phi_i[\rho(v,y)]| < \frac{\varepsilon}{2}$ whenever $y \in V_u$ and $v \in W_u$. Since K^* is compact, we may cover K^* with finitely many of the sets W_u; then the desired neighbourhood U of x is the intersection of the corresponding neighbourhoods V_u of x.

From (4.1) and (4.2) it follows easily that, for every $y \in U$,

$$|\Psi(x)\ (s,t) - \Psi(y)\ (s,t)| < \varepsilon$$

whenever $(s,t) \in K$. Hence $\Psi(U) \subseteq U_{\Psi(x)}\ (K,\varepsilon)$, and the continuity of Ψ at the point $x \in X$ is established.

L E M M A 4.3. <u>If we assume that</u> $\rho : G \times X \to X$ <u>has the additional property that</u> $\rho(e,x) = x$ <u>for every</u> $x \in X$, <u>then</u> Ψ <u>is relatively open and injective</u>.

P R O O F . It is sufficient to show that, for any $x \in X$ and any neighbourhood U of x in X there are a compact set $D \subseteq G \times G$ and an $\varepsilon > 0$ such that

$$\{y \mid y \in X \,\&.\, \Psi(y) \in U_{\Psi(x)}\ (D,\varepsilon)\} \subseteq U. \tag{4.3}$$

Indeed, if (4.3) is satisfied, $\Psi(O)$ is open in $\Psi(X)$ for every open $O \subseteq X$. In addition, if $x,y \in X$, $x \neq y$, then $U: = X \setminus \{y\}$ is an open neighbourhood of x in X, and it follows from (4.3) that some neighbourhood of $\Psi(x)$ does not contain $\Psi(y)$, so that $\Psi(x) \neq \Psi(y)$.

We proceed by proving (4.3). Let $x \in X$ and let U be an open neighbourhood of x in X. By the special choice of the functions ϕ_i there is an index $j \in I$ such that $\phi_j(x) > \frac{1}{2}$ and $\phi_j(y) = 0$ for every $y \in X \setminus U$. Hence for all $y \in X \setminus U$ we have

$$|\phi_j[\rho(e,x)] - \phi_j[\rho(e,y)]| > \frac{1}{2}. \tag{4.4}$$

Recall that we fixed $t_j \in C_j$ such that $\psi_j(t_j) = 1$. Now let $s_0 := t_j^{-1}$ and $t_0 := s_0^2$. Then we have $s_0 t_0^{-1} s_0 = e$, and $s_0 t_0^{-1} = t_j$, so that $(s_0, t_0) \in B_j$ and $\psi_j(s_0 t_0^{-1}) = 1$. Consequently, it follows from (4.4) that

$$(4.5) \qquad |\Psi(x)\,(s_0,t_0) - \Psi(y)\,(s_0,t_0)| > \frac{1}{2}$$

for every $y \in X \setminus U$. Hence to prove (4.3) it suffices to take $D = \{(s_0,t_0)\}$ and $\varepsilon = \frac{1}{2}$.

R E M A R K . In the preceding Lemma's we have not used the local compactness of G explicitly , we used only the fact that G has a suitable disjoint family $\{C_i \mid i \in I\}$ of non-empty open subsets such that $|I| = w(X)$. It should also be noticed that Ψ is continuous and (in the case of Lemma 4.3) relatively open and injective, i.e. a topological embedding, if $C(G\times G)$ is endowed with the point-open topology (in Lemma 4.3 we used only a one-point compact set D). So in this case <u>the compact-open topology and the point-open topology coïncide on</u> $\Psi(X)$.
However, the mapping $\tau : G \times C(G\times G) \to C(G\times G)$ is, in general, not continuous if G is non-discrete and if $C(G\times G)$ is endowed with the point-open topology. On the other hand, for equicontinuous invariant subsets Ω of $C(G\times G)$ we have that $\tau|_{G\times\Omega} : G \times \Omega \to \Omega$ is continuous when Ω is endowed with the point-open topology, so that $(G,\Omega,\tau|_{G\times\Omega})$ is a ttg. But it is well-known that in this case the compact-open topology and the point-open topology on Ω coïncide. So the most usefull topology on $C(G\times G)$ seems to be the compact-open topology. With respect to this topology τ is continuous if G is locally compact. This is the reason why, in the following theorem, local compactness of G seems to be essential.
One final remark: equicontinuïty of $\Psi(X)$ is, in spite if these remarks, of some importance in connection with boundedness (cf. Lemma 4.4 and 4.5 below).

