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J.C.S.P. VAN DER WOUDE

HYPERTRANSFORMATION GROUPS AND RECURSIVENESS: SOME REMARKS ON AN ARTICLE OF S.C. KOO

2e boerhaavestraat 49 amsterdam

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Hypertransformation groups and recursiveness: some remarks on an article of S.C. Koo

by

Jaap van der Woude

ABSTRACT

We present here a study about hypertransformation groups $(T,2^X)$, induced by a topological transformation group (T,X). In particular this note is concerned with recursive properties, following the article of S.C. KOO on this subject. However, we skip his requirement of all phase spaces being compact T_2 and so we obtain generalization of his results.

KEYWORDS & PHRASES: *Hyperspace, recursivity, almost periodicity.*

0. INTRODUCTION

In [4] KOO studies recursive properties in hypertransformation groups, induced by topological transformation groups with compact T_2 phase space. In doing so, he uses the uniform structure on 2^X, induced by the uniformity on X. This paper is a collection of thoughts after [4], and the intention is two-fold. First, we shall give simpler proofs of some of his results, using as much as possible the less complicated Vietoris topology on 2^X, instead of its uniformity. Second, we skip the requirement of all phase spaces being compact T_2.

The first section is a brief summary of useful aspects of hyper spaces. The second section is concerned with the orbit closure relation and the space of orbit closures as a subspace of 2^X. In the third section we introduce hypertransformation groups and give a generalization of [4], Theorem 1.1, showing the elegancy of the Vietoris topology on 2^X. Sections 4 and 5 are concerned with recursiveness and in majority they provide generalizations and two-fold proofs.

For a more detailed study of hyperspaces we refer to [5]. The results of the Theorems 2.3, 2.5 and 4.4(b) seem to be essentially new.

CONVENTION: ALL TOPOLOGICAL SPACES UNDER CONSIDERATION ARE ASSUMED TO BE T_1 (except for quotient spaces and the underlying topological spaces of the acting groups).

1. HYPERSPACES

For a topological space X define

$$C(X) = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ compact}\},$$

$$2^X = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed}\}.$$

Observe that $\{x\} \in C(X)$, and $\{x\} \in 2^X$ for all $x \in X$ and $C(X) \subseteq 2^X$ if X is Hausdorff. We may topologize $C(X)$ and 2^X by the *Vietoris topology* as follows. For $\mathcal{A} = C(X)$ or $\mathcal{A} = 2^X$ and open subsets $U_1,\dots,U_n$ of X, set

$$\langle U_1,\dots,U_n\rangle = \{E \in \mathcal{A} \mid E \subseteq \bigcup_{i=1}^{n} U_i \text{ and } E \cap U_i \neq \emptyset \quad \text{for } i \in \{1,\dots,n\}\}.$$

Then the basis for the Vietoris topology on $\mathcal{A}$ is formed by the collection

$$\{\langle U_1,\ldots,U_m\rangle \subseteq \mathcal{A} \mid m \in \mathbb{N} \text{ and } U_i \text{ open in } X \text{ for } i \in \{1,\ldots,m\}\}.$$

Let $(X,\mathcal{U})$ be a uniform space. Then $\mathcal{U}$ induces a uniform structure $\mathcal{U}^*$ on 2^X. Define for all $\alpha \in \mathcal{U}$ and $E \in 2^X$

$$\alpha(E) = \cup\{\alpha(x) \mid x \in E\} = \{y \in X \mid \exists x \in E \wedge (x,y) \in \alpha\}$$

and

$$\alpha^* = \{(A,B) \in 2^X \times 2^X \mid A \subseteq \alpha(B) \wedge B \subseteq \alpha(A)\}.$$

Then the collection $\{\alpha^* \mid \alpha \in \mathcal{U}\}$ constitutes a basis for the uniform structure $\mathcal{U}^*$ on 2^X. We shall write 2^X_u or 2^X_f if we consider 2^X with the uniform topology or the Vietoris topology, respectively. Since the topologies coincide on $\mathcal{C}(X)$, there is no need to distinguish between $\mathcal{C}(X)_u$ and $\mathcal{C}(X)_f$. If X is compact Hausdorff, then $2^X = \mathcal{C}(X)$ and $2^X_u = 2^X_f$. For proofs of the following facts we refer to [5].

<u>THEOREM 1.1.</u>

a. 2^X_f *and* 2^X_u *are* T_1;
b. X *is* T_3 *iff* 2^X_f *is* T_2;
c. X *is* $T_{3\frac{1}{2}}$ *iff* $\mathcal{C}(X)$ *is* $T_{3\frac{1}{2}}$;
d. X *is compact iff* 2^X_f *is compact*
e. X *is compact* T_2 *iff* 2^X *is compact* T_2.

