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(DEPARTMENT OF PURE MATHEMATICS)

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J. DE VRIES

COSEPARATORS IN CATEGORIES OF TOPOLOGICAL TRANSFORMATION GROUPS

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Coseparators in categories of topological transformation groups

by

J. de Vries

ABSTRACT

The main result in this note is that the category $COMP^G$ of all compact G-spaces has a coseparator, provided G is a locally compact Hausdorff topological group. This provides a partial solution to an open question raised earlier by the author.

KEY WORDS & PHRASES: *compact topological transformation group, coseparator in a category,* G-*space,* G-*compactification.*

In certain parts of mathematics the question is of interest whether all members of a given class of objects can be embedded in one "comprehensive" object. See for example [1]. This sort of problem, when studied in a categorical context, leads to the concept of a *coseparator*. See [2; 19.6] or [3; 24.6.5].

As to the existence of "comprehensive objects" for certain classes of topological transformation groups (ttg's) we refer to [4; Chap. III]. These results are derived independently of coseparators, but nevertheless it is interesting to know which categories of ttg's have a coseparator. In [4; section 6.4] we obtained some results in this direction as consequences of a general theorem about "preservation of coseparators" by certain functors. However, the question of whether the category of all compact Hausdorff G-spaces has a coseparator was left open. In this note we give an affirmative answer for the case that G is a locally compact Hausdorff group.

In the sequel, G shall always denote a locally compact Hausdorff topological group with identity element e. Recall that *a G-space* is a pair $\langle X,\pi \rangle$ in which X a topological space and $\pi: G\times X \to X$ is a continuous mapping satisfying the conditions: $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all s, $t \in G$ and $x \in X$. If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a *morphism of G-spaces* $f: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ is a continuous mapping $f: X \to Y$ such that $f(\pi(t,x)) = \sigma(t,f(x))$ for all $t \in G$, $x \in X$. In this way we obtain the category of all G-spaces and all morphisms of G-spaces, denoted $\mathcal{TOP}^G$. If $\mathcal{B}$ is a full subcategory of $\mathcal{TOP}$, then the corresponding full subcategory of $\mathcal{TOP}^G$ is denoted $\mathcal{B}^G$. As a general reference for categories of ttg's, see [4] and [5]. If X is any topological space, then $C_c(G,X)$ denotes the space of all continuous functions from G into X, endowed with the compact-open topology. If $\tilde{\rho}_X: G\times C_c(G,X) \to C_c(G,X)$ is defined by

$$\tilde{\rho}_X(t,f)(s) := f(st)$$

for $f \in C_c(G,X)$ and t, $s \in G$ (so each $\tilde{\rho}_X^{\,t}$ is a right-translation of functions), then $\langle C_c(G,X),\tilde{\rho}_X\rangle$ is a G-space ($\tilde{\rho}_X$ is continuous because G is locally compact). In the sequel we shall always omit the subscript X in $\tilde{\rho}_X$.

<u>PROPOSITION 1</u>. *Let* $\mathcal{B}$ *denote a full subcategory of* $\mathcal{TOP}$ *which has a coseparator* X *such that* $C_c(G,X)$ *is an object in* $\mathcal{B}$. *Then* $\langle C_c(G,X),\tilde{\rho}\rangle$ *is a*

coseparator in $\mathcal{B}^G$.

PROOF. Let $<Y,\sigma>$ be any object in $\mathcal{B}^G$ and let $y_1, y_2 \in Y$, $y_1 \neq y_2$. It is sufficient to show that there exists a morphism of G-spaces f: $<Y,\sigma> \to <C_c(G,X), \tilde{\rho}>$ with $f(y_1) \neq f(y_2)$. Since X is een coseparator in $\mathcal{B}$ there exists a continuous function g: $Y \to X$ such that $g(y_1) \neq g(y_2)$. Define f: $Y \to C_c(G,X)$ by

$$f(y)(t) := g(\sigma(t,y)), \quad y \in Y, \; t \in G.$$

It is easily checked that f: $Y \to C_c(G,X)$ is continuous. Moreover, by direct computation one can verify that f: $<Y,\sigma> \to <C_c(G,X),\tilde{\rho}>$ is a morphism of G-spaces. Since

$$f(y_1)(e) = g(y_1) \neq g(y_2) = f(y_2)(e)$$

we have $f(y_1) \neq f(y_2)$, as desired. □

EXAMPLES (cf. also [4; section 6.4]).

1. The indiscrete two-point space E_2 is a coseparator in *TOP*. Hence $<C_c(G,E_2),\tilde{\rho}>$ is a coseparator in TOP^G.
2. Let F_2 denote the two-point space $\{0,1\}$ with the T_0-topology $\{\phi,\{0\},\{0,1\}\}$. Then $<C_c(G,F_2),\tilde{\rho}>$ is a coseparator in the full subcategory of TOP^G, determined by all T_0 G-spaces.
3. The discrete two-point space D_2 is a coseparator in the full subcategory TOP_0 of all zero-dimensional Hausdorff spaces. Since $C_c(G,D_2)$ is also zero-dimensional, it follows that $C_c(G,D_2)$ is a coseparator in TOP^G.
4. The closed unit interval I is a coseparator in the category *TYCH* of all Tychonoff (= completely regular Hausdorff) spaces. Since $C_c(G,I)$ is a Tychonoff space, $<C_c(G,I),\tilde{\rho}>$ is a coseparator in $TYCH^G$.
5. Observe that $C_c(G,I)$ is compact iff G is discrete. [If G is discrete, $C_c(G,I) = I^G$ is compact by the Tychonoff-theorem. Conversely, if $C_c(G,I)$ is compact, then $C_p(G,I)$ is compact. But $C_p(G,I)$ is dense in I^G, hence it coincides with I^G.] Consequently, *unless G is discrete, our method does not provide a coseparator for* $COMP^G$ (*COMP* is the category of compact Hausdorff spaces).

THEOREM 2. *For any locally compact Hausdorff group* G *the category* $COMP^G$ *has a coseparator.*

PROOF. In [5] it is shown that every G-space has a G-compactification (G locally compact). For the G-space $<C_c(G,I),\tilde{\rho}>$ this means that there exists a morphism h: $<C_c(G,I),\tilde{\rho}> \to <Z,S>$ in TOP^G such that Z is a compact Hausdorff space and h is a topological embedding of $C_c(G,I)$ in Z. In particular, k is injective. It follows immediately from EXAMPLE 4 above and the injectivity of h, that <Z,G> is a coseparator in $TYCH^G$. Because <Z,G> is an object in $COMP^G$, it follows that it is a coseparator in $COMP^G$. □

REMARK. It can be shown that the weight $w(Z)$ of the space Z mentioned in the above proof equals $w(G)$, the weight of G. So if G is a separable metrizable group (and, of course, locally compact) then Z is second-countable, hence compact and metrizable; therefore, in this case <Z,S> is also a coseparator for the category of all compact metrizable G-spaces. It would be of interest to find a coseparator for this category under weaker conditions on G. [For example: G sigma-compact. In this context, observe that for a sigma-compact group G, $C_c(G,I)$ is metrizable, so that $<C_c(G,I),\tilde{\rho}>$ in a coseparator in the category of all metrizable G-spaces.] Another open question is, whether the category $COMP^G$ has *injective* coseparator.

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