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J. VAN DE LUNE

A NOTE ON EULER'S (INCOMPLETE) Γ-FUNCTION

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A note on Euler's (incomplete) Γ-function

by

J. van de Lune

KEY WORDS & PHRASES: *Gamma function, Laplace transform.*

1. The subject of this section of this note was inspired by the following observation

Let

$$(1) \qquad f(t) = \sum_{n=1}^{N} a_n \cos(t\lambda_n), \quad (t\in R)$$

where all a's and λ's are real.

Clearly f(t) is bounded and continuous so that we may consider the (one-sided) Laplace transform $\check{f}$ of f for s > 0:

$$(2) \qquad \check{f}(s) = \int_0^\infty e^{-st} f(t)\, dt =$$

$$= \sum_{n=1}^{N} a_n \int_0^\infty e^{-st} \cos(t\lambda_n)\, dt =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \int_0^\infty e^{-st}(e^{it\lambda_n} + e^{-it\lambda_n})\, dt =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \left\{\frac{1}{s - i\lambda_n} + \frac{1}{s + i\lambda_n}\right\}.$$

Differentiating k times with respect to s we obtain

$$(3) \qquad \int_0^\infty e^{-st} t^k f(t)\, dt = \frac{k!}{2} \sum_{n=1}^{N} a_n \left\{\frac{1}{(s-i\lambda_n)^{k+1}} + \frac{1}{(s+i\lambda_n)^{k+1}}\right\},$$

or, equivalently

$$(4) \qquad \frac{1}{k!} \int_0^\infty e^{-u} u^k f(\tfrac{u}{s})\, du = \frac{1}{2} \sum_{n=1}^{N} a_n \left\{\frac{1}{\left(1-\frac{i\lambda_n}{s}\right)^{k+1}} + \frac{1}{\left(1+\frac{i\lambda_n}{s}\right)^{k+1}}\right\}$$

Substituting $s = \frac{k}{t}$, (t>0) in (4) and letting k tend to infinity we arrive at

$$(5) \qquad \lim_{k\to\infty} \int_0^\infty \frac{e^{-u} u^k}{k!} f\left(\frac{tu}{k}\right) du =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \left(e^{it\lambda_n} + e^{-it\lambda_n} \right) = f(t).$$

The main purpose of this section is to prove the validity of (5), or rather a generalization of it, for quite a large class of functions $f\colon R^+ \to \mathbb{C}$. In the meanwhile we will obtain some interesting results concerning Euler's (incomplete) Γ-function.

We start with stating the following:

THEOREM. *Let* $f\colon R^+ \to \mathbb{C}$ *be such that*

(i) f *is integrable over every interval of the form* $(0,T)$ *where* $T > 0$,

(ii) *for some* $t_0 > 0$ *we have that the limits*

(6) $$\lim_{x \uparrow t_0} f(x) = L \text{ and } \lim_{x \downarrow t_0} f(x) = R$$

both exist and are finite,

(iii) *there exists a real constant* A *such that*

(7) $$f(x) = O(e^{Ax}), \quad (x \to \infty).$$

Under these conditions we have

(8) $$\lim_{s \to \infty} \int_0^\infty \frac{e^{-u} u^s}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du = \frac{L + R}{2}.$$

Before proving this theorem we will prove some lemmas.

LEMMA 1. *If* $0 < \alpha < 1$ *then*

(9) $$\lim_{s \to \infty} \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} du = 0.$$

PROOF. For any fixed $s > 0$ the function $e^{-u} u^s$, $(u \in R^+)$, is increasing on the interval $(0,s)$ so that

(10) $$0 < \int_0^{\alpha s} \frac{e^{-u} u^s}{\Gamma(s+1)} du < \alpha s \, \frac{e^{-\alpha s} (\alpha s)^s}{\Gamma(s+1)} =$$

$$= \alpha\, s\, \frac{e^{-\alpha s}\, \alpha^s s^s}{s^s e^{-s} \sqrt{2\pi s}\, e^{\mu(s)}} =$$

$$= \alpha\sqrt{s}\, \frac{(\alpha\, e^{1-\alpha})^s}{\sqrt{2\pi}\, e^{\mu(s)}}$$

where $\mu(s)$ is Binet's function which may be represented by

(11) $$\mu(s) = \int_0^\infty \frac{e^{-st}}{t} \left\{ \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right\} dt, \quad (s>0).$$

(See, for example, G. SANSONE & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen, (1960) p. 216.)

