DEPARTMENT OF PURE MATHEMATICS

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THE BLOCKING NUMBER OF AN AFFINE SPACE

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ABSTRACT

It is proved that the minimum cardinality of a subset of AG(k,n) which intersects all hyperplanes is \( k(n-1) + 1 \). In case \( k = 2 \) this settles a conjecture of J. Doyen.

KEY WORDS & PHRASES: affine space, blocking
J. DOYEN [1] proved that the minimum cardinality of a subset of $\text{PG}(2,n)$ intersecting all lines equals $n + 1$, where this minimum is attained only if such a subset is a line. He also showed that in each affine plane $\text{AG}(2,n)$ there is a subset of cardinality $2n - 1$, intersecting all lines (by taking e.g. the union of two intersecting lines) and that for some small values of $n$ there are no such subsets with fewer points. He conjectured that for all values of $n$ there is no subset of $\text{AG}(2,n)$, intersecting all lines and with fewer than $2n - 1$ points. This is shown by the following theorem.

**THEOREM.** Let $\text{AG}(k,n)$ be the $k$-dimensional affine space over $\text{GF}(n)$. Then the minimum cardinality of a subset of $\text{AG}(k,n)$ which intersects all hyperplanes is $k(n-1) + 1$.

(Note that we do not have any results on non-Desarguesian affine planes.)

**PROOF.** Let $n$ be a prime-power and let $\text{AG}(k,n)$ be the $k$-dimensional affine space over $\text{GF}(n)$. We first observe that there is always a subset of cardinality $k(n-1) + 1$ intersecting all hyperplanes. For the union of $k$ independent lines through one given point intersects all hyperplanes and has cardinality $k(n-1) + 1$. Secondly suppose $A \subset \text{AG}(k,n)$ intersects all hyperplanes. We may suppose that $\underline{0} = (0, \ldots, 0) \in A$; let $B = A \setminus \underline{0}$. Then $B$ intersects all hyperplanes not through $\underline{0}$. A hyperplane not through $\underline{0}$ is determined by an equation

$$w_1 x_1 + \ldots + w_k x_k = 1,$$

for some $w_1, \ldots, w_k$ in $\text{GF}(n)$, not all zero.

Hence for all $(w_1, \ldots, w_k) \neq \underline{0}$ there exists a $\underline{b} = (b_1, \ldots, b_k)$ in $B$ such that $w_1 b_1 + \ldots + w_k b_k = 1$. Therefore, if we let

$$F(x_1, \ldots, x_k) = \bigwedge_{\underline{b} \in B} (b_1 x_1 + \ldots + b_k x_k - 1),$$

then $F(w_1, \ldots, w_k) = 0$ for all $k$-tuples $(w_1, \ldots, w_k) \neq \underline{0}$.

Now a well-known theorem says that if $P(x_1, \ldots, x_k)$ is a polynomial which only assumes the value zero then $P(x_1, \ldots, x_k) \in (x_1^n - x_1, \ldots, x_k^n - x_k)$, that is, there are polynomials $P_i(x_1, \ldots, x_k)$ (for $i = 1, \ldots, k$) such that

$$P(x_1, \ldots, x_k) = P_1(x_1, \ldots, x_k)(x_1^n - x_1) + \ldots + P_k(x_1, \ldots, x_k)(x_k^n - x_k).$$
Now let

\[ F(x_1, \ldots, x_k) = F_1(x_1, \ldots, x_k)(x_1^{n-1} - x_1) + \ldots + F_k(x_1, \ldots, x_k)(x_k^{n-1} - x_k) + J(x_1, \ldots, x_k), \]

such that the highest degree of \( x_i \) in \( J(x_1, \ldots, x_k) \) is at most \( n-1 \) \((1 \leq i \leq k)\).

Since for each \( i = 1, \ldots, k \) the polynomial \( x_iF(x_1, \ldots, x_k) \) only assumes the value zero, also for each \( i = 1, \ldots, k \) the polynomial \( x_iJ(x_1, \ldots, x_k) \) only assumes the value zero. Applying the above-mentioned theorem and using the fact that the highest degree of each \( x_i \) in \( J(x_1, \ldots, x_k) \) is at most \( n-1 \), it follows that for each \( i = 1, \ldots, k \):

\[ (x_i^{n-1} - 1) \mid J(x_1, \ldots, x_k), \]

or

\[ \prod_{i=1}^{k} (x_i^{n-1} - 1) \mid J(x_1, \ldots, x_k). \]

Since \( F(0, \ldots, 0) \neq 0 \) and hence \( J(0, \ldots, 0) \neq 0 \), it follows that the degree of \( J(x_1, \ldots, x_k) \) is \( k(n-1) \). This implies that the degree of \( F(x_1, \ldots, x_k) \) is at least \( k(n-1) \). Now, by definition, the degree of \( F(x_1, \ldots, x_k) \) equals \( |B| \). Hence \( |B| \geq k(n-1) \) and \( |A| \geq k(n-1) + 1 \), proving the theorem. \( \Box \).

REFERENCE