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A. SCHRIJVER

A SHORT PROOF OF MINC's CONJECTURE

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A short proof of Minc's conjecture*)

by

A. Schrijver

ABSTRACT

A short proof is given of the following conjecture of Minc, proved in 1973 by Brègman. Let A be a n×n-(0,1)-matrix with r_i ones in row i. Then

$$\text{per } A \leq \prod_{i=1}^{n} r_i!^{1/r_i}.$$

KEY WORDS & PHRASES: *permanent, Minc's conjecture.*

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1. INTRODUCTION

Let $A = (a_{ij})$ be a nonnegative n×n-matrix. The *permanent* per A is by definition

$$\text{per } A = \sum_{\nu \in S_n} \prod_{i=1}^{n} a_{i\nu_i},$$

where S_n is the set of all permutations on $\{1,\dots,n\}$. Minc [3] conjectured and Brègman [1] proved the following upper bound for permanents of (0,1)-matrices A:

$$\text{per } A \leq \prod_{i=1}^{n} r_i!^{\,1/r_i};$$

here r_i is the number of ones in row i of A. In his proof Brègman uses the duality theorem of convex programming and some theory on doubly stochastic matrices. We give a short proof of his result using only elementary counting and the following easy lemma.

LEMMA. *If* $t_1,\dots,t_r$ *are nonnegative real numbers then*

$$\left(\frac{t_1+\dots+t_r}{r}\right)^{t_1+\dots+t_r} \leq t_1^{\,t_1}\dots t_r^{\,t_r}.$$

PROOF. Taking logarithms of both sides and dividing by r we have to show:

$$\left(\frac{t_1+\dots+t_r}{r}\right)\log\left(\frac{t_1+\dots+t_r}{r}\right) \leq \frac{t_1\log t_1 + \dots + t_r\log t_r}{r},$$

which is true by the convexity of the function $x\log x$. □

Our proof is based on the nice last step of Brègman's proof. In section 3 we discuss some generalizations of the upper bound to arbitrary nonnegative matrices.

We use the notation A_{ij} for the minor obtained from A by deleting row i and column j. Furthermore $0^0 = 1$ and $0!^{1/0} = 0$. For a survey on bounds for permanents we refer to Van Lint [2] p.54-62.

2. MINC'S CONJECTURE

We give an elementary proof of the following theorem of Brègman [1].

THEOREM. (Minc's conjecture [3], Brègman [1]). *Let* A *be a* n×n-(0,1)-*matrix with* r_i *ones in row* i $(1 \le i \le n)$; *then*

$$\text{per } A \le \prod_{i=1}^{n} r_i!^{1/r_i}.$$

PROOF. We use induction on n; for n = 1 the theorem is trivial. Suppose the theorem has been proved for (n-1) × (n-1)-matrices. We shall prove

$$(\text{per } A)^{n \text{ per } A} \le \left(\prod_{i=1}^{n} r_i!^{1/r_i}\right)^{n \text{ per } A},$$

which implies the above inequality. The proof consists of a number of steps (equalities and inequalities); we first give these steps and after that we justify each step. The variables i,j and k range from 1 to n. Let S be the set of all permutations ν of {1,...,n} for which $a_{i\nu_i} = 1$ for all i = 1,...,n. So |S| = per A.

$$(\text{per } A)^{n \text{ per } A} \overset{(1)}{=} \prod_i (\text{per } A)^{\text{per } A}$$

$$\overset{(2)}{\le} \prod_i \Big(r_i^{\text{per } A} \prod_{\substack{k \\ a_{ik}=1}} \text{per } A_{ik}^{\text{per } A_{ik}}\Big)$$

$$\overset{(3)}{=} \prod_{\nu\in S} \Big(\big(\prod_i r_i\big).\big(\prod_i \text{per } A_{i\nu_i}\big)\Big)$$

$$\overset{(4)}{\leq} \prod_{\nu\in S} \Big(\big(\prod_i r_i\big)\cdot\Big(\prod_i \Big(\prod_{\substack{j \\ j\neq i \\ a_{j\nu_i}=0}} r_j!^{1/r_j}\Big)\cdot\Big(\prod_{\substack{j \\ j\neq i \\ a_{j\nu_i}=1}} (r_j-1)!^{1/r_j-1}\Big)\Big)\Big) =$$