Let X and Y be topological spaces and f: X → Y a surjective map. If f is closed, define $f^*: 2^X \to 2^Y$ by $f^*(E) = f[E]$ for all $E \in 2^X$. If f is continuous, we may define $f^{\leftarrow *}: Y \to 2^X$ by $f^{\leftarrow *}(y) = f^{\leftarrow}(y)$ for all $y \in Y$ and $f^{\leftarrow **}: 2^Y \to 2^X$ by $f^{\leftarrow **}(D) = f^{\leftarrow}[D]$ for all $D \in 2^Y$. Then:

<u>THEOREM 1.2.</u>

a. $f^*: 2^X_f \to 2^Y_f$ *is continuous (topological) iff* f *is continuous (topological)*;
b. $f^*: 2^X_u \to 2^Y_u$ *is uniform continuous (topological) iff* f *is uniform continuous (topological)*;
c. $f^{\leftarrow **}: 2^Y_f \to 2^X_f$ *is continuous iff* $f^{\leftarrow *}: Y \to 2^X_f$ *is continuous iff* f *is open and closed*.

2. THE SPACE OF ORBIT CLOSURES AND 2^X_f

A *topological transformation group* (*ttg* for short) is a triple (T,X,π), with T a topological group, X a topological space and $\pi\colon T\times X \to X$ a continuous map, such that

a. $\pi(e,x) = x$ for all $x \in X$, and

b. $\pi(s,\pi(t,x)) = \pi(st,x)$ for all $s,t \in T$, $x \in X$.

We shall write $\pi^t(x) = \pi(t,x) = \pi_x(t)$; then $\pi^t\colon X \to X$ is a homeomorphism for every $t \in T$. Denote the orbit $\{\pi(t,x) \mid t \in T\}$ of x in X by $\Gamma(x)$, let $C(x) = \Gamma(x)$ be the orbit closure of x in X and define $f\colon X \to 2^X$ by $x \mapsto C(x)$. Then, in general, f fails to be continuous. However, f is always lower semi-continuous (that is, $\{x \in X \mid f(x) \cap U \neq \emptyset\}$ is open for every open U in X). Remember that for a ttg (T,X,π) a subset $A \subseteq X$ is called *minimal*, if A is nonempty, closed, invariant and A does not admit a proper subset with those properties.

THEOREM 2.1. *Let* (T,X,π) *be a ttg and let* $f\colon X \to 2^X_f$ *be continuous. Then every orbit closure is minimal. (In particular:* X *is pointwise almost periodic, if* X *is compact and* f *is continuous.)*

PROOF. Let $x \in X$ and suppose $C(x)$ is not minimal. Then there is a $y \in C(x)$ with $C(y) \neq C(x)$. Since 2^X is T_1 (Theorem 1.1(a)), there is a nbhd V of $C(y)$ in 2^X_f, such that $C(x) \notin V$. The continuity of f gives us a nbhd V_y of y in X, with $f[V_y] \subseteq V$. Now $y \in C(x)$, so $V_y \cap \Gamma(x) \neq \emptyset$, say $\pi(s,x) \in V_y$. Then $C(x) = C(\pi(x,s)) = f(\pi(s,x)) \in f[V_y] \subseteq V$, a contradiction. □

If every orbit closure in X is minimal, we may define an equivalence relation C on X by $xCy \iff x \in C(y)$. Denote the quotient space X/C, endowed with the quotient topology, by $(X/C)_q$ and define $(X/C)_f$ as the collection $\{C(x) \mid x \in X\} \subseteq 2^X_f$ with the relative topology. Remark that if $(X/C)_q$ exists, then it is (set-theoretic) isomorphic to $(X/C)_f$.

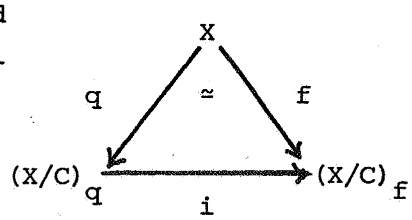
LEMMA 2.2. *The quotient topology on* X/C *is weaker than the Vietoris topology.*

PROOF. Let $q\colon X \to (X/C)_q$ be the quotient map, and let $U \subseteq (X/C)_q$ be open. Then $q^{\leftarrow}[U] = \{y \in X \mid C(y) \in U\}$ is open in X, so $\langle q^{\leftarrow}[U]\rangle$ is open in 2^X_f.

