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(DEPARTMENT OF PURE MATHEMATICS)

J. DE VRIES

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# Bounds for a cardinal function on G-spaces

by

J. de Vries

ABSTRACT

Let G be a locally compact topological group. For every Tychonoff G-space $\langle X,\pi\rangle$ we define $b\langle X,\pi\rangle$ as the least cardinal number of a base of a uniformity for X with respect to which $\pi$ is motion-equicontinuous. We show in this note that $\ell w(G) \leq b\langle X,\pi\rangle \leq w(X)$, where $\ell w$ and $w$ denote the local weight and the weight function, respectively.

KEY WORDS & PHRASES: G-*space, locally compact topological group, motion-equicontinuity, boundedness for* G-*spaces, weight, local weight, uniform weight*

## 1. INTRODUCTION

In this note the letter G will always denote a locally compact topological group with unit e. Recall that a G-*space* is an ordered pair $\langle X,\pi\rangle$, where X is a topological space and $\pi: G \times X \to X$ is a continuous mapping such that $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all $t,s \in G$ and $x \in X$. We shall use the following notation: $\pi^t x := \pi(t,x) =: \pi_x t$ for $(t,x) \in G \times X$. The G-space $\langle X,\pi\rangle$ is called *effective* whenever $\pi^t \neq \pi^e$ for $t \neq e$.
In the sequel we shall use only Tychonoff G-spaces, i.e. G-spaces $\langle X,\pi\rangle$ where X is a Tychonoff (= completely regular Hausdorff) space. If $\mathcal{U}$ is an admissible uniformity for X then $\langle X,\pi\rangle$ is called *$\mathcal{U}$-bounded* [)1] whenever the subset $\{\pi_x : x \in X\}$ of C(G,X) is equicontinuous at e (with respect to the uniformity $\mathcal{U}$ in X, of course). In [2], Proposition 7.3.12 it has been shown that this concept of boundedness is closely related to the possible existence of a G-compactification of $\langle X,\pi\rangle$, that is, an equivariant embedding of $\langle X,\pi\rangle$ in a compact Hausdorff G-space. According to the main result in [3], there exists always a uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded, provided G is locally compact. In that case, the least cardinal number of a base for a uniformity $\mathcal{U}$ of X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded will be denoted $b\langle X,\pi\rangle$. We shall derive bounds for $b\langle X,\pi\rangle$ in terms of the local weight $\ell w(G)$ of G, the weight $w(X)$ and the uniform weight $u(X)$ of X. In addition, we touch the question whether there is any relationship between the existence of a metrizable G-compactification of $\langle X,\pi\rangle$ (in the case that X is separable and metrizable) and the value of $b\langle X,\pi\rangle$.

## 2. RESULTS

PROPOSITION. *Let* G *be a locally compact topological group. Then for every Tychonoff* G*-space* $\langle X,\pi\rangle$ *the following inequalities hold:*

$$\max\{\ell w(G),\ u(X)\} \leq b\langle X,\pi\rangle \leq w(X)\ .$$

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)1 Also called *motion-equicontinuous* by some authors.

PROOF. It is obvious that $u(X) \leq b\langle X,\pi\rangle$, so it is sufficient to prove that $\ell w(G) \leq b\langle X,\pi\rangle \leq w(X)$. First, we show that $\ell w(G) \leq b\langle X,\pi\rangle$ provided $\langle X,\pi\rangle$ is effective. To this end, consider an admissible uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded, and which has a base $\mathcal{B}$ such that $|\mathcal{B}| = b\langle X,\pi\rangle$. Define, for every $x \in X$ and $\alpha \in \mathcal{B}$,

$$V_{x,\alpha} := \{t \in G \,\vdots\, (x,\pi_x t) \in \alpha\}.$$

Since the mapping $t \mapsto (x,\pi_x t) : G \to X \times X$ is continuous and each $\alpha \in \mathcal{B}$ is a neighbourhood of the diagonal in $X \times X$, each $V_{x,\alpha}$ is a neighbourhood of of e in G. Setting $V_\alpha := \bigcap\{V_{x,\alpha} \,\vdots\, x \in X\}$, the fact that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded implies that $V_\alpha$ is a neighbourhood of e in G for every $\alpha \in \mathcal{B}$. Moreover, $\bigcap\{V_\alpha \,\vdots\, \alpha \in \mathcal{B}\} = \{e\}$ because $\langle X,\pi\rangle$ is effective. It follows, that G is a Hausdorff group. However, G is locally compact, and now the fact that $\bigcap\{V_\alpha \,\vdots\, \alpha \in \mathcal{B}\} = \{e\}$ implies that $\{V_\alpha \,\vdots\, \alpha \in \mathcal{B}\}$ is a local subbase at e. Therefore, $\ell w(G) \leq |\mathcal{B}| = b\langle X,\pi\rangle$.

