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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

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A.E. BROUWER

TWO NEW NEARLY KIRKMAN TRIPLE SYSTEMS

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Two new Nearly Kirkman Triple Systems

by

A.E. Brouwer

ABSTRACT

Baker and Wilson constructed Nearly Kirkman Triple Systems $NKTS(6t) = RDG(3,1,2;6t)$ for all $t \geq 3$ except $t = 14, 17, 29$. Here we give a direct construction of a $NKTS(6t)$ for all odd $t, t \geq 11$. In particular this solves the cases $t = 17, 29$ so that $t = 14$ remains the last unsolved case.

KEY WORDS & PHRASES: *Nearly Kirkman Triple System, resolvable group divisible design.*

INTRODUCTION

A Nearly Kirkman Triple System NKTS(v) (nomenclature of KOTZIG & ROSA [3]) or a resolvable group divisible design RGD(3,1,2; v) (HANANI-like notation) is a resolvable pairwise balanced design on v points, with one parallel class consisting of pairs and the remaining parallel classes consisting of triples. Obviously a necessary condition for the existence of a NKTS(v) is $v \equiv 0 \pmod{6}$ or $v = 2$, and KOTZIG & ROSA [3] showed that a NKTS($6t$) does not exist for $t = 1, 2$ while it does exist for $t = 3$. BAKER & WILSON [1] proved the existence of a NKTS($6t$) for all $t \geq 3$ except for $t \in \{14, 17, 29\}$. In this note we give a direct construction of a NKTS($6t$) for t odd, $t \geq 11$. This means that $6 \cdot 14 = 84$ is the only remaining order for which the existence of a NKTS(v) is not yet decided.

The ingredients

We need some auxiliary designs:

- (a) A NKTS(18), which can be found in [3] or constructed directly by doubling the affine plane AG(2,3).
- (b) A group divisible design GD(4,1,6;6m) for all $m \geq 5$. These designs are constructed by BROUWER, HANANI & SCHRIJVER [2].
- (c) A pairwise balanced design on a set V of 18 points with 6 parallel classes of triples and 1 block Z of size 6 and on the remaining 12 points (i.e. on $V \setminus Z$) 2 parallel classes of triples and one parallel class of pairs.

[Note that indeed $\binom{18}{2} = 153 = 6 \cdot 18 + \binom{6}{2} + 2 \cdot 12 + 6$ so that this is possible numerically.]

This design was found by PDP11 and can be described as follows:

Let $V = Z \cup I_{12}$, where $I_{12} = \{0, 1, \dots, 11\}$ and $Z = \{a, b, c, d, e, f\}$.

Take the parallel classes on V :

1. $\{a, 0, 11\}$, $\{b, 1, 3\}$, $\{c, 4, 8\}$, $\{d, 6, 10\}$, $\{e, 2, 7\}$, $\{f, 5, 9\}$
2. $\{a, 7, 9\}$, $\{b, 0, 10\}$, $\{c, 1, 5\}$, $\{d, 2, 4\}$, $\{e, 8, 11\}$, $\{f, 3, 6\}$
3. $\{a, 4, 6\}$, $\{b, 2, 11\}$, $\{c, 0, 9\}$, $\{d, 5, 7\}$, $\{e, 3, 10\}$, $\{f, 1, 8\}$

4. $\{a,1,10\}$, $\{b,5,6\}$, $\{c,2,3\}$, $\{d,0,8\}$, $\{e,4,9\}$, $\{f,7,11\}$
5. $\{a,3,8\}$, $\{b,4,7\}$, $\{c,6,11\}$, $\{d,1,9\}$, $\{e,0,5\}$, $\{f,2,10\}$
6. $\{a,2,5\}$, $\{b,8,9\}$, $\{c,7,10\}$, $\{d,3,11\}$, $\{e,1,6\}$, $\{f,0,4\}$

And the parallel classes on V/Z :

- i. $\{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$, $\{9,10,11\}$
- ii. $\{0,3,7\}$, $\{1,4,11\}$, $\{2,6,9\}$, $\{5,8,10\}$
- I. $\{0,6\}$, $\{1,7\}$, $\{2,8\}$, $\{3,9\}$, $\{4,10\}$, $\{5,11\}$.