T H E O R E M 4.1. *Let G be an infinite locally compact group. The topological transformation group $\mathcal{F}_G$ is universal for the class of all topological transformation groups (G,X,ρ) with X a completely regular topological space of weight $w(X) \leq L(G)$.*

P R O O F . Let (G,X,ρ) be a ttg, X completely regular with $w(X) \leq L(G)$. Let $\Psi : X \to C(G\times G)$ be as before. Since ρ satisfies the conditions of Lemma 4.3, Ψ is a topological embedding of X into $C(G\times G)$. So we have only to show that, for every $t \in G$,

$$\tau^t \circ \Psi = \Psi \circ \rho^t.$$

Let $x \in X$ and $t \in G$. For any $(s_1,s_2) \in B_i$ $(i\in I)$ we have also $(s_1t,s_2t) \in B_i$, so that

$$\begin{aligned}\tau^t[\Psi(x)](s_1,s_2) &= \Psi(x)(s_1t,s_2t)\\ &= \psi_i(s_1tt^{-1}s_2^{-1}).\ \ \phi_i[\rho(s_1tt^{-1}s_2^{-1}s_1t,x)]\\ &= \psi_i(s_1s_2^{-1}).\ \ \phi_i[\rho(s_1s_2^{-1}s_1t,x)]\\ &= \psi_i(s_1s_2^{-1}).\ \ \phi_i[\rho(s_1s_2^{-1}s_1,\rho^tx)]\\ &= \Psi(\rho^tx)\ (s_1,s_2),\end{aligned}$$

and for $(s_1,s_2) \in G \times G \setminus \bigcup\{B_i \mid i \in I\}$ we have $\psi_i(s_1s_2^{-1}) = 0$, for each $i \in I$, hence $\Psi(\rho^tx)\ (s_1,s_2) = 0$. On the other hand, we also have $(s_1t,s_2t) \in G \times G \setminus \bigcup\{B_i \mid i \in I\}$, so that $\Psi(x)\ (s_1t,s_2t) = 0$, that is, $\tau^t[\Psi(x)]\ (s_1,s_2) = 0$. Consequently, in this case we also have the equality $\tau^t\ \Psi(x)\ \ (s_1,s_2) = \Psi(\rho^tx)\ (s_1,s_2)$.

R E M A R K . The class of ttgs for which F_G is universal comprises all ttgs (G,X,ρ) with X a separable metrizable space. Indeed, such a space has $w(X) \leq \aleph_0$, and for every infinite locally group G we have $L(G) \geq \aleph_0$.
If G is not only locally compact, but also σ-compact we have in Theorem 4.1 only separable metrizable spaces X. For in this case $L(G) = \aleph_0$, and every completely regular space with $w(X) \leq \aleph_0$ is

separable and metrizable (use the well-known metrization theorem of Urysohn, [13], p.125).

Bearing in mind the remarks preceding Definition 4.1, our next lemma is an obvious consequence of Proposition 3.2.

L E M M A 4.4. <u>Let Ω be a uniformly bounded invariant subset of</u> $C(G\times G)$. <u>Then the subsystem</u> (G,Ω,τ) <u>of</u> F_G <u>is embeddable in a ttg</u> (G,Y,τ') <u>with compact phase space</u> Y <u>if Ω is equicontinuous on</u> $G \times G$. <u>In that case, if</u> G <u>is σ-compact</u>, Y <u>may supposed to be metrizable</u>.