Since

(12) $$0 < \alpha\, e^{1-\alpha} < 1, \quad (0<\alpha<1)$$

and

(13) $$\lim_{s\to\infty} \mu(s) = 0$$

the lemma follows easily. □

LEMMA 2. *If* $\beta > 1$ *then*

(14) $$\lim_{s\to\infty} \int_{\beta s}^\infty \frac{e^{-u} u^s}{\Gamma(s+1)}\, du = 0$$

PROOF. Observe that for $s > 0$ we have

(15) $$0 < \int_{\beta s}^\infty \frac{e^{-u} u^s}{\Gamma(s+1)}\, du = \int_0^\infty \frac{e^{-x-\beta s}}{\Gamma(s+1)} (x+\beta s)^s\, dx =$$

$$= \frac{e^{-\beta s}\, \beta^s s^s}{s^s e^{-s} \sqrt{2\pi s}\, e^{\mu(s)}} \cdot \int_0^\infty e^{-x} \left(1 + \frac{x}{\beta s}\right)^s dx =$$

$$= \frac{(\beta\, e^{1-\beta})^s}{\sqrt{2\pi s}\, e^{\mu(s)}} \int_0^\infty e^{-x} \left(1 + \frac{x}{\beta s}\right)^s dx <$$

$$< \frac{(\beta e^{1-\beta})^s}{\sqrt{2\pi s}\, e^{\mu(s)}} \int_0^\infty e^{-x} e^{\frac{x}{\beta}} dx = \frac{(\beta e^{1-\beta})^s}{\sqrt{2\pi s}\, e^{\mu(s)}} \frac{1}{1 - \frac{1}{\beta}}$$

Since for $\beta > 1$ we also have

$$(16) \qquad 0 < \beta\, e^{1-\beta} < 1$$

the lemma follows easily. □

LEMMA 3. *The function*

$$(17) \qquad \frac{1}{\Gamma(s+1)} \int_0^s e^{-u} u^s du, \quad (s \in \mathbb{R}^+)$$

tends (increasingly) to $\frac{1}{2}$ *when* $s \to \infty$

PROOF. By the substitution $u = s - x\sqrt{s}$ we obtain

$$(18) \qquad \frac{1}{\Gamma(s+1)} \int_0^s e^{-u} u^s du = \frac{1}{\Gamma(s+1)} \int_0^{\sqrt{s}} e^{-s+x\sqrt{s}} (s-x\sqrt{s})^s \sqrt{s}\, dx =$$

$$= \frac{e^{-\mu(s)}}{\sqrt{2\pi}} \int_0^{\sqrt{s}} \exp\left\{x\sqrt{s} + s \log\left(1 - \frac{x}{\sqrt{s}}\right)\right\} dx =$$

$$= \frac{e^{-\mu(s)}}{\sqrt{2\pi}} \int_0^{\sqrt{s}} \exp\left\{-\frac{x^2}{2} - \sum_{n=3}^{\infty} \frac{x^n}{n} \frac{1}{s^{\frac{n}{2}-1}}\right\} dx =$$

Now observe that

$$(19) \qquad \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} > 0, \quad (t>0)$$

so that, using (11), it follows that $\mu(s)$ is decreasing on R^+. Consequently $e^{-\mu(s)}$ is increasing on R^+.

Also note that

$$(20) \qquad \sum_{n=3}^{\infty} \frac{x^n}{n} \frac{1}{s^{\frac{n}{2}-1}}, \quad (0<x<\sqrt{s};\ s>0)$$

is decreasing in s for any fixed x subject to the conditions stated in (20).

Consequently we have established the "increasing part" of the lemma.

Using Lebesgue's dominated convergence theorem it also follows from (18) that

$$\lim_{s\to\infty} \frac{1}{\Gamma(s+1)} \int_0^s e^{-u} u^s du = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{1}{2}, \tag{21}$$

completing the proof of the lemma. □

<u>LEMMA 4</u>. *The function*

$$\frac{1}{\Gamma(s+1)} \int_s^\infty e^{-u} u^s du, \quad (s \in \mathbb{R}^+) \tag{22}$$

tends (decreasingly) to $\frac{1}{2}$ *when* $s \to \infty$.

