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# A PROOF OF TOTAL DUAL INTEGRALITY OF MATCHING POLYHEDRA

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# A proof of total dual integrality of matching polyhedra

by

A. Schrijver & P.D. Seymour*

ABSTRACT

The Tutte-Berge theorem gives a min-max formula for the maximum cardinality of a matching in a graph. Edmonds' matching polyhedron theorem characterizes the convex hull of the set of (characteristic vectors of) matchings in a graph. We prove a common generalization of these theorems; this generalization was proved earlier, in a different (algorithmic) way, by CUNNINGHAM & MARSH.

KEY WORDS & PHRASES: *matching, linear programming, integrality.*

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## 1. INTRODUCTION

[In this paper a *graph* is a finite undirected graph without loops or parallel edges; an edge is, by definition, a *set* of two vertices.]
A famous theorem of TUTTE [7] asserts the following.

(1.1) *A graph* $G = (V,E)$ *has a* 1*-factor if and only if for each subset* $V'$ *of* $V$ *the number of odd components of* $\langle V\setminus V'\rangle$ *does not exceed* $|V'|$.

[Here $\langle V\setminus V'\rangle$ is the subgraph of G induced by $V \setminus V'$, and an *odd* component is a component with an odd number of vertices. A 1*-factor* is a collection of edges, pairwise disjoint and covering all points.]
This theorem has turned out to be fundamental for subsequent investigations in matching theory. [A *matching* is a collection of pairwise disjoint edges.] For example, by adding new vertices one can deduce the following theorem of BERGE [1].

(1.2) *The maximum cardinality of a matching in a graph* $G = (V,E)$ *equals*

$$\min_{V'\subseteq V} \frac{|V|+|V'|-o(V\setminus V')}{2}.$$

[In this formula $o(V\setminus V')$ denotes the number of odd components of $\langle V\setminus V'\rangle$.] This result is known as the *Tutte-Berge theorem.*

EDMONDS [3] studied maximum *weighted* matchings, and he gave a good algorithm for finding one (given a weighting of the edges). An interesting theoretical by-product is his *matching polyhedron theorem*:

(1.3) *Let* $G = (V,E)$ *be a graph and let* $f$ *be a nonnegative real-valued function defined on* $E$. *Then* $f$ *is expressible as a convex combination of (characteristic functions of) matchings in* $G$ *if and only if*

(i) $\sum_{e\ni v} f(e) \le 1$, *for each vertex* $v$, *and*

(ii) $\sum_{e\subseteq V'} f(e) \le \lfloor \frac{|V'|}{2} \rfloor$, *for each subset* $V'$ *of* $V$.

[$\lfloor x \rfloor$ denotes the lower integer part of a real number x, and $\lceil x \rceil$ its upper integral part.] Clearly, the inequalities (i) and (ii) are satisfied by any convex combination of matchings, since each matching itself satisfies them - the content of the theorem is the converse. Edmonds' theorem gives the faces of the convex hull of the matchings;
the faces of the convex hull of a set S of points in a euclidean space is that, given a linear functional, one may apply linear programming methods to find an optimum point of S. (1.3) makes it possible to express explicitly the convex hull P of the matchings in the form

$$P = \{x \mid x \geq 0,\ Ax \leq b\}$$

for a certain matrix A and vector b, derived in the obvious way from the inequalities (i) and (ii) of (1.3). Since the matchings are the extreme points of P we have that

$$\max \{wx \mid x \geq 0,\ x \text{ integer-valued},\ Ax \leq b\} = \max \{wx \mid x \geq 0,\ Ax \leq b\}$$

for each weight function $w \in \mathbb{R}^E$. [wx denotes the inner product of w and x. By expressions such as wx and $Ax \leq b$ we implicitly assume that the vectors and matrices are of the correct sizes.] The left hand side is the maximum weight of a matching; the duality theorem of linear programming states that

$$\max \{wx \mid x \geq 0,\ Ax \leq b\} = \min \{yb \mid y \geq 0,\ yA \geq w\}.$$