$$\overset{(5)}{=} \prod_{\nu\in S}\Big(\big(\prod_i r_i\big)\cdot\Big(\prod_j \Big(\prod_{\substack{i \\ i\neq j \\ a_{j\nu_i}=0}} r_j!^{1/r_j}\Big)\cdot\Big(\prod_{\substack{i \\ i\neq j \\ a_{j\nu_i}=1}} (r_j-1)!^{1/r_j-1}\Big)\Big)\Big) =$$

$$\overset{(6)}{=} \prod_{\nu\in S} \Big(\big(\prod_i r_i\big)\cdot\Big(\prod_j r_j!^{\frac{n-r_j}{r_j}} (r_j-1)!^{\frac{r_j-1}{r_j-1}}\Big)\Big) =$$

$$\overset{(7)}{=} \prod_{\nu\in S}\big(\prod_i r_i!^{n/r_i}\big) \quad \overset{(8)}{=} \Big(\prod_i r_i!^{1/r_i}\Big)^n \text{ per } A .$$

Explanation of the steps:

(1) trivial;

(2) apply the lemma (note that r_i is the number of k such that $a_{ik} = 1$ and $\text{per } A = \sum_{\substack{k \\ a_{ik}=1}} \text{per } A_{ik}$);

(3) the number of factors r_i equals per A on both sides, while the number of factors per A_{ik} equals the number of $\nu \in S$ for which $\nu_i = k$ (this is per A_{ik} in case $a_{ik} = 1$, and 0 otherwise);

(4) apply the induction hypothesis to each $A_{i\nu_i}$ $(i = 1,\ldots,n)$;

(5) change the order of multiplication;

(6) the number of i such that $i \neq j$ and $a_{j\nu_i} = 0$ is $n-r_i$, whereas the number of i such that $i \neq j$ and $a_{j\nu_i} = 1$ is r_j-1 (note that $a_{j\nu_j} = 1$ and that the equality is proved for all fixed ν and j separately);

(7) and (8) are trivial. □

3. GENERALIZATIONS TO ARBITRARY NONNEGATIVE MATRICES.

Using essentially the same method one can find an upper bound for permanents of arbitrary nonnegative matrices. To this end define the function f on vectors by

$$f(\underline{a}) = \Big(\prod_{\mu \varepsilon S_k} b_0(b_0+b_{\mu_1}) \;....\; (b_0+b_{\mu_1}+\ldots+b_{\mu_k}) \Big)^{\frac{1}{(k+1)!}}$$

where $(b_0,\ldots,b_k)$ is a permutation of (the entries of) the vector $\underline{a}$ such that $b_0 \geq b_i$ for $i = 1,\ldots,k$.
Now f is stable under permutation of arguments and $f(a_1,\ldots,a_n,0) = f(a_1,\ldots,a_n)$. Repeating the arguments of the proof in section 2 then yields the bound

$$\text{per } A \leq \prod_{i=1}^{n} f(\underline{a}_i),$$

in which $\underline{a}_i$ is the i-th row of A. This upper bound applied to (0,1)-matrices produces Minc's upper bound, since $f(1,\ldots,1,0,\ldots,0) = r!^{1/r}$, where r is the number of ones. However, this bound is not very good. A.E. Brouwer observed that a sharper upper bound for permanents of nonnegative matrices can be obtained directly from Minc's bound by the following method. First remark that the permanent function is linear on the rows (just as the determinant is), and that each row of a nonnegative matrix is a (non-negative) linear combination of (0,1)-vectors. Now define

$$g(\underline{a}) := \sum_{i=1}^{n} (b_i - b_{i+1})(i!)^{1/i}$$

in which $(b_1,\ldots,b_n)$ is a permutation of the entries of the vector $\underline{a}$ such that $b_1 \geq \ldots \geq b_n$ and $b_{n+1} = 0$. Then

$$\text{per } A \leq \prod_{i=1}^{n} g(\underline{a}_i)$$

in which $\underline{a}_i$ is the i-th row of A. Again, specialized to (0,1)-matrices this upper bound passes into Minc's upper bound.

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