<u>PROOF</u>. Since

$$\int_0^s e^{-u} u^s du + \int_s^\infty e^{-u} u^s du = \int_0^\infty e^{-u} u^s du = \Gamma(s+1) \tag{23}$$

the lemma is a direct consequence of lemma 3. □

After these preparations we proceed by proving the theorem stated before:

<u>PROOF</u>. For any given $\varepsilon > 0$ determine δ such that

$$0 < \delta < t_0 \tag{24}$$

$$|f(x) - L| < \varepsilon, \quad (t_0 - \delta < x < t_0) \tag{25}$$

and

$$|f(x) - R| < \varepsilon, \quad (t_0 < x < t_0 + \delta). \tag{26}$$

Write

$$\alpha = \frac{t_0 - \delta}{t_0} \quad \text{and} \quad \beta = \frac{t_0 + \delta}{t_0} \tag{27}$$

so that $0 < \alpha < 1$ and $\beta > 1$.

Also write

$$(28)\qquad \int_0^\infty \frac{e^{-u}u^s}{\Gamma(s+1)}\, f\left(\frac{t_0 u}{s}\right) du = \left\{\int_0^{\alpha s} + \int_{\alpha s}^{s} + \int_s^{\beta s} + \int_{\beta s}^{\infty}\right\} \frac{e^{-u}u^s}{\Gamma(s+1)}\, f\left(\frac{t_0 u}{s}\right) du$$

Step 1. We first show that

$$(29)\qquad \lim_{s\to\infty} \int_0^{\alpha s} \frac{e^{-u}u^s}{\Gamma(s+1)}\, f\left(\frac{t_0 u}{s}\right) du = 0.$$

In order to see this we observe that

$$(30)\qquad \left|\int_0^{\alpha s} \frac{e^{-u}u^s}{\Gamma(s+1)}\, f\left(\frac{t_0 u}{s}\right) du\right| \le$$

$$\le \frac{e^{-\alpha s}(\alpha s)^s}{\Gamma(s+1)} \int_0^{\alpha s} \left| f\left(\frac{t_0 u}{s}\right)\right| du =$$

$$= \frac{(\alpha\, e^{1-\alpha})^s}{\sqrt{2\pi s}\; e^{\mu(s)}}\, \frac{s}{t_0} \int_0^{\alpha t_0} |f(x)|\, dx$$

so that (29) follows easily.

Step 2. If $\alpha\, s < u < s$ then

$$(31)\qquad t_0 - \delta < \frac{t_0 u}{s} < t_0$$

so that

$$(32)\qquad \left|f\left(\frac{t_0 u}{s}\right) - L\right| < \varepsilon .$$

Hence

$$(33)\qquad \left|\int_{\alpha s}^{s} \frac{e^{-u}u^s}{\Gamma(s+1)}\, f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} L\right| =$$

$$= \left| \int_{\alpha s}^{s} \frac{e^{-u}u^{s}}{\Gamma(s+1)} \left\{ f\left(\frac{t_0 u}{s}\right) - L \right\} du + L \int_{\alpha s}^{s} \frac{e^{-u}u^{s}}{\Gamma(s+1)}\, du - \frac{1}{2} L \right| \leq$$

$$\leq \varepsilon \int_{\alpha s}^{s} \frac{e^{-u}u^{s}}{\Gamma(s+1)}\, du + |L| \left| \int_{0}^{s} \frac{e^{-u}u^{s}}{\Gamma(s+1)}\, du - \frac{1}{2} \right| + |L| \int_{0}^{\alpha s} \frac{e^{-u}u^{s}}{\Gamma(s+1)}\, du,$$

from which it is clear that

$$(34) \qquad \limsup_{s\to\infty} \left| \int_{\alpha s}^{s} \frac{e^{-u}u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} L \right| \leq \frac{\epsilon}{2}.$$

Step 3. Similarly as in step 2 one may show that

$$(35) \qquad \limsup_{s\to\infty} \left| \int_{s}^{\beta s} \frac{e^{-u}u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du - \frac{1}{2} R \right| \leq \frac{\varepsilon}{2}\ .$$

Step 4. We have

$$(36) \qquad \lim_{s\to\infty} \int_{\beta s}^{\infty} \frac{e^{-u}u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du = 0.$$