Moreover, $U = \langle q^{\leftarrow}[U]\rangle \cap (X/C)$; for if $q(y) = C(y) \in U$, then $C(y) \subseteq q^{\leftarrow}[U]$ and $C(y) \in \langle q^{\leftarrow}[U]\rangle$, so $U \subseteq \langle q^{\leftarrow}[U]\rangle \cap X/C$. Conversely, if $q(z) = C(z) \in$ $\in \langle q^{\leftarrow}[U]\rangle$, then $C(z) \in q^{\leftarrow}[U]$, so $z \in q^{\leftarrow}[U]$ and $q(z) \in U$. Hence $\langle q^{\leftarrow}[U]\rangle \cap$ $\cap\ (X/C) \subseteq U$. □

<u>THEOREM 2.3</u>. *Let* (T,X,π) *be a ttg and let* $f\colon X \to 2^X$ *be continuous* $(x \mapsto C(x))$. *Then* $(X/C)_q \simeq (X/C)_f$.

<u>PROOF</u>. Observe that $(X/C)_q$ exists (see Theorem 2.1). Let $i\colon (X/C)_q \to (X/C)_f$ be the set-theoretic isomorphism and let $f'\colon X \to (X/C)_f$ be the corestriction of f to $(X/C)_f$. Then f' is continuous and $f' = i \circ q$. Since q is a quotient map, it follows that i is continuous. In view of Lemma 2.2 this proves our theorem. □



<u>COROLLARY 2.4</u>. *For a ttg* (T,X,π) *the following statements are equivalent:*

1. $f\colon X \to 2^X$ *is continuous;*
2. C *is an equivalence relation and* $(X/C)_q \subseteq 2^X_f$.

<u>THEOREM 2.5</u>. *Let* (T,X,π) *be a ttg with compact phase space. Then* f *is continuous, if* $(X/C)_q$ *is* T_2.

<u>PROOF</u>. Choose $x \in X$ and let $\langle U_1,\ldots,U_n\rangle$ be a basis open nbhd of $f(x)$ in 2^X_f, i.e., $C(x) \subseteq \bigcup_{i=1}^{n} U_i = U$ and $C(x) \cap U_i \neq \emptyset$ for all $i \in \{1,\ldots,n\}$ (U_i open in X).

First we show that

a. there exists a nbhd O_x of x in X, such that $f(z) \subseteq U$ for every $z \in O_x$. Let $y \notin U$; then $C(x) \neq C(y)$ and there are open nbhds V_x^y and V_y of $C(x)$ and $C(y)$ in $(X/C)_q$ with $V_x^y \cap V_y = \emptyset$. Then $O_y = q^{\leftarrow}[V_y]$ and $O_x^y = q^{\leftarrow}[V_x^y]$ are disjoint open nbhds of y and x in X and both are the union of orbit closures. Since $\{O_y \mid y \notin U\}$ is an open covering of X/U and X/U is compact, there are an $m \in \mathbb{N}$ and $y_1,\ldots,y_m$ in X/U, such that $X/U \subseteq \bigcup_{i=1}^{m} O_{y_i} = O$. Now $O_x = \bigcap_{i=1}^{m} O_x^{y_i}$ is an open nbhd of x in X with $O_x \cap O = \emptyset$ and O_x is the union of orbit closures. For every $z \in O_x$ we clearly have $f(z) \subseteq O_x \subseteq U$.

Next we show that

b. there exists a nbhd V_x of x in X, such that for every $z \in V_x$ and every

$i \in \{1,\dots,n\}$, $f(z) \cap U_i \neq \phi$. For every $i \in \{1,\dots,n\}$ U_i is open and $C(x) \cap U_i \neq \phi$, so there exists a t_i in T with $\pi(t_i,x) \in U_i$. Now $\pi^{t_i^{-1}}[U_i]$ is an open nbhd of x and for every $z \in \pi^{t_i^{-1}}[U_i]$ we have $C(z) \cap U_i \neq \phi$. Define $V_x = \bigcap_{i=1}^{n} \pi^{t_i^{-1}}[U_i]$. Then V_x is an open nbhd of x in X, such that $f(z) \cap U_i \neq$ $\neq \phi$ for all $z \in V_x$.

Furthermore,

c. Define $W_x = O_x \cap V_x$. Then $f(z) \cap U_i \neq \phi$ and $f(z) \subseteq U$, so $f(z) \in \langle U_1,\dots,U_n\rangle$ for every $z \in W_x$, and f is continuous. □

<u>COROLLARY 2.6</u>. *Let* (T,X,π) *be a ttg with compact* T_2 *phase space. The following statements are equivalent:*

1. $f\colon X \to 2^X$ *is continuous;*
2. C *is an equivalence relation and* $(X/C)_q \subseteq 2^X$ $(= 2^X_f = 2^X_u)$;
3. C *is an equivalence relation and* $(X/C)_q$ *is* T_2.