Hence, the last right hand side equals the maximum weight of a matching. For the case $w \equiv 1$ we have, by the Tutte-Berge theorem (1.2), a stronger result, since (1.2) may be formulated as

$$\max \{\underline{1}x \mid x \geq 0,\ x \text{ integer-valued},\ Ax \leq b\} = \min \{yb \mid y \geq 0,\ y \text{ integer-valued},\ yA \geq \underline{1}\}$$

[$\underline{1}$ denotes an all-one vector], that is, the minimum is also attained by an integer-valued vector. We shall prove that this is true for *each* integer-valued weight-function, that is, following EDMONDS & GILES [4], the matching polyhedron is *totally dual integral*. This was proved earlier, in a different way, by CUNNINGHAM & MARSH [2]. This result appears to be not obtainable from (1.1) and (1.3) by elementary constructions, despite the self-refining nature of these theorems. It is itself self-refining, and in section 4 we mention some of its stronger forms.

## 2. A LEMMA ON LINEAR PROGRAMMING

In this section we prove a useful lemma, which is a combination of results of EDMONDS & GILES [4] and LOVÁSZ [5,6]. For convenience we shall identify functions in $\mathbb{R}^X$ (where X is a set) with the corresponding vectors. Similarly, an $X \times Y$-matrix is a matrix with rows indexed by X and columns indexed by Y. Moreover, subsets of X are identified with their corresponding characteristic functions in $\mathbb{R}^X$. For vectors v, the vectors $\lfloor v \rfloor$ and $\lceil v \rceil$ arise from v by taking coordinate-wise lower and upper integral parts, respectively. $\mathbb{Z}_+$ and $\mathbb{R}_+$ are the sets of nonnegative integers and real numbers, respectively. For nonnegative numbers r, $r\mathbb{Z}_+$ denotes the set of nonnegative integer multiples of r. Finally, given a vector y, $|y|$ denotes the sum of its coordinates.

Observe that for $m \times n$-matrices C and vectors $d \in \mathbb{Z}^m$, $w \in \mathbb{Z}^n$ we always have:

$$(1) \qquad \begin{aligned} &\max \{wx \mid x \in \mathbb{Z}^n_+,\ Cx \le d\} \le \max \{wx \mid x \in \mathbb{R}^n_+,\ Cx \le d\} = \\ &\min \{yd \mid y \in \mathbb{R}^m_+,\ yC \ge w\} \le \min \{yd \mid y \in \tfrac{1}{2}\mathbb{Z}^m_+,\ yC \ge w\} \le \\ &\min \{yd \mid y \in \mathbb{Z}^m_+,\ yC \ge w\}. \end{aligned}$$

Hence, if the first and the last expression are equal then also the last two minima are equal. The lemma asserts that also the converse holds: if, *for each* $w \in \mathbb{Z}^n$, the last two minima are equal, then all five optima are the same (for each $w \in \mathbb{Z}^n$).

<u>LEMMA</u>. *Let* C *be an* $m \times n$*-matrix and let* $d \in \mathbb{Z}^m$. *Then:*

(2) *for each* $w \in \mathbb{Z}^n$ *both sides of the linear programming duality equation* $\max \{wx \mid x \in \mathbb{R}^n_+,\ Cx \le d\} = \min \{yd \mid y \in \mathbb{R}^m_+,\ yC \ge w\}$ *are attained by integer-valued vectors* x *and* y,

*if and only if*

(3) *for each* $w \in \mathbb{Z}^n$, $\min \{yd \mid y \in \frac{1}{2}\mathbb{Z}^m_+,\ yC \geq w\}$ *is attained by an integer-valued vector* y.

PROOF. By (1) it is sufficient to prove that (3) implies (2). So suppose (3) holds. Then for each natural number k

(4) $$\min \{yd \mid y \in 2^{-(k+1)}\mathbb{Z}^m_+,\ yC \geq w\} = \min \{yd \mid y \in 2^{-k}\mathbb{Z}^m_+,\ yC \geq w\},$$

since this is equivalent to

(5) $$2^{-k} \cdot \min \{yd \mid y \in \tfrac{1}{2}\mathbb{Z}^m_+,\ yC \geq 2^k \cdot w\} = 2^{-k} \cdot \min \{yd \mid y \in \mathbb{Z}^m_+,\ yC \geq 2^k \cdot w\},$$

which is true by (3). Therefore also for each natural number k

(6) $$\min \{yd \mid y \in 2^{-k}\mathbb{Z}^m_+,\ yC \geq w\} = \min \{yd \mid y \in \mathbb{Z}^m_+,\ yC \geq w\}.$$