Now let (G,X,ρ) be a ttg with X a completely regular space such that $w(X) \leq L(G)$. Then $\Psi(X)$ is a uniformly bounded subset of $C(G\times G)$, which is obviously invariant (cf. the proof of Theorem 4.1: we have $\tau^t[\Psi(x)] = \Psi(\rho^t x) \in \Psi(X)$ for every $x \in X$ and $t \in G$). Thus the question may be raised when $\Psi(X)$ is an equicontinuous family of functions on $G \times G$. A close inspection of the proof of Lemma 4.1 reveals that a sufficient condition is the following:

(P) (G,X,ρ) is bounded (i.e. $\{\rho_x \mid x \in X\}$ is equicontinuous on G) with respect to some uniformity $\mathcal{U}$ for X, and, for every $i \in I$, the function $\phi_i : X \to [0,1]$ is uniformly continuous with respect to this uniformity $\mathcal{U}$.

However, the second half of this condition can always be fulfilled whenever the first half is. Indeed, the completely regular spaces are exactly the uniformizable spaces ([10], Theorem I.15), and, in addition, in a uniform space X, for any $x \in X$ and any closed $S \subseteq X$ such that $x \notin S$, there is a function $f : X \to [0,1]$ with the following properties: $f(x) = 1$, $f(y) = 0$ for all $y \in S$, and f is <u>uniformly</u> continuous. Consequently, if (G,X,ρ) is bounded with respect to a uniformity $\mathcal{U}$, the functions $\phi_i (i\in I)$ may supposed to be uniformly continuous with respect to $\mathcal{U}$ (cf. the proof of Theorem 2.3.8 in [5], where the arguments for the existence of the collection $\{\phi_i \mid i \in I\}$ with the desired properties, except uniform continuity, are given). Resuming, we have

L E M M A 4.5. Let (G,X,ρ) be a bounded ttg. Then the embedding $\Psi : X \to C(G\times G)$ may supposed to be such that $\Psi(X)$ is an equicontinuous subset of $C(G\times G)$.

R E M A R K . Conversely, if $\Psi(X)$ is equicontinuous, it follows from Proposition 3.2 and Theorem 4.1 that (G,X,ρ) is bounded. We state our results in a somewhat different form:

T H E O R E M 4.2. Let (G,X,ρ) be a topological transformation group with G an infinite locally compact group and X a completely regular space of weight $w(X) \le L(G)$. The following conditions are equivalent:

(i) (G,X,ρ) is bounded

(ii) (G,X,ρ) is embeddable in a ttg (G,Y,ρ') with compact phase space Y.

If G is σ-compact and X is separable metrizable, then Y' may supposed to be metrizable.

P R O O F (i) $\Longrightarrow$ (ii). Cf. Lemma 4.4 and Lemma 4.5

(ii) $\Longrightarrow$ (i). This is Proposition 3.1.

C O R O L L A R Y 4.1. Let G be an infinite, σ-compact locally compact group, let X be a separable metrizable space, and let $\rho : G \times X \to X$ be such that (G,X,ρ) is a ttg. The following conditions are equivalent:

(i) (G,X,ρ) is bounded

(ii) (G,X,ρ) is metrically bounded.

C O R O L L A R Y 4.2. Let G be a countable discrete group, let X be a separable metrizable space, and let $\rho : G \times X \to X$ be such that (G,X,ρ) is a ttg. Then (G,X,ρ) is embeddable in a ttg (G,Y,ρ') with a compact metrizable phase space Y.

P R O O F . Since G is discrete, (G,X,ρ) is bounded. In addition, G is σ-compact, so the result follows directly from Theorem 4.2.

R E M A R K . The essential fact in Corollary 4.2 is the metrizability of Y. Indeed, if G is discrete, one may take $Y = \beta X$, the Stone-Čech compactification of X, and one may define an action of G on Y in the obvious way, taking as $(\rho')^t$ the canonical extension of ρ^t over $Y = \beta X$ $(t \in G)$. Since G is discrete, $\rho' : G \times \beta X \to \beta X$ is continuous, and it is easy to see that $(G, \beta X, \rho')$ is a ttg in which (G,X,ρ) is embeddable. However, βX is never metrizable (unless X is a compact metrizable space). Originally, Corollary 4.2 is due to J. de Groot and R.H. McDowell [7]. Another proof is given in [2], Corollary 3.4.11, and the case $G = \mathbb{Z}$ is also handled in [1], Theorem 2.4.