The following provides an example of a situation in which f is continuous. Remember that in a ttg (T,X,π) with a uniform phase space a point $x \in X$ is called *equicontinuous* whenever, for every $\alpha \in \mathcal{U}$ (uniformity on X), there exists a $\beta \in \mathcal{U}$, such that $\pi(t,y) \in \alpha(\pi(t,x))$ for every $y \in \beta(x)$ and every $t \in T$.

<u>EXAMPLE 2.7</u>. Let (T,X,π) be a ttg, with compact T_2 phase space and let $x \in X$ be an equicontinuous point. Then f is continuous in x.

<u>PROOF</u>. Remark that $2^X = 2^X_f = 2^X_u$. Let $\mathcal{U}$ be the unique uniform structure on X and let $\alpha \in \mathcal{U}$ be closed and symmetric. Then $\alpha^*(f(x))$ is a nbhd of $f(x)$ in 2^X. We have to prove that there exists a $\beta \in \mathcal{U}$, such that $f(\beta(x)) \subseteq$ $\subseteq \alpha^*(f(x))$ or, equivalently, that $C(y) \subseteq \alpha(C(x))$ and $C(x) \subseteq \alpha(C(y))$ for every $y \in \beta(x)$. Since x is equicontinuous, there exists a $\beta \in \mathcal{U}$, such that for every $y \in \beta(x)$ and $t \in T$ we have $\pi(t,y) \in \alpha(\pi(t,x))$, so $\pi(t,x) \in$ $\in \alpha^{-1}(\pi(t,y)) = \alpha(\pi(t,y))$. Now $\{\pi(t,y) \mid t \in T\} \subseteq \bigcup\{\alpha(\pi(t,x)) \mid t \in T\} =$ $\alpha(\Gamma(x)) \subseteq \alpha(C(x))$ and also $\Gamma(x) \subseteq \alpha(C(y))$. Since α is closed, it follows that $C(y) \subseteq \alpha(C(x))$ and $C(x) \subseteq \alpha(C(y))$ for all $y \in \beta(x)$. □

<u>COROLLARY 2.8</u>. *If* X *is equicontinuous, then* f *is continuous and* $(X/C)_q$ *is* T_2.

3. HYPERTRANSFORMATION GROUPS

Every ttg (T,X,π) induces a ttg $(T_d,2^X_f,\tilde{\pi})$ and in case X is a uniform space, also a ttg $(T_d,2^X_u,\tilde{\pi})$, where T_d stands for the topological group T with the discrete topology. The action $\tilde{\pi}: T_d \times 2^X \to 2^X$ is defined by $\tilde{\pi}(t,A) = \pi^t[A]$. Since every π^t is a homeomorphism, it follows that every $\tilde{\pi}^t = \pi^{t*}$ is a homeomorphism and it is easy to verify that $\tilde{\pi}^e = i_{2^X}$ and $\tilde{\pi}^s \circ \tilde{\pi}^t = \tilde{\pi}^{st}$.

<u>THEOREM 3.1</u>. *Let* (T,X,π) *be a ttg with arbitrary phase group* T. *Then* $(T,\mathcal{C}(X),\tilde{\pi})$ *is a ttg.*

