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A NOTE ON THE COVERING OF ALL TRIPLES ON 7 POINTS WITH STEINER TRIPLE SYSTEMS

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A note on the covering of all triples on 7 points with Steiner triple systems

by

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ABSTRACT

We give a partition of the thirty Steiner triple systems on seven points into three sets of ten, each set covering all triples exactly twice. In the language of designs, considering the 35 triples on seven points as points, we look for block designs with the restriction that each block is a Steiner triple system. In particular $k = 7$ and $b = 5\lambda$. It is well known since Cayley that such a design does not exist for $\lambda = 1$, while it was known for $\lambda = 3$. Here we give it for $\lambda = 2$ thus proving the existence of the design for each $\lambda > 1$.

KEY WORDS & PHRASES: *Steiner triple system*

0. INTRODUCTION

If E is a design (i.e. a collection of subsets of a finite set) and $\mathcal{D}$ is a collection of designs then a λ-cover of E by $\mathcal{D}$, called a $C_\lambda(E,\mathcal{D})$, is a collection $\{\mathcal{D}_j | j \in J\}$ of designs such that

(i) each $\mathcal{D}_j$ is isomorphic to an element of $\mathcal{D}$
(ii) $U\{\mathcal{D}_j | j \in J\} = E$
(iii) each $E \in E$ occurs in exactly λ of the $\mathcal{D}_j$.

If $\lambda = 1$ we will drop the subscript.
Specializing $\mathcal{D}$ and E we get all kinds of familiar designs some of which are needed below.

1. Let I_n be a fixed set of size n, and let $S_k(n)$ be the collection of all k-subsets of I_n. Then a $C_\lambda(S_2(\nu), \{S_2(k) | k \in K\})$ is a pairwise balanced design with block sizes in K: a PBD(K,λ;ν). In particular a $C_\lambda(S_2(\nu), \{S_2(k)\})$ is a BIBD(b,ν,r,k,λ). More generally, a $C_\lambda(S_t(\nu), \{S_t(k)\})$ is a t-design.

2. Let $\mathcal{D}$ be the collection of all Steiner triple systems on ν points: $\mathcal{D} := S_\nu := \mathrm{STS}(\nu)$ and E the collection of all triples on ν points: $E = S_3(\nu)$, then an $A_\lambda(\nu) := C_\lambda(E,\mathcal{D})$ is a λ-fold cover of all triples with Steiner triple systems.
   In particular an $A_1(\nu)$ is a partition of all triples in disjoint STS(ν)'s. Such partitions have been found among others by T.P. Kirkman [3], R.H.F. Denniston [2], A. Rosa [5] and L. Teirlinck [6]; it is conjectured that they exist for each $\nu \neq 7$ (for which an STS(ν) exists, that is, for $\nu \equiv 1$ or 3 (mod 6)), and they are known for all $\nu < 100$ except $\nu = 37, 85, 97$. For $\nu = 7$ however the maximum number of pairwise disjoint STSs is 2 (Cayley [1]), so it is impossible to cover $S_3(7)$ with 5 disjoint STS(7). Lindner & Rosa [4] showed the existence of an $A_3(7)$, and in this note we will show the existence of exactly two different $A_2(7)$ so that for each $\lambda > 1$ an $A_\lambda(7)$ exists.

3. Finally, take $\mathcal{D} = A_\lambda(7)$ and let S_7 be the collection of all STS(7). Note that $|A| = 5\lambda$ for each $A \in A_\lambda(7)$ and $|S_7| = 30$. Obviously a $C(S_7, A_\lambda(7))$ can exist only if $5\lambda | 30$ i.e. if $\lambda \in \{1,2,3,6\}$. For $\lambda = 6$

we have the trivial $C(S_7,A_6(7))$ obtained by taking $\{S_7\}$. Lindner & Rosa showed the existence of a $C(S_7,A_3(7))$ i.e. a partition of the 30 STS(7) in two $A_3(7)$'s, and here we will show the existence of exactly two $C(S_7,A_3(7))$ one containing $A_2(7)$'s of different types while the other contains three isomorphic $A_2(7)$. Finally, since $A_1(7)$ does not exist neither does $C(S_7,A_1(7))$. This settles all cases.

1. TRIPLES

On 7 points there are $\binom{7}{3} = 35$ triples. These triples can be divided into five cyclic 1-designs: (represent a triple by a characteristic vector of length 7)

A: circ(1110000)
B: circ(1001100)
C: circ(1010100)
D: circ(1011000)
E: circ(1101000)

In this way each triple gets a name: A_0 = (1110000), A_1 = (0111000), C_4 = (0100101) etc.