In order to see this we proceed as follows: Determine the constants K, A and x_0 such that

$$(37) \qquad x_0 > \beta\, t_0,$$

$$(38) \qquad x_0 > 2\, t_0$$

and

$$(39) \qquad |f(x)| \leq K\, e^{Ax},\ (x>x_0).$$

Since

$$(40) \qquad \left| \int_{\beta}^{\frac{x_0}{t_0}s} \frac{e^{-u}u^{s}}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du \right| \leq$$

$$\leq \frac{e^{-\beta s}(\beta s)^{s}}{\Gamma(s+1)} \frac{s}{t_0} \int_{\beta t_0}^{x_0} |f(x)|\, dx$$

it follows as before that

(41) $$\lim_{s\to\infty} \int_{\beta s}^{\frac{x_0}{t_0}s} \frac{e^{-u}u^s}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du = 0.$$

Hence, it suffices to show that

(42) $$\lim_{s\to\infty} \int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du = 0.$$

If $u > \frac{x_0}{t_0} s$ then $\frac{t_0 u}{s} > x_0$ so that, if in addition $s \geqq 2At_0$, we have

(43) $$\left|\int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} f\left(\frac{t_0 u}{s}\right) du\right| \leqq$$

$$\leqq K \int_{\frac{x_0}{t_0}s}^{\infty} \frac{e^{-u}u^s}{\Gamma(s+1)} e^{A\frac{t_0 u}{s}} du =$$

$$= \frac{K}{\Gamma(s+1)} \int_{\frac{x_0}{t_0}s}^{\infty} e^{-u\left(1-\frac{At_0}{s}\right)} u^s du =$$

$$= \frac{K}{\Gamma(s+1)} \int_{\frac{x_0 s}{t_0}\left(1-\frac{At_0}{s}\right)}^{\infty} e^{-x}\left(\frac{x}{1-\frac{At_0}{s}}\right)^s \frac{dx}{1-\frac{At_0}{s}} \leqq$$

$$\leqq \frac{K}{\left(1-\frac{At_0}{s}\right)^{s+1}} \int_{\frac{x_0 s}{2t_0}}^{\infty} \frac{e^{-x}x^s}{\Gamma(s+1)} dx .$$

Since $\frac{x_0}{2t_0} > 1$ by (38) it follows from lemma 2 that (42) holds true, completing the proof of the theorem. □

REMARK. From lemma 4 one may derive very simply that the sequence

$$\left\{e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}\right\}_{n=0}^{\infty} \tag{44}$$

tends *decreasingly* to its limit $(=\frac{1}{2})$.

In order to see this we note that for any non negative integer n we have

$$\frac{1}{\Gamma(n+1)}\int_n^\infty e^{-u}u^n\,du = \frac{1}{n!}\int_0^\infty e^{-(n+x)}(n+x)^n\,dx = \tag{45}$$

$$= \frac{1}{n!}\int_0^\infty e^{-n-x}\sum_{k=0}^{n}\binom{n}{k}\,n^k x^{n-k}\,dx =$$

$$= \frac{e^{-n}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\;n^k\int_0^\infty e^{-x}x^{n-k}\,dx =$$

$$= e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!},$$

so that our assertion follows from lemma 4.

2. The subject of this section of this note was inspired by the following observation.

For any positive integer n one has

$$\log\left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = \frac{1}{n}\log\left(\frac{n}{1}\,\frac{n}{2}\,\frac{n}{3}\cdots\frac{n}{n}\right) = \tag{46}$$

$$= -\frac{1}{n}\log\left(\frac{1}{n}\,\frac{2}{n}\,\frac{3}{n}\cdots\frac{n}{n}\right) =$$

$$= -\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{n-k}{n} = -\frac{1}{n}\sum_{k=0}^{n-1}\log\left(1-\frac{k}{n}\right) =$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{k}{n}\right)^m =$$

$$= \sum_{m=1}^{\infty}\frac{1}{m}\left\{\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^m\right\} =$$

$$= \sum_{m=1}^{\infty}\frac{1}{m}L_n(m),$$

where $L_n(m)$ denotes the n-th canonical lower Riemann sum corresponding to the function x^m, $0 \leq x \leq 1$.

It may be shown (see the author's Mathematical Centre Report ZW 39/75) that for any fixed $m > 0$, $L_n(m)$ is increasing in n. From this fact and (46) it then follows that the sequence

$$(47) \qquad \left\{\left(\frac{n^n}{n!}\right)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$$

is increasing.