Hence, since

(7) $$\min \{yd \mid y \in \mathbb{R}^m_+,\ yC \geq w\} = \inf_k \left(\min\{yd \mid y \in 2^{-k} \cdot \mathbb{Z}^m_+,\ yC \geq w\}\right),$$

we have that

(8) $$\min \{yd \mid y \in \mathbb{R}^m_+,\ yC \geq w\} = \min \{yd \mid y \in \mathbb{Z}^m_+,\ yC \geq w\}.$$

By the duality theorem of linear programming

(9) $$\max \{wx \mid x \in \mathbb{R}^n_+\ Cx \leq d\} = \min \{yd \mid y \in \mathbb{R}^m_+,\ yC \geq w\}.$$

Since d is integer-valued, it follows from (8) and (9) that $\max \{wx \mid x \in \mathbb{R}^n_+,\ Cx \leq d\}$ is an integer, for each $w \in \mathbb{Z}^n$. Therefore, as can be seen easily, the vertices of the polyhedron $\{x \in \mathbb{R}^n_+ \mid Cx \leq d\}$ are integer-valued. Since each minimal (nonempty) face of this polyhedron is a vertex we have that

(10) $$\max\{wx \mid x\in\mathbb{Z}^n_+,\ Cx\le d\} = \max\{wx \mid x\in\mathbb{R}^n_+,\ Cx\le d\}$$

for each $w \in \mathbb{Z}^n$ (and hence for each $w \in \mathbb{R}^n$). (8), (9) and (10) together imply (2). □

In the proof we have made use of the fact that each minimal face of the polyhedron $\{x\in\mathbb{R}^n_+ \mid Cx\le d\}$ is a vertex. The lemma can be extended to linear programming problems in which this is not necessarily the case - see EDMONDS & GILES [4].

## 3. MAXIMUM WEIGHTED MATCHINGS

Now we come to the common generalization of the Tutte-Berge theorem and Edmonds' matching polyhedron theorem. This generalization has been proved earlier, using different methods, by CUNNINGHAM & MARSH [2].

Let $G = (V,E)$ be a graph, and let $A$ be its $V \times E$-incidence matrix, i.e.

$$A_{v,e} = 1 \text{ if } v \in e, \text{ and } A_{v,e} = 0 \text{ if } v \notin e,$$

for $v \in V$ and $e \in E$. Let $\mathcal{P}(V)$ be the power-set of $V$; define $B$ as the $\mathcal{P}(V) \times E$-matrix given by

$$B_{V',e} = 1 \text{ if } e \subseteq V', \text{ and } B_{V',e} = 0 \text{ if } e \nsubseteq V',$$

for $V' \subseteq V$ and $e \in E$. So the rows of $B$ are the collections of edges of induced subgraphs of $G$. The function (or vector) $f \in \mathbb{R}_+^{\mathcal{P}(V)}$ is defined by

$$f(V') = f_{V'} = \lfloor \tfrac{1}{2}|V'| \rfloor,$$

for $V' \subseteq V$.

We first prove a theorem which shows the existence of certain nice integral solutions for our main linear programming problem, but which is also useful in proving the existence of integral solutions; so theorem 1 may be considered also as a lemma for theorem 2.

We call a collection $\mathcal{F}$ of subsets of $V$ *nested* if $V_1 \cap V_2 = \emptyset$, or $V_1 \subseteq V_2$, or $V_2 \subseteq V_1$, whenever $V_1, V_2 \in \mathcal{F}$. Similarly (by our convention this is an extension) a vector $t \in \mathbb{R}^{\mathcal{P}(V)}$ is called *nested* if the collection $\{V' \subseteq V \mid t_{V'} \neq 0\}$ is nested.

Nested collections have a certain tree-like structure (the Venn-diagram is "planar"); one can split up a nested collection $F$ in *levels*. The first level consists of all maximal sets (under inclusion) of $F$. The (i+1)-th level consists of all maximal sets properly contained in some set of i-th level. The sets of any level are pairwise disjoint.

