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AFDELING ZUIVERE WISKUNDE ZN 77/77 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER

THE WORST COVERING OF POINTS BY PERMUTATIONS

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The worst covering of points by permutations

by

A.E. Brouwer

ABSTRACT

We show that for $n \geq 3$ the cardinality of a largest minimal cover of points by permutations is $n(n-2)$.

KEY WORDS & PHRASES: *permutation-design, covering.*

INTRODUCTION

M. Deza introduced the concepts of packing and covering of permutations, analogues of the corresponding concepts for sets.

A collection $\mathcal{P}$ of permutations of $I_n = \{1,2,\ldots,n\}$ is called a t-packing (resp. t-cover) if for each injection $f: T \to I_n$ (where $|T| = t$ and $T \subset I_n$) there is at most (resp. at least) one $\pi \in \mathcal{P}$ such that $\pi|T = f$.

A minimal cover is a cover such that none of its elements can be removed; a worst covering is a minimal cover with maximal cardinality. (And likewise we have the concepts of maximal packing and worst packing.) If we represent the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \pi_1 & \pi_2 & \ldots & \pi_n \end{pmatrix}$$

by the row $\pi_1\pi_2\ldots\pi_n$, then we are looking for $N\times n$ matrices with N as large as possible such that each column contains all the numbers $1,2,\ldots,n$ while each row contains an element that is unique in its column. Deza told me that $n(n-2)$ is an upper bound for N while $n(n-2)$ can be achieved for $n = 3,4,5$. We shall see that indeed $N = n(n-2)$ is possible for all $n \geq 3$.

Now we know everything about permutation 1-designs: The worst packing, best packing and best covering all have n elements (the corresponding matrix being a Latin square).

About permutation 2-designs we have less information; Deza showed that a perfect permutation 2-design (an optimal 2-packing that is at the same time an optimal 2-cover) is essentially the same object as a projective plane of order n.

For worst designs one can prove that for $n = 4$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$$

is (the unique) worst packing, and the 4! − 4 = 20 remaining permutations form (the unique) worst covering. (Note that the worst 2-packing for n = 4 has less elements than the worst 2-packing for n = 3 !)

<u>The worst 1-cover</u>

For $n \le 3$ we have:

$$n = 2: \quad \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \qquad\qquad n = 3: \quad \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$

(in these cases the best and the worst 1-cover coincide).

For $n > 3$ we have:

If a column contains n unique elements, then there are only n rows.

If a column contains n−1 unique elements then by induction there are at most $(n-1) + (n-1)(n-3) = (n-1)(n-2) < n(n-2)$ rows.

If each column contains at most n−2 unique elements then there are at most $n(n-2)$ rows.

Hence for $n \ge 3$: $N \le n(n-2)$

and if equality holds (and $n > 3$) then each column contains exactly n−2 unique elements.

Example for n = 5 (the unique elements are underlined):

$\underline{2}$ 1 3 4 5	π_{21}
1 $\underline{3}$ 2 4 5	π_{22}
1 2 $\underline{4}$ 3 5	..
1 2 3 $\underline{5}$ 4	..
5 2 3 4 $\underline{1}$	π_{25}
$\underline{3}$ 1 2 4 5	π_{31}
1 $\underline{4}$ 2 3 5	..
1 2 $\underline{5}$ 3 4	..
5 2 3 $\underline{1}$ 4	..
5 1 3 4 $\underline{2}$	π_{35}
$\underline{4}$ 1 2 3 5	π_{41}
1 $\underline{5}$ 2 3 4	..
5 2 $\underline{1}$ 3 4	..
5 1 3 $\underline{2}$ 4	..
5 1 2 4 $\underline{3}$	π_{45}

Generally for $2 \le i \le n-1$ and $1 \le j \le n$ we define the permutation π_{ij} by

$$\pi_{ij}(k) = \begin{cases} k & \text{if } i \le k-j \le n-1 \\ k-1 & \text{if } 1 \le k-j \le i-1 \\ \underline{i+j-1} & \text{if } j = k \end{cases}$$

where all arithmetic is done mod n.

It is easily seen that

(i) each π_{ij} is a permutation of I_n, and

(ii) in column k all permutations have k or k-1 except the permutations π_{ik} which have

$$\pi_{ik}(k) = k+i-1 \quad (i=2,\ldots,n-1)$$

so that each of them is necessary.

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