<u>PROOF</u>. Since $\mathcal{C}(X) \subseteq 2^X$ is invariant in $(T_d,2^X,\tilde{\pi})$, we only have to check the continuity of $\tilde{\pi}: T \times \mathcal{C}(X) \to \mathcal{C}(X)$. Choose $(t,A) \in T \times \mathcal{C}(X)$ and let $\langle U_1,\dots,U_n \rangle$ be a basis open nbhd of $\tilde{\pi}(t,A) = \pi^t[A]$. Then $\pi^t[A] \subseteq \bigcup_{i=1}^n U_i$ and $\pi^t[A] \cap U_i \neq \emptyset$ for all $i \in \{1,\dots,n\}$. Since π is continuous and A is compact, there are open nbhds V_t^0 of t in T and O_A of A in X, such that $\pi[V_t^0 \times O_A] \subseteq \bigcup_{i=1}^n U_i$. Fix $x_i \in A$ with $\pi(t,x_i) \in U_i$ for $i = 1,\dots,n$. Then by the continuity of π there are open nbhds V_t^i of t in T and W_{x_i} of x_i in X, such that $\pi[V_t^i \times W_{x_i}] \subseteq U_i$ and $W_{x_i} \subseteq O_A$. Now $V_t := \bigcap_{i=0}^n V_t^i$ is an open nbhd of t in T, $\langle O_A,W_{x_1},\dots,W_{x_n} \rangle$ is an open nbhd of A in $\mathcal{C}(X)$ and $\tilde{\pi}[V_t \times \langle O_A,W_{x_1},\dots,W_{x_n} \rangle] \subseteq \langle U_1,\dots,U_n \rangle$. For let $s \in V_t$ and $E \in \langle O_A,W_{x_1},\dots,W_{x_n} \rangle$, then $E \subseteq O_A$, so $\tilde{\pi}(s,E) \subseteq \pi[V_t^0 \times O_A] \subseteq$ $\subseteq \bigcup_{i=1}^n U_i$. Also $E \cap W_{x_i} \neq \emptyset$ for all $i \in \{1,\dots,n\}$. Choose $e_i \in E \cap W_{x_i}$; then $\pi(s,e_i) \in \tilde{\pi}(s,E) \cap U_i$.

This proves the continuity of $\tilde{\pi}$. □

<u>COROLLARY 3.2</u> [KOO]. *Let* (T,X,π) *be a ttg with arbitrary phase group* T *and compact phase space* X. *Then* $(T,2^X,\tilde{\pi})$ *(=* $(T,2^X_u,\tilde{\pi}) = (T,2^X_f,\tilde{\pi})$*) is a ttg.*

In the sequel we assume the existence of $(T,2^X_f,\tilde{\pi})$ or $(T,2^X_u,\tilde{\pi})$ as soon as we discuss them. Also we shall skip the action-symbol and write the action as a left multiplication of elements (subsets) of X by elements of T: $tx := \pi(t,x)$, $tA := \tilde{\pi}(t,A)$.

4. RECURSIVENESS IN X AND 2^X

The following definitions are taken from [3]. Let T be a topological group and let $\mathcal{H}$ be a fixed collection of subsets of T, the so called *admissible sets*.

Let (T,X) be a ttg. A point $x \in X$ is *recursive*, if for every nbhd U of x in X there exists an admissible set H with $Hx \subseteq U$; $x \in X$ is *locally recursive*, if for every nbhd U of x in X there exist an $H \in \mathcal{H}$ and an open nbhd V of x in X with $HV \subseteq U$.

X is called *pointwise (locally) recursive*, if every $x \in X$ is (locally) recursive.

Let $(X,\mathcal{U})$ be a uniform space; then X is called *uniformly recursive*, if for every $\alpha \in \mathcal{U}$ there exists an $H \in \mathcal{H}$, such that $Hx \subseteq \alpha(x)$ for every $x \in X$.

If we choose $\mathcal{H}$ to be the collection of all right-syndetic subjects of T, then this special form of recursiveness is called *almost periodicity*.

In the following we find generalizations of [4] Theorems 2.3, 2.1, 2.2 in 4.2, 4.3 and 4.4(a), respectively. Theorem 4.4(b) seems new.

REMARK 4.1.

a. If $x \in X$ is locally recursive, then x is recursive;

b. if X is uniformly recursive, then X is pointwise locally recursive.

THEOREM 4.2. *Let* (T,X) *be a ttg and* $(X,\mathcal{U})$ *a uniform space, such that* $(T,2^X_u)$ *is a ttg. Then* 2^X_u *is uniformly recursive iff* X *is uniformly recursive.*

PROOF. [4] Theorem 2.3, since the compactness of X has not been used in the proof. □

THEOREM 4.3.

a. *Let* X *be* T_3. *If* 2^X_f *is pointwise recursive, then* X *is pointwise locally recursive;*

b. *let* $(X,\mathcal{U})$ *be a locally compact uniform space. If* 2^X_u *is pointwise recursive, then* X *is pointwise locally recursive.*

PROOF. Choose $x \in X$ and let U_x be an open nbhd of x in X. Then there exists an open nbhd V_x of x in X with $x \in V_x \subseteq \bar{V}_x \subseteq U_x$. Then $\bar{V}_x \in 2^X$ and $\langle U_x \rangle$ is an open nbhd of $\bar{V}_x$ in 2^X_f. Since $\bar{V}_x$ is a recursive point in 2^X_f, there exists an $H \in \mathcal{H}$ with $H\bar{V}_x \subseteq \langle U_x \rangle$. So $HV_x \subseteq U_x$, and x is locally recursive in X.