For ttgs (G,X,ρ) with <u>locally compact</u> phase space X, Theorem 4.2 and Corollary 4.2 do not really give any information: such ttgs are bounded <u>because</u> they are embeddable in ttgs (G,Y',ρ') with compact phase spaces $Y' = X \cup \{\infty\}$ (one-point compactification). In that case no conditions on G are required.

Observe, that the compact phase space Y in Theorem 4.2 may supposed to be the closure of $\Psi(X)$ in the space $C(G \times G)$. In that case Y is a <u>compactification</u> of X (i.e. X is homeomorphic with a dense subset of Y) and, in addition, $w(Y) \le w[C(G \times G)]$. Since for any locally compact space Z we have $w[C(Z)] = w(Z).\aleph_0$, (cf.[16]), we may suppose that $w(Y) \le w(G \times G) = w(G)$ (for infinite G we certainly have $w(G) \ge \aleph_0$). In our next theorem we remove the condition $w(X) \le L(G)$ and the non-finiteness of G from our hypothesis.

First we introduce the concept of <u>local weight</u> of a topological group H:

$$\lambda(H) = \min \{ |\mathcal{B}_e| \mid \mathcal{B}_e \text{ is a base for } \mathcal{V}_e \} .$$

Since any topological group H is completely regular it follows from [11], 2.27, that

$$w(H) = \lambda(H) \,.\, L(H).$$

([11], 2.27 applies only to infinite spaces, but for finite discrete spaces H we have $\lambda(H) = 1$ and $w(H) = |H| = L(H)$).

T H E O R E M 4.3. *Let (H,X,ρ) be a bounded ttg with H a locally compact group and X a completely regular space. Then (H,X,ρ) is embeddable in a ttg (H,Y,ρ') such that Y is a compactification of X of weight*

$$w(Y) \leq \max \{w(H), w(X)\}.$$

P R O O F . Since for compact X the result is trivial, we may suppose that X is an infinite space. Hence $w(X) \geq \aleph_0$, and $\lambda(H) \,.\, L(H) \,.\, w(X) = \max \{w(H), w(X)\}$.

Suppose $w(X) \leq L(H)$ and H non-compact. Then it follows from Theorem 4.2 that (H,X,ρ) is embeddable in a ttg (H,Y,ρ'), where Y is a compactification X of weight $w(Y) \leq w(H) = \lambda(H) \,.\, L(H)$ (cf. the preceding remarks). Since $w(X) \leq L(H)$, we have $L(H) = L(H) \,.\, w(X)$, so that, indeed, $w(Y) \leq \lambda(H) \,.\, L(H) \,.\, w(X)$.

Suppose $w(X) > L(H)$. Let H_0 be a discrete group such that $|H_0| = w(X)$ (e.g. H_0 is a free group with $w(X)$ generators). Then $G := H \times H_0$ is a non-compact locally compact group, and trivially $L(G) \geq L(H_0) = |H_0| = w(X)$. Define $\rho'' : G \times X \to X$ by

$$\rho''((t,s),x) = \rho(t,x) \qquad ((t,s) \in H \times H_0,\ x \in X).$$

Then ρ'' is continuous, and (G,X,ρ'') is a ttg. In addition, (G,X,ρ'') is bounded because (H,X,ρ) is bounded. So we may apply the preceding result: (G,X,ρ'') is embeddable in a ttg (G,Y,ρ^*) such that Y is a compactification of X and $w(Y) \leq w(G) = w(H) \,.\, w(H_0) = \lambda(H) \,.\, L(H) \,.\, |H_0| = \lambda(H) \,.\, L(H) \,.\, w(X)$. Now define $\rho' : H \times Y \to Y$ by

$$\rho'(t,y) = \rho^*((t,e_0),y) \qquad (t\in H, y\in Y)$$

(e_0 is the identity of H_0). Then ρ' is continuous, and (H,Y,ρ') is a ttg in which (H,X,ρ) is embeddable.

