ma
the
ma
tisch

cen
trum

AFDELING ZUIVERE WISKUNDE ZN 56/73 DECEMBER

A.E. BROUWER

ON THE NUMBER OF UNIQUE SUBGRAPHS OF A GRAPH

amsterdam 1973

stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZN 56/73 DECEMBER

A.E. BROUWER

ON THE NUMBER OF UNIQUE SUBGRAPHS OF A GRAPH

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 05C30

Abstract

A question of Entringer and Erdös concerning the number of unique subgraphs of a graph is answered.

Entringer and Erdös [1] call a subgraph H of a graph G unique if H is not isomorphic to any other subgraph of G. If f(n) is the largest number of unique subgraphs a graph on n vertices can have, they prove

$$f(n) > 2^{\frac{1}{2}n^2 - cn^{3/2}} \quad \text{for } c > 3/\sqrt{2} \text{ and n sufficiently large}$$

It will be proved below that

$$^2\log f(n) = \tfrac{1}{2}n^2 - n \cdot {}^2\log n + O(n).$$

Since the number of nonisomorphic graphs on n vertices is

$$\frac{2^{\binom{n}{2}}}{n!}\left(1 + \frac{n^2-n}{2^n-1} + O\left(\frac{n^3}{2^{3n/2}}\right)\right)$$

(see e.g. [2], p.196), we have

$$^2\log f(n) \le \tfrac{1}{2}n^2 - n \cdot {}^2\log n + O(n).$$

On the other hand, given n we construct a graph G_n on n vertices with $2^{\frac{1}{2}n^2-n \cdot {}^2\log n+O(n)}$ unique subgraphs as follows:

Let $m = \lceil {}^2\log n \rceil$ and $N = n - m - 2$. Then $N \le 2^m - m - 1$.

Let $G_n = A \cup B \cup C$ where

$A = K_N$, the complete graph on N points,
B is a rigid tree with m vertices (such a tree exists for each $m \ge 7$),
$C = K_2$, a single edge connecting points c_0 and c_1, connected as follows:

G_n contains all edges (c_1,b) for $b \in B$ and no other edges between C and $A \cup B$; furthermore, if we view A as a set of subsets of B each containing at least two points (which is possible since $N \le 2^m-m-1$), then G_n contains the edge (a,b) where $a \in A$ and $b \in B$ if and only if $b \in a$.

Now define the subgraph H_n of G_n as follows:

$H_n = A' \cup B \cup C$ where A' is the vertex graph on N vertices (that is, A' is totally disconnected) and A', B, C are interconnected like A, B, C in G_n. That is, H_n contains the same n points as G_n but has $\binom{N}{2}$ edges less.

If H is a subgraph of G_n such that $H_n \subset H \subset G_n$ then H is unique: First, c_0 is the only point of H with degree one, and therefore if we imbed H in G_n the point c_0 of H must go to the point c_0 of G_n. Next it follows that c_1 must go to c_1 and therefore that $B \subset H$ must map onto $B \subset G$. Since B is rigid the imbedding restricted to B must be the identity on B. Finally, since each point of A is coded by a subset of B, A too cannot be imbedded in any other way. Therefore H is unique.

The number of subgraphs H between H_n and G_n being $2^{\binom{N}{2}} =$ $= 2^{\frac{1}{2}n^2-n\cdot {}^2\log n+O(n)}$, this proves our assertion.

References

[1] R.C. Entringer and Paul Erdös, *On the number of unique subgraphs of a graph*, JCT (B) 13, 112-115 (1972).

[2] Frank Harary & Edgar Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.