If X is locally compact, we may choose V_x to be compact. Now there exists an $\alpha \in \mathcal{U}$, such that $\alpha(V_x) \subseteq U_x$. Since 2^X_u is pointwise recursive, there is an $H \in \mathcal{H}$ with $HV_x \subseteq \alpha^*(V_x)$. Then for every $h \in H$ we have $hV_x \subseteq \subseteq \alpha(V_x) \subseteq U_x$, so $HV_x \subseteq U_x$ and x is locally recursive. □

THEOREM 4.4. *Let T be an abelian group. Then the following statements hold, both for 2^X_f and 2^X_u:*

a. $x \in X$ *is recursive iff every finite subset of* $\Gamma(x)$ *is recursive in* 2^X;

b. $x \in X$ *is locally recursive iff every finite subset of* $\Gamma(x)$ *is locally recursive in* 2^X.

PROOF. Observe that in both cases the "iff" part is trivial. First we prove the theorem for 2^X_u. Case a. is Theorem 2.2 of [4].

b. Let $A = \{t_1x,\dots,t_xn\} \subseteq \Gamma(x)$ be a finite subset of $\Gamma(x)$ and let $\alpha^*(A)$ be a basis-open nbhd of A in 2^X_u for some symmetric $\alpha \in \mathcal{U}$. Since π^{t_i} is continuous for $i \in \{1,\dots,n\}$, there exists a $\beta \in \mathcal{U}$ with $t_i\beta(x) \subseteq \alpha(t_ix)$ for every $i \in \{1,\dots,n\}$. Because x is locally recursive, there are $H \in \mathcal{H}$ and $\delta \in \mathcal{U}$ with $H\delta(x) \subseteq \beta(x)$. By the continuity of every $\pi^{t_i^{-1}}$ we can find a symmetric $\gamma \in \mathcal{U}$ with $t_i^{-1}\gamma(t_ix) \subseteq \delta(x)$ for every $i \in \{1,\dots,n\}$. We shall prove that $H\gamma^*(A) \subseteq \alpha^*(A)$, so that A is a locally recursive point in 2^X_u.

Let $E \in \gamma^*(A)$, so $E \subseteq \gamma(A)$ and $A \subseteq \gamma(E)$. For every $e \in E$ there is an $i_e \in \{1,\dots,n\}$, such that $e \in \gamma(t_{i_e}x)$ and for every $i \in \{1,\dots,n\}$ there is an $e_i \in E$, such that $t_ix \in \gamma(e_i)$ and, by the symmetry of γ, $e_i \in \gamma(t_ix)$. If $e \in \gamma(t_ix)$, then for every $h \in H$ we have $he \in H\gamma(t_ix) \subseteq Ht_i\delta(x) = = t_iH\delta(x) \subseteq t_i\beta(x) \subseteq \alpha(t_ix)$ and also $t_ix \in \alpha(he)$. Now it follows that

$$hE = \cup\{he \mid e \in E\} \subseteq \cup\{\alpha(t_{i_e}(x)) \mid e \in E\} \subseteq \alpha(A)$$

and

$$A = \cup\{t_ix \mid i \in \{1,\dots,n\}\} \subseteq \cup\{\alpha(he_i) \mid i \in \{1,\dots,n\}\} \subseteq \alpha(hE),$$

so $hE \in \alpha^*(A)$. Since $h \in H$ and $E \in \gamma^*(A)$ were arbitrary, it follows that $H\gamma^*(A) \subseteq \alpha^*(A)$.

We now turn to 2^X_f. Let $A = \{t_1x,\dots,t_mx\} \subseteq \Gamma(x)$ be a finite subset of

$\Gamma(x)$ and let $\langle U_1,\dots,U_n\rangle$ be an open nbhd of A in 2^X_f. For every $j \in \{1,\dots,n\}$ choose an element $k_j \in \{1,\dots,m\}$, such that $t_{k_j}x \in U_j$, and for every $k \in K = \{1,\dots,m\}\setminus\{k_j \mid j = 1,\dots,n\}$ choose an $\ell_k \in \{1,\dots,n\}$, such that $t_k(x) \in U_{\ell_k}$. Then $O = \bigcap_{j=1}^{n} t_{k_j}^{-1} U_j \cap \bigcap\{t_k^{-1} U_{\ell_k} \mid k \in K\}$ is an open nbhd of x in X.