Finally, suppose $w(X) \leq L(H)$ and H compact. This case can be treated in a similar way by considering $G := H \times H_0$, where $H_0 = \mathbb{Z}$.

5. The universal topological transformation group $\mathcal{H}_G$.

Throughout this section G will denote an infinite , σ-compact locally compact group. Then $\mathcal{F}_G$ is a ttg with a Frechet space as a phase space, which is universal for all ttgs (G,X,ρ) with X a separable metrizable space. The ttg $\mathcal{H}_G$ which we shall define below has a Hilbert space as a phase space, but it is universal only for the bounded ttgs (G,X,ρ) with X a separable metrizable space.

Let w denoted a fixed weight function on G. Since $G \times G$ is σ-compact and locally compact, and since $v : (s_1,s_2) \mapsto w(s_1)\, w(s_2)$ is a weight function on $G \times G$, we may define a ttg $(G\times G, L^2(G\times G),\sigma)$ according to Proposition 2.3. If we apply the principle preceding Definition 4.1 then we get our system $\mathcal{H}_G$:

DEFINITION 5.1. Let $\pi : G \times L^2(G\times G) \to L^2(G\times G)$ be defined by

$$\pi(t,f)\ (s_1,s_2) = \frac{w(s_1)\ w(s_2)}{w(s_1t)w(s_2t)}\ f(s_1t,s_2t)$$

for $(t,f) \in G \times L^2(G\times G)$ and $(s_1,s_2) \in G \times G$. The topological transformation group $(G,L^2(G\times G),\pi)$ defined in this way will be denoted by $\mathcal{H}_G$.

LEMMA 5.1. Let H be a σ-compact locally compact group, and let v denote a weight function on H. In addition, let $\omega : C(H) \to L^2(H)$ be defined by

$$\omega(f)\ (t) = v(t)\ f(t) \qquad (f\in C(H), t\in H).$$

Then ω is injective, and for every $t \in H$ we have

$$(5.1) \qquad \sigma^t \circ \omega = \omega \circ \rho^t$$

where $\sigma : H \times L^2(H) \to L^2(H)$ and $\rho : H \times C(H) \to C(H)$ are as in the Propositions 2.3 and 2.4. In addition, ω is continuous on uniformly bounded subsets of C(H).

PROOF. Let $f,g \in C(H)$, $f \neq g$. Then there is an open set in H on which f and g differ from each other. Since open subsets of H have positive Haar measure and $v(t) > 0$ for every $t \in H$, it follows that

$\omega(f)$ cannot be equal to $\omega(g)$ almost everywhere, i.e. $\omega(f) \neq \omega(g)$. So ω is injective.
The equality (5.1) can easily be proved by computation.
Finally, let Ω be a uniformly bounded subset of $C(H)$, say $|f(t)| \leq M$ for every $f \in \Omega$ and $t \in H$, and fix $f_0 \in \Omega$. Because $v \in L^2(H)$ and H is σ-compact, there is, for every $\varepsilon > 0$, a compact set $C_\varepsilon \subseteq H$ such that

$$\int_{H\setminus C_\varepsilon} v(t)^2 dt < \frac{\varepsilon^2}{8M^2} \quad .$$

Now for every $f \in U_{f_0}(C_\varepsilon, \frac{\varepsilon}{2||v||}) \cap \Omega$ we have

$$||\omega(f) - \omega(f_0)||^2 = \int_{C_\varepsilon} v(t)^2 \, |f(t) - f_0(t)|^2 \, dt +$$

$$+ \int_{H\setminus C_\varepsilon} v(t)^2 \, |f(t) - f_0(t)|^2 \, dt$$

$$< \frac{\varepsilon^2}{4||v||^2} \, ||v||^2 + 4M^2 \, \frac{\varepsilon^2}{8M^2} < \varepsilon^2 \ .$$

This proves that $\omega|_\Omega$ is continuous.