As a generalization we prove the following

PROPOSITION. *The function* $f: \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$(48) \qquad f(s) = \left\{\frac{s^s}{\Gamma(s+1)}\right\}^{\frac{1}{s}}, \quad (s>0)$$

is increasing on $\mathbb{R}^+$.

PROOF. For $s > 0$ we have

$$(49) \qquad \Gamma(s+1) = s^s e^{-s}\sqrt{2\pi s}\, e^{\mu(s)}$$

where $\mu(s)$ may be written as in (11). Since $f(s) > 0$ for $s > 0$ we may just as well prove that $\log f(s)$ is increasing on $\mathbb{R}^+$.

Observing that

$$(50) \qquad \log f(s) = 1 - \frac{\mu(s) + \frac{1}{2}\log(2\pi s)}{s}, \quad (s>0)$$

it is clearly sufficient to show that

(51) $$\frac{d}{ds}\,\frac{\mu(s)+\frac{1}{2}\log(2\pi s)}{s} < 0,\quad (s>0).$$

First we compute the derivative in (51):

(52) $$\frac{d}{ds}\,\frac{\mu(s)+\frac{1}{2}\log(2\pi s)}{s} = \frac{s\{\mu'(s)+\frac{1}{2s}\} - \{\mu(s)+\frac{1}{2}\log(2\pi s)\}}{s^2}.$$

Writing

(53) $$\phi(s) = s\{\mu'(s)+\frac{1}{2s}\} - \{\mu(s)+\frac{1}{2}\log(2\pi s)\},\quad (s>0)$$

we will prove that

(54) $$\phi(s) < 0,\quad (s>0).$$

We first prove that

(55) $$\lim_{s\downarrow 0}\phi(s) = 0.$$

In order to see this we first note that

(56) $$e^{\mu(s)}\sqrt{2\pi s} = \frac{e^s\,\Gamma(s+1)}{s^s}$$

so that

(57) $$\lim_{s\downarrow 0}\{\mu(s)+\frac{1}{2}\log(2\pi s)\} = \lim_{s\downarrow 0}\log\frac{e^s\,\Gamma(s+1)}{s^s} = \log 1 = 0.$$

Next we observe that from the integral representation of $\mu(s)$ it follows that

(58) $$\mu'(s) = -\int_0^\infty e^{-st}\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}\,dt =$$

$$= -\frac{1}{2s} + \int_0^\infty e^{-st}\{\frac{1}{t} - \frac{1}{e^t-1}\}\,dt$$

so that

(59) $$\mu'(s) + \frac{1}{2s} = \int_0^\infty e^{-st}\left\{\frac{1}{t} - \frac{1}{e^t-1}\right\} dt.$$

Since

$$\lim_{t\to\infty} \{\frac{1}{t} - \frac{1}{e^t-1}\} = 0$$

it follows from the general theory of Laplace transforms that

(60) $$\lim_{s\downarrow 0} s\{\mu'(s) + \frac{1}{2s}\} = 0.$$

Combining (57) and (60) it follows that (55) holds.

In view of (55) we may prove (54) by showing that

(61) $$\phi'(s) < 0, \quad (s>0).$$

In order to see this we note that

(62) $$\phi'(s) = (\mu'(s)+ \frac{1}{2s}) + s\frac{d}{ds}(\mu'(s)+ \frac{1}{2s}) - (\mu'(s)+ \frac{1}{2s}) =$$

$$= s\frac{d}{ds}(\mu'(s)+ \frac{1}{2s})$$

so that, using (59),

(63) $$\phi'(s) = - s\int_0^\infty e^{-st}\{1 - \frac{t}{e^t-1}\} dt, \ (s>0).$$

Since

$$1 - \frac{t}{e^t-1} > 0, \quad (t>0)$$

it follows from (63) that (61) holds true, completing the proof. □

REMARK. From (46) and the fact that $L_n(m)$ tends increasingly to $\int_0^1 x^m dx = \frac{1}{m+1}$

we may obtain a very transparent alternative proof of the well-known fact that

(64) $$\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = e.$$

Indeed

(65) $$\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \exp\{\sum_{m=1}^{\infty} \frac{1}{m} L_n(m)\} =$$

(by uniform convergence in n of the series involved)

$$= \exp\{\sum_{m=1}^{\infty} \frac{1}{m} (\lim_{n\to\infty} L_n(m))\} =$$

$$= \exp\{\sum_{m=1}^{\infty} \frac{1}{m(m+1)}\} = \exp(1) = e.$$