2. STEINER TRIPLE SYSTEMS

On 7 points there are 30 Steiner triple systems (all isomorphic), partitioned into six orbits (with sizes 7,7,7,7,1,1) under the action of a cyclic shift.
We give from each of the orbits X the element X_0:

I_0:	1110000	i.e.	A_0	IV_0:	0001101	E_3
	1001100		B_0		0110001	D_6
	1000011		A_5		1000011	A_5
	0101010		C_1		0011010	E_2
	0100101		C_4		0100110	B_1
	0011001		B_6		1010100	C_0
	0010110		D_2		1101000	E_0

II_0:	0000111	A_4	V:	1101000	E_0
	0011001	B_6		0110100	E_1
	1100001	A_6		0011010	E_2
	0101010	C_1		0001101	E_3
	1010010	C_5		1000110	E_4
	1001100	B_0		0100011	E_5
	0110100	E_1		1010001	E_6
III_0:	1011000	D_0	VI:	1011000	D_0
	1000110	E_4		0101100	D_1
	1100001	A_6		0010110	D_2
	0101100	D_1		0001011	D_3
	0110010	B_5		1000101	D_4
	0010101	C_2		1100010	D_5
	0001011	D_3		0110001	D_6

That there are no more Steiner triple systems can be seen for instance from the fact that V has an automorphism group of order 168 = 7.6.4 (and the fact that all STS(7) are isomorphic, which is readily seen by looking at I_0).

Each triple occurs in six STS(7), i.e. we have the trivial $A_6(7) = S_7$ and therefore a unique $C(S_7, A_6(7))$.

Incidences between triples and STSs can be read off the following table:

	I	II	III	IV	V	VI
A	05	46	6	5	-	-
B	06	06	5	1	-	-
C	14	15	2	0	-	-
D	2	-	013	6	-	0-6
E	-	1	4	023	0-6	-

Here the entry 05 in row A, column I means that I_0 contains the triples A_0 and A_5 (and hence I_1 contains A_1 and A_6 etc.).

Intersections between STSs: two STS(7) have either 0 or 1 or 3 triples in common. If we take all STS(7) with intersectionnr 1 with a given STS(7), then it is found that these STS(7) also have mutual intersectionnr 1. In this

way we get the two sets $\{I_{0-6}, IV_{0-6}, VI\}$ and $\{II_{0-6}, III_{0-6}, V\}$; where two STS(7) have exactly one triple in common iff they are in the same set. Each of these two sets covers all triples exactly thrice, that is, we have here two isomorphic $A_3(7)$ covering S_7: an $C(S_7,A_3(7))$.
If two STS(7) have intersectionnr 3 then the three triples they have in common intersect in a singleton; for instance $I_0 \cap II_0 = \{B_0, B_6, C_1\}$ and $B_0 \cap B_6 = B_0 \cap C_1 = B_6 \cap C_1 = B_0 \cap B_6 \cap C_1 = \{3\}$:

$$B_0 = (1001100)$$
$$B_6 = (0011001)$$
$$C_1 = (0101010)$$
$$\{3\} = (0001000)$$

Conversely, given a singleton, then it is associated in this way with 90 pairs of STS(7) with intersectionnr 3.

The following table gives for each singleton and a representative from each orbit of STS(7)'s the uniquely determined STS(7) such that the singleton is associated with this pair of triple systems.

	0	1	2	3	4	5	6
I_0	II_1	II_3	III_1	II_0	III_2	III_6	II_6
II_0	I_1	IV_1	IV_5	I_0	IV_6	I_4	I_6
III_0	IV_1	I_6	I_5	VI	IV_2	IV_4	I_1
IV_0	II_6	III_3	III_5	V	II_2	II_1	III_6
V	IV_4	IV_5	IV_6	IV_0	IV_1	IV_2	IV_3
VI	III_4	III_5	III_6	III_0	III_1	III_2	III_3

Finally, each STS(7) is disjoint from (the remaining) 8 systems; for example I_0 is disjoint from II_2, II_4, II_5, III_0, III_3, III_4, III_5, V. Since these 8 systems have always mutual intersectionnr 1, it follows that there are no three mutually disjoint STS(7). In particular it is not possible to cover all triples with 5 mutually disjoint STS(7), i.e. an $A_1(7)$ does not exist.

3. NOTE ON THE $A_3(7)$.

The $A_3(7)$ indicated above is an interesting system:
Firstly, if we view the 35 triples as points, then the 15 STS(7) are vectors of length 35 and weight 7 and mutual distance 12. But by the Johnson bound the maximum number of vectors with these properties is 15, i.e. $A_3(7)$ is an optimal constant weight code showing that $A(35,12,7) = 15$.
Secondly, if we view the 15 STS(7) as points then each triple determines a triple of STS(7) [since the 15 STS(7) cover all triples thrice], while each pair of STS(7) determines one triple [their intersection].
Therefore we have a block design with $v = 15$, $b = 35$, $k = 3$, $r = 7$, $\lambda = 1$ in other words, an STS(15).