C O R O L L A R Y 5.1. <u>The mapping</u> $\omega_G : C(G\times G) \to L^2(G\times G)$ <u>defined by</u>

$$\omega_G(f)\ (s_1,s_2) = w(s_1)\ w(s_2)\ f(s_1,s_2)$$

($f\in C(G\times G)$, $(s_1,s_2)\in G\times G$) <u>is injective, it satisfies the equation</u>

$$(5.2) \qquad \pi^t \circ \omega_G = \omega_G \circ \tau^t \qquad (t\in G),$$

<u>where</u> τ^t <u>and</u> π^t <u>are as in the Definitions</u> 4.1 <u>and</u> 5.1, <u>and it is continuous on uniformly bounded subsets of</u> $C(G\times G)$.
P R O O F . Obvious.

C O R O L L A R Y 5.2. Let Ω be a compact subset of $C(G\times G)$, invariant under the action of G by τ. Then $\omega_G|_\Omega : \Omega \to L^2(G\times G)$ is an embedding of the ttg (G,Ω,τ) in the ttg $(G,L^2(G\times G),\pi)$.

P R O O F . Compact subsets of $C(G\times G)$ are uniformly bounded, so we may apply Corollary 5.1 to obtain that $\omega_G|_\Omega$ is injective and continuous. Since a continuous injection of a compact space into a Hausdorff space is a topological embedding and since ω_G satisfies (5.2), the proof is finished.

T H E O R E M 5.1. Let G be a non-compact, σ-compact and locally compact group. The ttg $\mathcal{H}_G$ is universal for the class of all bounded ttgs (G,X,ρ) with X a separable metrizable space.

P R O O F . Let (G,X,ρ) be a bounded ttg with X a separable metrizable space. By Theorem 4.2 we may suppose that X is compact. Let Ψ denote the embedding of the ttg (G,X,ρ) into the ttg $\mathcal{F}_G$ (cf. Theorem 4.1). Since $\Psi(X)$ is a compact invariant subset of $C(G\times G)$, it follows from Corollary 5.2 that $\omega_G \circ \Psi$ is an embedding of the ttg (G,X,ρ) in the ttg $\mathcal{H}_G$.

R E M A R K . It is possible to prove Theorem 5.1 directly, without any reference to the system $\mathcal{F}_G$ or the compactification theorem 4.2, by constructing the embedding mapping $\omega_G \circ \Psi$ directly following the lines, indicated in Section 4. Then the boundedness of (G,X,ρ) is needed to show that this mapping is relatively open (in the present proof of Theorem 5.1, boundedness of (G,X,ρ) is used in the assumption that X is compact; then the fact that the embedding mapping is relatively open follows from its continuity and its injectiveness).
It is clear from the proof of Theorem 5.1 which we presented above, that any bounded ttg (G,X,ρ) with X a separable metrizable space, can be embedded in $\mathcal{H}_G$ in such a way that X is mapped onto an invariant subset of $L^2(G\times G)$ with compact closure. Consequently, (G,X,ρ) may be identified with a subsystem of $\mathcal{H}_G$ which is bounded as a ttg with respect to the usual metric uniformity in $L^2(G\times G)$. However, it should be noticed that $\mathcal{H}_G$ itself is, in general, not bounded with respect to this uniformity. Indeed it is quite easy to indicate an unbounded subsystem of $\mathcal{H}_{\mathbb{R}\times\mathbb{R}}$. Concerning the question whether in Theorem 5.1 the condition

$w(X) \leq \aleph_0$ on X may be weakened we observe that

$$w[L^2(G\times G)] = w(G\times G) = w(G)$$

(cf. [17] for a proof; in [9], 28.2 only compact groups are treated). Hence X cannot be embedded in $L^2(G\times G)$ unless $w(X) \leq w(G)$. However, the author does not know whether this condition is sufficient.

R E F E R E N C E S .

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