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J. VAN DE LUNE

A NOTE ON THE FUNDAMENTAL THEOREM FOR RIEMANN-INTEGRALS

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A NOTE ON THE FUNDAMENTAL THEOREM FOR RIEMANN-INTEGRALS *)

by

Jan van de Lune

Usually the fundamental theorem for R-integrals

$$\int_a^b f'(x)dx = f(b) - f(a)$$

is proved under the assumption that f' is R-integrable over [a,b]. Compare, for example, [2, pg.115].

It appears that there is no answer in the literature to the question under what conditions the derivative f' of a differentiable function f is R-integrable (without making use of specific measure theoretic notions). Compare, for example, [1, pg.47].

In this note we prove the following

THEOREM. If $f : [a,b] \to \mathbb{R}$ is differentiable then the derivative f' is R-integrable over [a,b] if and only if there exists an R-integrable function $\phi : [a,b] \to \mathbb{R}$ such that

$$f(x) = f(a) + \int_a^x \phi(t)dt.$$

In order to prove this theorem we need two lemmas.

Lemma 1. Let $\phi : [a,b] \to \mathbb{R}$ be R-integrable such that

$$m \leqq \phi(x) \leqq M, \qquad \forall x \in [a,b].$$

Define

$$\Phi(x) = \int_a^x \phi(t)dt, \qquad \forall x \in [a,b],$$

*) This paper is not for review; it is meant for publication in a journal.

and assume that Φ is differentiable on $[a,b]$.
Then we have

$$m \leqq \Phi'(x) \leqq M, \qquad \forall x \in [a,b].$$

Proof. Define $K(x) = \Phi(x) - m \cdot (x-a)$.
Then $K(x)$ is differentiable such that

$$K'(x) = \Phi'(x) - m.$$

We also have

$$K(x) = \int_a^x \phi(t)dt - m \cdot (x-a) = \int_a^x (\phi(t) - m)dt.$$

Since $\phi(t) \geqq m$ we see that K is monotonically non-decreasing. Hence

$$K'(x) \geqq 0,$$

or

$$\Phi'(x) \geqq m, \qquad \forall x \in [a,b].$$

In a similar fashion one proves

$$\Phi'(x) \leqq M, \qquad \forall x \in [a,b].$$

Lemma 2. Under the same conditions as in Lemma 1 we have that Φ' is R-integrable over $[a,b]$.

Proof. Lemma 1 is applicable on every closed subinterval of $[a,b]$. From this it follows that the fluctuation of Φ' is not larger than the fluctuation of ϕ.
According to a well known criterion for R-integrability [2, pg.107] we may conclude that Φ' is R-integrable over $[a,b]$.

Proof of the theorem. It is clear that the given condition is necessary; take $\phi = f'$.

To prove the sufficiency we write

$$f(x) - f(a) = \int_a^x \phi(t)dt.$$

Then it is clear that f' is the derivative of a function of the form

$$\int_a^x \phi(t)dt.$$

According to Lemma 2 we obtain that f' is R-integrable over [a,b], completing the proof.

References

[1] H. Kestelman, Modern theories of integration, Clarendon Press, Oxford, 1937.

[2] W. Rudin, Principles of mathematical analysis, McGraw-Hill, 1964.