THEOREM 1. *For each* $w \in \mathbb{Z}^n$

(11) $$\min \{|y|+tf \mid y \in \mathbb{Z}_+^V,\ t \in \mathbb{Z}_+^{\mathcal{P}(V)},\ yA+tB \geq w\}$$

*is attained by some nested* t.

PROOF. Let $w \in \mathbb{Z}^n$, and choose $y \in \mathbb{Z}_+^V$, $t \in \mathbb{Z}_+^{\mathcal{P}(V)}$ such that y and t attain the minimum in (11) and such that

(12) $$\sum_{V' \subseteq V} t_{V'} \cdot |V'| \cdot (|V \setminus V'|+1)$$

is as small as possible. We prove that t is nested. Suppose t is not nested; then there exist $V_1, V_2 \subseteq V$ such that $t_{V_1} \geq t_{V_2} > 0$, $V_1 \cap V_2 \neq \emptyset$ and $V_1 \nsubseteq V_2 \nsubseteq V_1$. First suppose that $|V_1 \cap V_2|$ is odd. Define

$$t'_{V_1 \cap V_2} = t_{V_1 \cap V_2} + t_{V_2},$$
$$t'_{V_1 \cup V_2} = t_{V_1 \cup V_2} + t_{V_2},$$
$$t'_{V_2} = 0,$$
$$t'_{V_1} = t_{V_1} - t_{V_2},$$

and let t' be equal to t in the remaining coordinates, i.e.

$$t' = t + t_{V_2}\{V_1 \cap V_2, V_1 \cup V_2\} - t_{V_2}\{V_1, V_2\},$$

using our identification of subsets of $\mathcal{P}(V)$ with vectors in $\mathbb{R}^{\mathcal{P}(V)}$. It can be checked straightforwardly that

$$\begin{cases} |y| + t'f \le |y| + tf, \\ yA + t'B \ge yA + tB, \\ \sum_{V' \subseteq V} t'_{V'} \cdot |V'| \cdot (|V \setminus V'|+1) < \sum_{V' \subseteq V} t_{V'} \cdot |V'| \cdot (|V \setminus V'|+1), \end{cases}$$

contradicting the fact that y and t were chosen such that (12) is minimal.

Secondly assume that $|V_1 \cap V_2|$ is even. Let

$$y' = y + t_{V_2} \cdot (V_1 \cap V_2),$$
$$t' = t + t_{V_2} \{V_1 \setminus V_2, V_2 \setminus V_1\} - t_{V_2} \{V_1, V_2\}.$$

In this case

$$\begin{cases} |y'| + t'f \le |y| + tf, \\ y'A + t'B \ge yA + tB, \\ \sum_{V' \subseteq V} t'_{V'} \cdot |V'| \cdot (|V \setminus V'|+1) < \sum_{V' \subseteq V} t_{V'} \cdot |V'| \cdot (|V \setminus V'|+1) \end{cases}$$

again contradicting the fact that (12) is minimal. □

Now we are ready to prove

THEOREM 2. *For each* $w \in \mathbb{Z}^E$ *both sides of the linear programming duality equation*

(13) $$\max \{wx \mid x \ge 0,\ Ax \le \underline{1},\ Bx \le f\} = \min \{|y| + tf \mid y \ge 0,\ t \ge 0,\ yA + tB \ge w\}$$

*are attained by integer-valued* x, y, t, *where* t *is nested at the same time.*

PROOF. By the lemma and theorem 1 it is sufficient to prove that for each $w \in \mathbb{Z}_+^E$

(14) $$\min \{y + tf \mid y \in \tfrac{1}{2}\mathbb{Z}_+^V,\ t \in \tfrac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)},\ yA + tB \ge w\}$$

is attained by integer-valued y and t (since A and B are nonnegative we have to consider only $w \in \mathbb{Z}_+^E$). Suppose this is not true, and let $w \in \mathbb{Z}_+^E$ be a fixed counterexample to this, such that $|w|$ is as small as possible. Then each $y \in \frac{1}{2}\mathbb{Z}_+^V$, $t \in \frac{1}{2}\mathbb{Z}_+^{P(V)}$ which reach the minimum in (14) are such that $y \in \{0,\frac{1}{2}\}^V$, $t \in \{0,\frac{1}{2}\}^{P(V)}$, except, possibly, the (inessential) t-values on singletons and the empty set. If this were not the case, there would exist, as can be seen easily, a counterexample $w'$ with $|w'| < |w|$.