a. Let $x \in X$ be recursive. Then there is an $H \in \mathcal{H}$ with $Hx \subseteq O$, so $Hx \subseteq t_{k_j}^{-1} U_j$ for every $j \in \{1,\dots,n\}$ and $Hx \subseteq t_k^{-1} U_{\ell_k}$ for every $k \in K$. Since T is abelian, it follows that $Ht_{k_j}x \subseteq U_j$ and $Ht_k \subseteq U_{\ell_k}$ for every $j \in \{1,\dots,n\}$ and $k \in K$. But then $HA \subseteq \langle U_1,\dots,U_n\rangle$ and A is recursive in 2^X_f.

b. Let $x \in X$ be locally recursive. Then there are an $H \in \mathcal{H}$ and an open nbhd V_x of x in X with $HV_x \subseteq O$, so $HV_x \subseteq t_{k_j}^{-1} U_j$ and $Ht_{k_j}V_x \subseteq U_j$ for every $j \in \{1,\dots,n\}$ and $HV_x \subseteq t_k^{-1} U_{\ell_k}$, hence $Ht_kV_x \subseteq U_{\ell_k}$ for every $k \in K$. If we enumerate the elements of K as $k_{n+1},\dots,k_p$, then we may define $W = \langle t_{k_1}V_x,\dots,t_{k_p}V_x\rangle$. Then $A \in W$ and we shall prove that $HW \subseteq \langle U_1,\dots,U_n\rangle$, that is, the point $A \in 2^X_f$ is locally recursive in 2^X_f.

Let $B \in W$, so $B \subseteq \bigcup_{i=1}^{p} t_{k_i}V_x$ and $B \cap t_{k_i}V_x \neq \emptyset$ for every $i \in \{1,\dots,p\}$. For every $h \in H$ we have $hB \subseteq \bigcup\{ht_{k_i}V_x \mid i \in \{1,\dots,p\}\}$. But $ht_{k_i}V_x \subseteq$ $\subseteq Ht_{k_i}V_x \subseteq U_i$ for $i \in \{1,\dots,n\}$ and $ht_{k_i}V_x \subseteq Ht_{k_i}V_x \subseteq U_{\ell_{k_i}}$ for every $i \in \{n+1,\dots,p\}$, so $hB \subseteq \bigcup_{i=1}^{n} U_i$. Also $hB \cap ht_{k_i}V_x \neq \emptyset$, so $hB \cap U_i \neq \emptyset$ for every $i \in \{1,\dots,n\}$. It follows that $hB \in \langle U_1,\dots,U_n\rangle$. □

<u>LEMMA 4.5</u>. *Let* X *be point transitive, and let* $x \in X$ *be such that* $X = C(x)$. *Then* $\{E \in 2^X \mid E \subseteq \Gamma(x)$ and E is finite$\}$ *is a dense subset of* 2^X_f.

<u>PROOF</u>. Let $\langle U_1,\dots,U_n\rangle$ be an open basis set in 2^X_f. Every U_i is open in X and so it contains an element from $\Gamma(x)$, $t_ix \in U_i$ say. Then $A = \{t_1x,\dots,t_nx\} \in \langle U_1,\dots,U_n\rangle$. □

<u>COROLLARY 4.6</u>. *Let* T *be abelian and* $X = C(x)$ *for a (locally) recursive* $x \in X$. *Then* 2^X_f *has a dense subset of (locally) recursive points.*

5. ALMOST PERIODICITY

We shall apply and refine Section 4 for the special case of almost periodicity, that is recursiveness where the admissible sets are the right-syndetic subsets of T.

We shall call two points x and y in X *topologically distal*, whenever either $x = y$ or there does not exist a net $\{t_i\}$ in T, such that $\lim t_i x = z = \lim t_i y$. Equivalently, x and y are topologically distal iff $C(x,y) \cap \Delta_X = \phi$ in $X \times X$, where Δ_X denotes the diagonal in $X \times X$. For compact T_2 spaces X with uniformity $\mathcal{U}$ this is equivalent to the existence of an $\alpha \in \mathcal{U}$, with $(tx,ty) \notin \alpha$ for every $t \in T$, and so x and y are topologically distal iff they are distal. We shall call X topologically distal, if every x and y in X are topologically distal.

The following result generalizes [4] Lemma 4.2. Also compare [4] Lemma 4.1.

<u>THEOREM 5.1</u>. *Let* X *be a* T_3 *space (uniform space) and let* $\{x,y\}$ *be an almost periodic point in* 2^X_f (2^X_u). *Then* x *and* y *are topologically distal points in* X.