It is easy to see that there are other $A_3(7)$'s besides the one indicated above. In fact if one fixes a triple (say D_6) then the three STS(7) from $M_1 := \{I_{0-6}, IV_{0-6}, VI\}$ containing D_6(sc.I_4, IV_0, VI) and the three STS(7) from $M_2 := \{II_{0-6}, III_{0-6}, V\}$ containing D_6 (sc. III_3, III_5, III_6) contain the same triples, namely besides D_6 both sets of three STS(7) cover exactly those triples which intersect D_6 in a single point. Therefore one might exchange $\{I_4, IV_0, VI\}$ and $\{III_3, III_5, III_6\}$, thus obtaining two new $A_3(7)$: M_1' and M_2' (which of course together form a $C(S_7, A_3(7))$).
This process may be repeated, exchanging three STS(7) between M_1' and M_2' giving M_1'' and M_2''. In this way we obtain at least three non-isomorphic types of $A_3(7)$ namely M_1, M_1' and M_1''. I do not know whether there are any other types besides these.

4. DESCRIPTION OF THE TWO TYPES OF $A_2(7)$.

In order to examine all possibilities for an $A_2(7)$ we first need to know how many different (i.e. non-isomorphic) pairs of STS(7) there are. Of course the cardinality of the intersection of the pair is an invariant; on the other hand, by examining a few permutations it is readily seen that this is the only invariant: two pairs of STS(7) are isomorphic iff they have the same intersectionnr.

Now considering the number of triples of each of the types A, B, C, D and E contained in STSs of each of the orbits it is found that an $A_2(7)$ must contain two disjoint STS(7), say V and VI. Now it follows in the same way that we need 3 systems of types I and II each and 1 system of types III and IV each. [Note that an $A_2(7)$ contains 10 STS(7).] By using an appropriate cyclic shift we may ensure the the occurrence of III_0.
The collection $\{V, VI, III_0\}$ can be completed to an $A_2(7)$ in four ways, giving the systems

1. $V, VI, III_0, I_0, I_2, I_3, II_0, II_4, II_5, IV_0$
2. $V, VI, III_0, I_0, I_2, I_4, II_2, II_4, II_6, IV_6$
3. $V, VI, III_0, I_0, I_3, I_4, II_1, II_2, II_5, IV_5$
4. $V, VI, III_0, I_2, I_3, I_4, II_0, II_1, II_6, IV_3$

The permutation (016)(245) maps 2. onto 3. and (015)(246) maps 3. onto 4., hence the last three systems are isomorphic. The first two systems however are not isomorphic as can be seen as follows:
Both systems can be partitioned in five pairs of systems with intersectionnr 3 (we write the point associated with such a pair in front of it):

1. $3: (I_0,II_0)$, $3:(I_2,II_5)$, $3:(I_3,II_4)$, $3:(V,IV_0)$, $3:(VI,III_0)$
2. $6: (I_0,II_6)$, $5:(I_2,II_2)$, $0:(I_4,II_4)$, $2:(V,IV_6)$, $3:(VI,III_0)$

That is, the first type of $A_2(7)$ has associated with it a unique point (in the above case the point $\{3\}$) while the second type of $A_2(7)$ has associated with it a pair of points (those not occurring as point associated with a pair with intersectionnr 3, in the above case the pair $\{1,4\}$).
The structure of an $A_2(7)$ of the first kind - called a centered $A_2(7)$ - can be described as follows:

1. Colour of the edges of a K_6 with five colours (three edges of each colour).
2. Add a seventh point $\{\infty\}$ to each of these edges; this gives 15 triples divided in five groups of three.
3. Each group of three triples can be completed to an STS(7) in exactly

two ways. This gives five pairs of STS(7).

It is easily seen that these ten STS(7) form a centered $A_2(7)$, and since all centered $A_2(7)$ are isomorphic each $A_2(7)$ is obtained in this way. Since there are 6 ways to colour a K_6 there are 6 $A_2(7)$ with a given center, and 42 $A_2(7)$ in all.

Two centered $A_2(7)$ with the same center have exactly one pair (with intersectionnr 3) in common; given an $A_2(7)$ with center $\{i\}$ and a point $j \neq i$ then there is exactly one $A_2(7)$ with center $\{j\}$ disjoint from the given one. Two such disjoint centered $A_2(7)$ determine a unique $C(S_7,A_2(7))$ the third $A_2(7)$ simply consisting of all STS(7) not occurring in the first two. This third one is of the second kind, with associated pair $\{i,j\}$. Therefore there are 6 $A_2(7)$ of the second kind associated with a given pair $\{i,j\}$, and 126 $A_2(7)$ in all.

Given an $A_2(7)$ associated with the pair $\{i,j\}$ then an $A_2(7)$ of the second kind disjoint with it must be associated with a pair $\{i,k\}$ or $\{j,k\}$ with $k \notin \{i,j\}$. Indeed, given an $A_2(7)$ associated with $\{i,j\}$ and a point $\{k\}$ there is exactly one $A_2(7)$ associated with $\{i,k\}$ disjoint with it. Two such disjoint $A_2(7)$ of the second kind determine a unique $C(S_7,A_2(7))$, the third $A_2(7)$ being associated with $\{j,k\}$. Therefore there are two different $C(S_7,A_2(7))$, the first containing 2 centered $A_2(7)$ and the second containing no centered $A_2(7)$; there are 126 of the first type and 210 of the second type.

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