Since (14) is equal to

$$\tfrac{1}{2}\min \{|y|+tf \mid y\in\mathbb{Z}_+^V,\ t\in\mathbb{Z}_+^{P(V)},\ yA+tB\geq 2w\}$$

it follows from theorem 1 that (14) is attained by some half-integer-valued y and t where t is nested. We may assume that t equals zero on singletons and on the empty set. We may also assume that y and t are chosen such that $|y|$ is as large as possible, under the condition that t is nested.

Now we define the nested collection $F$ as

$$F = \{V'\subseteq V \mid t_{V'}=\tfrac{1}{2}\},$$

and let

$$S = \{v\in V \mid y_v=\tfrac{1}{2}\}.$$

First suppose $F = \emptyset$, i.e. $t \equiv 0$. Define

$$\begin{cases} y' \equiv 0, \\ t' = \{S\}. \end{cases}$$

Then, as can be checked easily,

$$|y'| + t'f \leq |y| + tf,$$
$$y'A + t'B \geq \lfloor yA + tB \rfloor \geq w,$$

so $y'$, $t'$ reach the minimum in (14). This contradicts the assumption that for w there does not exist integer-valued y,t attaining the minimum in (14).

If $F \neq \emptyset$ there are sets on an odd level of the nested collection $F$;

let V' be a minimal set (under inclusion) in $\mathcal{F}$ on an odd level, i.e. V' is a minimal set such that $|\{V''\in\mathcal{F}|V'\subset V''\}|$ is odd. Let $W_1,\ldots,W_k$ be the sets in $\mathcal{F}$ properly contained in V' (possibly k = 0). So $W_1,\ldots,W_k$ are pairwise disjoint. It is easy to see that either

$$(15)\qquad \lfloor\tfrac{1}{2}|V'|\rfloor+\lfloor\tfrac{1}{2}|W_1|\rfloor+\ldots+\lfloor\tfrac{1}{2}|W_k|\rfloor \geq |V'\cap S| + 2(\lfloor\tfrac{1}{2}|W_1\setminus S|\rfloor+\ldots+\lfloor\tfrac{1}{2}|W_k\setminus S|\rfloor)$$

or

$$(16)\qquad \lfloor\tfrac{1}{2}|V'|\rfloor+\lfloor\tfrac{1}{2}|W_1|\rfloor+\ldots+\lfloor\tfrac{1}{2}|W_k|\rfloor \geq |V'\setminus S| + 2(\lfloor\tfrac{1}{2}|W_1\cap S|\rfloor+\ldots+\lfloor\tfrac{1}{2}|W_k\cap S|\rfloor).$$

If (15) is true let

$$y' = y + \tfrac{1}{2}(V'\cap S),$$
$$t' = t - \tfrac{1}{2}\{V',W_1,\ldots,W_k\} + \{W_1\setminus S,\ldots,W_k\setminus S\}.$$

Since, as can be checked straightforwardly, (the first inequality follows from (15))

$$(17)\qquad \begin{cases} |y'| + t'f \leq |y| + tf \\ y'A + t'B \geq \lfloor yA+tB\rfloor \geq w \end{cases}$$

we have that y', t' reach the minimum in (14). Hence y', t' are $\{0,\tfrac{1}{2}\}$-valued (except, possibly, the t'-values in singletons and the empty set), which implies that the right hand side of (15) equals zero. Since the left hand side of (15) is not zero this would yield a strict inequality

$$|y'| + t'f < |y| + tf$$

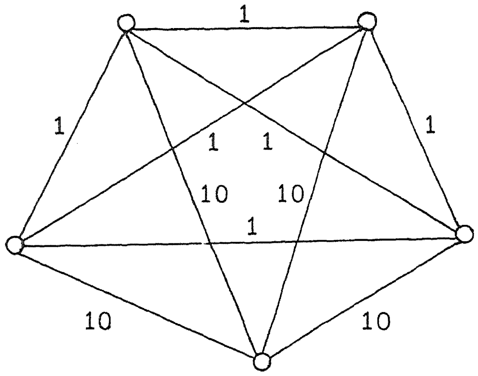
contradicting the minimality of $|y| + tf$.

