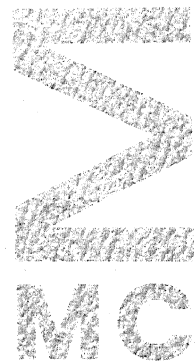


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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

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NOVEMBER

A.E. BROUWER

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OF ORDER FIVE

amsterdam

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**stichting
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The uniqueness of the truncated affine plane of order five

by

A.E. Brouwer

ABSTRACT

We prove that the only pairwise balanced design on 24 points with blocks of size 4 or 5 (and $\lambda = 1$) is the truncation of the affine plane of order five.

KEY WORDS & PHRASES: *Pairwise balanced design.*

Trying to construct non-isomorphic block designs with certain parameters I was led to the question whether there exist non-isomorphic $PBD(\{4,5\},1;24)$ designs. Unfortunately this turns out not to be the case.

Let (X, \mathcal{B}) be a pairwise balanced design with $|X| = 24$ and $B \in \mathcal{B} \rightarrow |B| \in \{4,5\}$. Let $B_i = \mathcal{B} \cap P_i(x)$ ($i = 4,5$). If a point x is in c_4 blocks of size 4 and in c_5 blocks of size 5 then $3c_4 + 4c_5 = 23$ so that $c_4 = 1, c_5 = 5$ or $c_4 = 5, c_5 = 2$. In the former case x is called of type I, and in latter - of type II. Let there be a points of type I and b points of type II, then we have

$$\begin{aligned} a + b &= 24 \\ 5a + 2b &= 5b_5 \\ a + 5b &= 4b_4 \end{aligned}$$

(where $b_i = |B_i|$, $i = 4,5$) and we have the following five possibilities:

Case 1: $b = 0, a = 24, b_4 = 6, b_5 = 24$.

Here each point is in a unique block of size 4 so that the blocks of size 4 form a parallel class. Completing this parallel class by adjoining a point ∞ produces (the unique) affine plane of order five. Hence this is the case of $AG(2,5) \setminus \{p\}$.

From now on we suppose that $b > 0$.

Case 2: $b = 5, a = 19, b_4 = 11, b_5 = 21$.

Case 3: $b = 10, a = 14, b_4 = 16, b_5 = 18$.

Case 4: $b = 15, a = 9, b_4 = 21, b_5 = 15$.

Case 5: $b = 20, a = 4, b_4 = 26, b_5 = 12$.

Suppose that some block B of size 4 contains three or four points of type I. If $y \notin B$ then the blocks containing y and the points of type I in B all have size 5 so that y must be of type I. But then there can be at most one point of type II. Contradiction.

In the same way it follows that if two blocks B and B' , both of size 4, each contain two points of type I, then there are no points of type II

outside $B \cup B'$, so that $b \leq 4$. Contradiction.

But this shows that $a \leq b_4 + 1$ so that case 2 is impossible.

Let $X_1 = \{x \in X \mid x \text{ of type I}\}$ and $X_2 = X \setminus X_1$. Counting the number of pairs in X_2 covered by blocks of size 4 we find either

$$(b_4 - a) \cdot \binom{4}{2} + a \cdot \binom{3}{2},$$

or

$$(b_4 - a + 1) \cdot \binom{4}{2} + (a - 2) \cdot \binom{3}{2} + \binom{2}{2}$$

i.e.

$$6b_4 - 3a + (0 \text{ or } 1).$$

This must be less than the total number of edges $\binom{b}{2}$. In case 3 we find $54 \leq 45$. Contradiction.

Next suppose that some block B of size 4 contains two points x_1, x_2 of type I, and let B' be another block of size 4 containing a point x_3 of type I. The blocks B_i containing x_i and x_3 ($i = 1, 2$) are of size 5 and we have

$$|B \cup B' \cup B_1 \cup B_2| = 13 \text{ or } 14.$$

If $y \notin B \cup B' \cup B_1 \cup B_2$ then y must be of type I, since each of the three blocks containing y and x_i ($i = 1, 2, 3$) is of size 5. But then $X_2 \subset B \cup B' \cup B_1 \cup B_2$ and $b = |X_2| \leq 11$. Contradiction.

Hence each block of size 4 contains at most one point of type I.

Let x_1, x_2, x_3 be three points of type I, and let B_i be the block of size 4 containing x_i ($i = 1, 2, 3$). Furthermore, let B_i' be the block (of size 5) containing x_j and x_k ($\{i, j, k\} = \{1, 2, 3\}$). As before, if $y \notin B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'$ then y must be of type I, and since

$$|B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'| \leq 21$$

it follows that

$$a \geq 3 + (24 - 21) = 6.$$

This disposes of case 5.

Since only case 4 is left we know that

$$|\{B \in \mathcal{B}_4 \mid B \cap X_1 = \emptyset\}| = 12$$

and

$$|\{B \in \mathcal{B}_4 \mid B \cap X_1 \neq \emptyset\}| = 9.$$

Let

$$p_i = |\{B \in \mathcal{B}_5 \mid |B \cap X_1| = i\}| \quad (i = 0, 1, 2, 3, 4, 5).$$

We have:

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 15.$$

Counting edges in X_2 not covered by blocks of size 4:

$$10p_0 + 6p_1 + 3p_2 + p_3 = 6.$$

Counting edges in X_1 :

$$p_2 + 3p_3 + 6p_4 + 10p_5 = \binom{9}{2} = 36.$$

It follows that

$$6p_0 + 2p_1 + 2p_4 + 6p_5 = 42 - 60 = -18.$$

Contradiction.

Hence also case 4 is impossible.