The construction

I got the idea of trying this construction when J.-C. BERMOND told me how he constructed resolvable decompositions of the complete directed graph into directed 3-circuits. The construction is a modification of the one used by BAKER & WILSON: we use 6 'points at infinity' instead of two. At first it would seem that this requires a NKTS(18) with subdesign NKTS(6) which cannot exist, but on closer inspection one finds that the existence of ingredient c suffices.

Starting with a group divisible design $GD(4,1,6;6m)$ $(X, \mathcal{B}, \mathcal{G})$ where X is the point-set ($|X| = 6m$), \mathcal{B} is the collection of blocks (all of size 4) and $\mathcal{G} = \{X_0, X_1, \dots, X_{m-1}\}$ is the collection of groups (all of size 6) we form a NKTS($12m + 6$) with pointset $(X \times \{0,1\}) \cup Z$ as follows:

Let $Z = Y \times \{0,1\}$ where $|Y| = 3$ and take for the parallel class consisting of pairs the collection of pairs $\{(u,0), (u,1)\}$ for $u \in X \cup Y$. We need $6m + 2$ parallel classes of triples; for each point $x \in X$ we shall find one parallel class C_x and furthermore we have two other parallel classes \mathcal{D} and \mathcal{D}' .

Let $((X_0 \times \{0,1\}) \cup Z, \mathcal{B}_0, \mathcal{G}_0)$ be a NKTS(18) as given by ingredient a, where $\mathcal{G}_0 = \{\{(u,0), (u,1)\} \mid u \in X_0 \cup Y\}$ and \mathcal{B}_0 is partitioned into parallel classes P_i ($0 \leq i \leq 7$).

Let for $1 \leq j \leq m-1$ $((X_j \times \{0,1\}) \cup Z, \mathcal{B}_j)$ be a design as given by ingredient c, where each time Z is the block of size 6, $V = (X_j \times \{0,1\}) \cup Z$,

$\{(u,0), (u,1)\} \mid u \in X_j\}$ is the parallel class of pairs on $V \setminus Z$, \mathcal{D}_j and \mathcal{D}'_j are the parallel classes of triples on $V \setminus Z$, and $Q_{j,i}$ ($0 \leq i \leq 5$) are the parallel classes of triples on V .

For each block $B \in \mathcal{B}$ construct a $\text{GD}(3,1,2;8)$ on $B \times \{0,1\}$ by deleting the point ∞ from an $\text{AG}(2,3)$ constructed on $(B \times \{0,1\}) \cup \{\infty\}$ in such a way that it contains the lines $\{\infty, (u,0), (u,1)\}$ for $u \in B$. For $u \in B$ let $C_{u,B}$ denote the pair of the blocks in the $\text{GD}(3,1,2;8)$ that were lines parallel to $\{\infty, (u,0), (u,1)\}$ in the $\text{AG}(2,3)$.

Now define C_x as follows:

If $x \in X_0$ then let

$$C_x = \cup\{C_{x,B} \mid x \in B \in \mathcal{B}\} \cup P_{i(x)}$$

and if $x \in X_j$ ($j > 0$) then let

$$C_x = \cup\{C_{x,B} \mid x \in B \in \mathcal{B}\} \cup Q_{j,i(x)}$$

where for each j we have fixed some bijection $i : X_j \rightarrow \{0,1,2,3,4,5\}$.

Finally let

$$\mathcal{D} = \cup\{\mathcal{D}_j \mid j > 0\} \cup P_6$$

and

$$\mathcal{D}' = \cup\{\mathcal{D}'_j \mid j > 0\} \cup P_7.$$

This completes the construction of a $\text{NKTS}(12m + 6)$.

REMARK: Similar methods work in the case of a $\text{RGD}(k,1,k-1;v)$ for general k .

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