Next, we show that $b\langle X,\pi\rangle \leq w(X)$. Remember from the first part of the proof that G is Hausdorff. Since G is also locally compact, and G acts effectively on X, it follows that $\ell w(G) \leq w(X)$; see [4]. In [3], we constructed a uniformity $\mathcal{U}$ for X such that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded. This uniformity was generated by a set $\{g_j \,\vdots\, j \in J\}$ of continuous, $[0,1]$-valued functions, whence $b\langle X,\pi\rangle \leq |J|$. In the construction, the index set J was, in fact, the set $\mathcal{B}_e \times C(X,[0,1])$, where $\mathcal{B}_e$ is a local base at e in G. So we may assume that $|\mathcal{B}_e| = \ell w(G) \leq w(X)$. However, the construction in [3] works equally well if we replace $C(X,[0,1])$ by any of its subsets which separates points and closed subsets of X. Since X can topologically be embedded in a product of $w(X)$ copies of $[0,1]$, there exists such a subset of $C(X,[0,1])$ of cardinality $w(X)$. Thus we may assume that $|J| \leq w(X)$, whence $b\langle X,\pi\rangle \leq w(X)$. □

REMARKS. Let $\langle X,\pi\rangle$ be a G-space.

1. If $\mathcal{U}$ is an admissible uniformity for X and if $\mathcal{B}$ is a base for $\mathcal{U}$, then we can define, for every $x \in X$ and every $\alpha \in \mathcal{B}$, as in the above proof

$$V_{x,\alpha} := \{t \in G \,\vdots\, (x,\pi_x t) \in \alpha\};$$

$$V_\alpha := \bigcap_{x \in X} V_{x,\alpha}.$$

*Obviously,* $\langle X,\pi\rangle$ *is* $\mathcal{U}$*-bounded iff* $V_\alpha$ *is a neighbourhood of* e *in* G *for every* $\alpha \in \mathcal{B}$.
The following is easy to prove: if $\alpha \in \mathcal{B}$ and $\alpha$ is closed in $X \times X$, and if A is a dense subset of X, then

$$V_\alpha = \bigcap_{x\in A} V_{x,\alpha}.$$

If G is non-discrete, and the cardinal number $p(G)$ is defined as the least cardinal number of a collection of neighbourhoods of e in G whose intersection is *not* a neighbourhood of e, then the following statement is clear: *if* $d(X) < p(G)$, *then* $\langle X,\pi\rangle$ *is* $\mathcal{U}$*-bounded for every admissible uniformity* $\mathcal{U}$ *of* X (here $d(X)$ denotes the density of X). This generalizes the trivial observation that $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded for every admissible uniformity if G is discrete.

2. In the second part of the proof of our proposition, i.e. the proof that $b\langle X,\pi\rangle \le w(X)$, we used the fact that G was Hausdorff (shown in the first part of the proof) and that $\langle X,\pi\rangle$ was effective. Both assumptions can be removed. Indeed, if G is locally compact (but possibly not Hausdorff) and $\langle X,\pi\rangle$ is not effective, then $H := \{t \in G : \pi^t = \pi^e\}$ is a closed, normal subgroup of G. Hence G/H is a locally compact Hausdorff group. Moreover, G/H acts effectively on X by $\sigma(tH,x) := \pi(t,x)$ $(t\in G,\ x\in X)$. So we have an effective G/H-space $\langle X,\sigma\rangle$. It is easy to see that for every admissible uniformity $\mathcal{U}$ of X the G-space $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded iff the G/H-space $\langle X,\sigma\rangle$ is $\mathcal{U}$-bounded, so that $b\langle X,\pi\rangle = b\langle X,\sigma\rangle$. But our proposition applies to the G/H-space $\langle X,\sigma\rangle$ to the effect that $b\langle X,\sigma\rangle \le w(X)$. Hence $b\langle X,\pi\rangle \le w(X)$.

3. In a similar way one shows, that if $\langle X,\pi\rangle$ is not effective, and G possibly not Hausdorff (but still locally compact), then $\ell w(G/H) \le b\langle X,\pi\rangle$.

