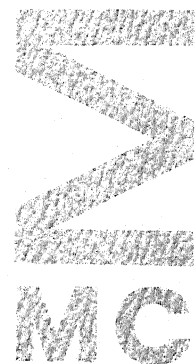


**ma
the
ma
tisch**

**cen
trum**



AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 86/78 JULI

J. VAN DE LUNE

A CONVEXITY THEOREM FOR SEQUENCES

amsterdam

1978

**stichting
mathematisch
centrum**



AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 86/78

JULI

J. VAN DE LUNE

A CONVEXITY THEOREM FOR SEQUENCES

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

A convexity theorem for sequences

by

J. van de Lune

ABSTRACT

It is shown that if a sequence is convex then certain derived weighted average sequences are also convex.

KEY WORDS & PHRASES: *convexity, inequalities.*

In [2] it was shown by Ozeki (also see [1; p. 202]) that if a sequence $\{a_n\}_{n=0}^{\infty}$ is convex then also the sequence of corresponding Cesàro means

$$\left\{ \frac{a_0 + a_1 + \dots + a_{n-1} + a_n}{n+1} \right\}_{n=0}^{\infty}$$

is convex.

In this note we give a generalization of this result.

THEOREM. *If the sequence $\{a_n\}_{n=0}^{\infty}$ is convex, i.e.*

$$a_{n-1} + a_{n+1} \geq 2a_n, \quad (n = 1, 2, 3, \dots),$$

then for any fixed $k \in \{0, 1, 2, 3, \dots\}$ the sequence

$$(*) \quad \left\{ \frac{\binom{k}{k} a_k + \binom{k+1}{k} a_{k+1} + \dots + \binom{n}{k} a_n}{\binom{n+1}{k+1}} \right\}_{n=k}^{\infty}$$

is also convex.

PROOF. The convexity of the sequence (*) may be expressed as

$$\frac{\binom{k}{k} a_k + \dots + \binom{n}{k} a_n}{\binom{n+1}{k+1}} + \frac{\binom{k}{k} a_k + \dots + \binom{n+2}{k} a_{n+2}}{\binom{n+3}{k+1}} \geq 2 \frac{\binom{k}{k} a_k + \dots + \binom{n+1}{k} a_{n+1}}{\binom{n+2}{k+1}}$$

where $k \in \{0, 1, 2, 3, \dots\}$ is fixed and $n \geq k$.

After multiplication by the factor

$$(n+3)(n+2) \dots (n-k+3)(n-k+2)(n-k+1)/(k+1)!$$

we obtain the equivalent inequality

$$\begin{aligned}
& (n+2)(n+3)\left\{\binom{k}{k}a_k + \dots + \binom{n}{k}a_n\right\} + \\
& + (n-k+2)(n-k+1)\left\{\binom{k}{k}a_k + \dots + \binom{n+2}{k}a_{n+2}\right\} + \\
& - 2(n+3)(n-k+1)\left\{\binom{k}{k}a_k + \dots + \binom{n+1}{k}a_{n+1}\right\} \geq 0.
\end{aligned}$$

For $r = k, k+1, \dots, n$ the coefficient c_r of the term a_r is

$$\begin{aligned}
c_r &= \{(n+2)(n+3) + (n-k+2)(n-k+1) - 2(n+3)(n-k+1)\}\binom{r}{k} = \\
&= (k+1)(k+2)\binom{r}{k}.
\end{aligned}$$

The coefficient c_{n+1} of a_{n+1} is

$$\begin{aligned}
c_{n+1} &= -2(n+3)(n-k+1)\binom{n+1}{k} + (n-k+2)(n-k+1)\binom{n+1}{k} = \\
&= -(n-k+1)(n+k+4)\binom{n+1}{k}
\end{aligned}$$

whereas the coefficient c_{n+2} of a_{n+2} is

$$c_{n+2} = (n-k+1)(n-k+2)\binom{n+2}{k}.$$

Now observe that from the convexity of $\{a_n\}_{n=0}^{\infty}$ it follows that for $r = k, k+1, \dots, n$

$$\begin{aligned}
& (r-k+1)(r-k+2)\binom{r+2}{k}a_r + (r-k+1)(r-k+2)\binom{r+2}{k}a_{r+2} + \\
& - 2(r-k+1)(r-k+2)\binom{r+2}{k}a_{r+1} \geq 0.
\end{aligned}$$

Add these inequalities and observe that (in the sum) for $r = k, k+1, \dots, n$ the coefficient c_r^* of a_r is

$$\begin{aligned}
c_r^* &= (r-k+1)(r-k+2) \binom{r+2}{k} + (r-k-1)(r-k) \binom{r}{k} - 2(r-k)(r-k+1) \binom{r+1}{k} \\
&= (k+1)(k+2) \left\{ \binom{r+2}{k+2} + \binom{r}{k+2} - 2 \binom{r+1}{k+2} \right\}.
\end{aligned}$$

Since, as one may verify

$$\binom{r+2}{k+2} + \binom{r}{k+2} - 2 \binom{r+1}{k+2} = \binom{r}{k}$$

we find that $c_r^* = c_r$ for $r = k, k+1, \dots, n$.

Assigning the obvious meaning to c_{n+1}^* and c_{n+2}^* we have

$$c_{n+1}^* = (n-k)(n-k+1) \binom{n+1}{k} - 2(n-k+1)(n-k+2) \binom{n+2}{k}$$

and

$$c_{n+2}^* = (n-k+1)(n-k+2) \binom{n+2}{k}.$$

Since

$$c_{n+2}^* = c_{n+2}$$

our proof will be complete if we can show that

$$c_{n+1}^* = c_{n+1}$$

or, equivalently (after division by $n-k+1$)

$$(n-k) \binom{n+1}{k} - 2(n-k+2) \binom{n+2}{k} = -(n+k+4) \binom{n+1}{k}.$$

This last inequality simplifies to

$$(n+2) \binom{n+1}{k} = (n-k+2) \binom{n+2}{k}$$

or, equivalently (after multiplication by $k!$)

$$(n+2)(n+1)n(n-1)\dots(n-k+2) = (n-k+2)(n+2)(n+1)n(n-1)\dots(n-k+3).$$

Since this equality holds true indeed, our proof is complete.

REFERENCES

- [1] MITRINOVIC, D.S., *Analytic inequalities*, Springer, (1970).
- [2] OZEKI, N., *On some inequalities*, J. College of Arts and Sci. Chiba Univ.,
4 No.3 (1965) pp. 211-214 (in Japanese).