Similarly we can deal with the case that (16) holds. Now let

$$y' = y + \tfrac{1}{2}(V'\setminus S),$$
$$t' = t - \tfrac{1}{2}\{V', W_1,\ldots,W_k\} + \{W_1\cap S,\ldots,W_k\cap S\}.$$

Again (17) holds; since t' is nested we have that $|y'| \leq |y|$; moreover, t' is $\{0,\frac{1}{2}\}$-valued. Hence the right hand side of (16) equals zero. This leads in the same way as before to a contradiction. □

As corrolaries of theorem 2 one has, straightforwardly, (1.1), (1.2) and (1.3). It is not true that there always exists a solution y,t for the minimum in (13) such that if $t_{V'} \neq 0$ and $v \in V'$ then $y_v = 0$, i.e. that we can split up V in a "y-part" and a "t-part" (as is the case if $w \equiv 1$, i.e. the Tutte-Berge theorem). As a counterexample consider the weighted graph




No solution y,t has $y_a = 0$ or $y_a = 10$.

## 4. DEGREE-CONSTRAINED SUBGRAPHS

Theorem 2 is of a self-refining nature: by means of elementary constructions as splitting up points or edges, adjoining new points or edges, taking large weights one may derive successively stronger results. We mention one result in this direction, giving an answer to the question: given a graph G = (V,E) and vectors $\ell, u \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$, what is the maximal value of

$$\sum_{e \in E} w(e) \cdot \phi(e)$$

where $\phi \in \mathbb{Z}^E$ such that for all $v \in V$

$$\ell(v) \leq \sum_{e \ni v} \phi(e) \leq u(v)?$$

Clearly, if $\ell \equiv 0$ and $u \equiv 1$ such function $\phi$ is a matching. So suppose

$\ell, u \in \mathbb{Z}_+^V$, where $G = (V,E)$ is a graph. Let $\mathcal{P}$ be the collection of all pairs $(X,Y)$ of subset of $V$ such that $X \cap Y = \emptyset$. Define $g \in \mathbb{Z}_+^{\mathcal{P}}$ by

$$g_{(X,Y)} = \lfloor \frac{u(X)-\ell(Y)}{2} \rfloor,$$

for $(X,Y) \in \mathcal{P}$. [Here $u(X) = \sum_{x \in X} u(x)$ and $\ell(Y) = \sum_{y \in Y} \ell(y)$.] Furthermore, define the $\mathcal{P} \times E$-matrix $C$ by

$$\begin{aligned} C_{(X,Y),e} &= 1, \text{ if } e \subseteq X, \\ &= -1, \text{ if } e \cap X = \emptyset \neq e \cap Y, \\ &= 0, \text{ otherwise,} \end{aligned}$$

for $(X,Y) \in \mathcal{P}$ and $e \in E$. A subcollection $\mathcal{F}$ of $\mathcal{P}$ is called *nested* if for all $(X_1,Y_1)$, $(X_2,Y_2) \in \mathcal{F}$

$$(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = \emptyset, \text{ or } (X_1 \subseteq X_2,\ Y_1 \subseteq Y_2), \text{ or } (X_2 \subseteq X_1,\ Y_2 \subseteq Y_1).$$

As an extension, a vector $t \in \mathbb{R}^{\mathcal{P}}$ is called *nested* if the collection $\{(X,Y) \in \mathcal{P} \mid t_{(X,Y)} \neq \emptyset\}$ is nested. Again nested collections of pairs have a nice structure. Now it is possible to prove: *for each* $w \in \mathbb{Z}^E$ *both sides of the LP-duality equation*

$$\max \{wx \mid x \geq 0,\ \ell \leq Ax \leq u,\ Cx \leq g\} = \min \{yu - z\ell + tg \mid y,z,t \geq 0,\ (y-z)A + tC \geq w\}$$

*are attained by integer-valued* $x,y,z,t$ *where* $t$ *is nested at the same time.*

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