4. In [2], 7.3.2, we defined a G-space $\langle X,\pi\rangle$ to be *metrically bounded* if X is metrizable and $\langle X,\pi\rangle$ is $\mathcal{U}$-bounded for some metric uniformity $\mathcal{U}$ (equivalently: a uniformity with a countable base). So a G-space $\langle X,\pi\rangle$ is metrically bounded iff $b\langle X,\pi\rangle \le \aleph_0$. It was shown that if G is locally compact and sigma-compact, then $\langle X,\pi\rangle$ is metrically bounded if X is separable and metrizable. Using Remark 2 above, it is clear that sigma-compactness of G can be removed from the hypothesis: *if* G *is locally compact then every separable metrizable* G*-space* $\langle X,\pi\rangle$ *is metrically bounded.*

5. In a sense, the bounds for $b\langle X,\pi\rangle$ given in our proposition are best possible. Indeed, if G is discrete and X is metrizable but not separable, then $b\langle X,\pi\rangle = u(X) = \aleph_0$ and $b\langle X,\pi\rangle < w(X)$. On the other hand, in [2], 7.3.5 (iii) is an example of a locally compact (even sigma-compact) group G and a non-separable metrizable space X for which $b\langle X,\pi\rangle = w(X)$. Finally, if we consider a suitable locally compact group G acting on itself by left translations, we obtain a G-space $\langle G,\rho\rangle$ with $b\langle G,\rho\rangle = \ell w(G) < w(G)$ (start with a group G for which $\ell w(G) < w(G)$, and observe that $(G,\rho)$ is $\mathcal{U}$-bounded for the right uniformity $\mathcal{U}$ of G; hence $b\langle G,\rho\rangle \leq \ell w(G)$).

## 3. RELATION OF $b\langle X,\pi\rangle$ TO THE SIZE OF G-COMPACTIFICATIONS

Recall from [3] that a G-*compactification* of $\langle X,\pi\rangle$ is an equivariant dense embedding of $\langle X,\pi\rangle$ in a compact Hausdorff G-space $\langle Y,\sigma\rangle$. If $\langle Y,\sigma\rangle$ is a G-compactification of $\langle X,\pi\rangle$, then clearly $b\langle X,\pi\rangle \leq u(Y)$. Indeed, since Y is compact, a straightforward compactness argument shows that $\langle Y,\sigma\rangle$ is bounded with respect to its unique uniformity $\mathcal{U}$. Then $\langle X,\pi\rangle$ is, of course, bounded with respect to the relativation of $\mathcal{U}$ to X, and $b\langle X,\pi\rangle \leq u(Y)$. However, for the compact space Y, we have $u(Y) = w(Y)$, hence $b\langle X,\pi\rangle \leq w(Y)$.
In [3], the existence of a G-compactification $\langle Y,\sigma\rangle$ of $\langle X,\pi\rangle$ has been shown such that $w(Y) \leq \max\{w(G),w(X)\}$, under the assumptions that G is locally compact and X is a Tychonoff space. Obviously, this is consistent with our proposition, but it gives no additional information about the value of $b\langle X,\pi\rangle$.

So we ask the question the other way round: can the weight of a possible G-compactification be estimated in terms of $b\langle X,\pi\rangle$? In particular, has $\langle X,\pi\rangle$ a metrizable G-compactification if $b\langle X,\pi\rangle = \aleph_0$ ?
The following example (which is esentially due to the late professor J. DE GROOT [1] ) answers the second question in the negative, thus leaving completely open the first one.

EXAMPLE. Let X be the space of the rationals with its usual topology, and let G be the group of all homeomorphisms of X onto itself, provided with the discrete topology, the action of G on X being the obvious one.

Then $b\langle X,\pi\rangle = u(X) = \aleph_0$. We shall show that no G-compactification of $\langle X,\pi\rangle$ can be metrizable.

Let Y be an arbitrary metrizable compactification of X. Then the metric of Y induces a metric in X, and if all members of G where extendable to Y, they would be all uniformly continuous with respect to this metric. This, however, is not true: there exists a Cauchy sequence $\{x_n\}_n$ in X with respect to this metric which does not converge (X is not topologically complete). If $\{a_n\}_n$ and $\{b_n\}_n$ are sequences converging to 0 and 1 respectively, then there exists $h \in G$ such that $h(x_n) = a_n$ if n is odd and $h(x_n) = b_n$ if n is even. Then h is not uniformly continuous.

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