<u>PROOF</u>. a. Let X be T_3 and assume $x \neq y$. Then there are closed nbhds U and V of x and y in X, with $U \cap V = \phi$, so $(U \times V) \cap \Delta_X = \phi$. Since $\{x,y\} \in \langle U^\circ, V^\circ \rangle$ and $\{x,y\}$ is almost periodic in 2^X_f, there exists a right-syndetic subset H of T, such that $H\{x,y\} \subseteq \langle U^\circ, V^\circ \rangle$. It follows that $H(x,y) \subseteq U^\circ \times V^\circ \cup V^\circ \times U^\circ$ and so $\overline{H(x,y)} \subseteq U \times V \cup V \times U$ and also $\overline{H(x,y)} \cap \Delta_X = \phi$. Let $K \subseteq T$ be compact, such that $KH = T$. Then $K\,\overline{H(x,y)} \cap \Delta_X = \phi$. Since $K\,\overline{H(x,y)} = \overline{KH(x,y)} = C(x,y)$, this shows that x and y are topologically distal.

b. Let $(X,\mathcal{U})$ be a uniform space and $x \neq y$. Choose a symmetric $\beta \in \mathcal{U}$, such that $\beta(x) \cap \beta(y) = \phi$ and choose a closed index $\omega \in [\mathcal{U}^*]$ (the uniform structure on 2^X_u induced by $\mathcal{U}$) with $\omega \subseteq \beta^*$. Since $\{x,y\}$ is an almost periodic point in 2^X_u, there exists a right-syndetic set $H \subseteq T$ with $H\{x,y\} \subseteq \omega(\{x,y\})$, so $\overline{H\{x,y\}} \subseteq \omega(\{x,y\}) \subseteq \beta^*(\{x,y\})$. We shall prove that $\overline{H(x,y)} \cap \Delta_X = \phi$, so that, similar to part a, x and y are topologically distal in X.

Suppose $(z,z) \in \overline{H(x,y)}$, then for every $\alpha \in \mathcal{U}$ there is an $h \in H$, with $(hx,hy) \in \alpha(z) \times \alpha(z)$, and so $h\{x,y\} \subseteq \alpha(z)$. For symmetric $\alpha \in \mathcal{U}$ it follows, that $h\{x,y\} \in \alpha^*(z)$. Since $\mathcal{U}$ has a basis consisting of symmetric indexes, it follows that $\{z\} \in \overline{H\{x,y\}} \in \beta^*(\{x,y\})$. But then $\{x,y\} \subseteq \beta(z)$ and $z \in \beta(x) \cap \beta(y)$, which contradicts our assumption about $\beta \in \mathcal{U}$. □

COROLLARY 5.2. *Let* X *be a* T_3*-space (uniform space). Then* X *is topologically distal, if* 2_f^X *(*2_u^X*) is pointwise almost periodic. If* X *is compact* T_2*, then* X *is distal, if* 2^X *is pointwise almost periodic (*[4], Corollary 4.2*).*

LEMMA 5.3. *Let* X *be a topological space (uniform space) and* $n \in \mathbb{N}$*. Then* $\{x_1,\ldots,x_n\}$ *is almost periodic in* 2_f^X *(*2_u^X*), if* $(x_1,\ldots,x_n)$ *is almost periodic in* X^n*.*

PROOF. a. Let $\langle U_1,\ldots,U_m\rangle$ be an open nbhd of $\{x_1,\ldots,x_n\}$. Choose for every $i \in \{1,\ldots,m\}$ an element $j_i \in \{1,\ldots,n\}$, such that $x_{j_i} \in U_i$ and for every $k \in \{1,\ldots,n\}\backslash\{j_i \mid i \in \{1,\ldots,m\}\}$ an $i_k \in \{1,\ldots,m\}$, with $x_k \in U_{i_k}$. Define for every $\ell \in \{1,\ldots,n\}$ a nbhd V_ℓ of x_ℓ as follows:

$$\text{If } \ell \in \{j_i \mid i \in \{1,\ldots,m\}\} \text{ then } V_\ell := \bigcap\{U_i \mid j_i = \ell\},$$
$$\text{else } V_\ell := U_{i_\ell}.$$

Now $V_1 \times \ldots \times V_n$ is a nbhd of $(x_1,\ldots,x_n)$ in X^n, so there exists a right-syndetic subset H of T, with $H(x_1,\ldots,x_n) \subseteq V_1 \times \ldots \times V_n$ and obviously, $H\{x_1,\ldots,x_n\} \subseteq \langle U_1,\ldots,U_n\rangle$.

b. Straightforward. □

THEOREM 5.4. *Let* X *be a compact* T_2 *space. Then the following statements are equivalent:*

a. X *is distal;*

b. *every doubleton in* X *is almost periodic in* 2^X*;*

c. *every finite subset of* X *is almost periodic in* 2^X*.*

PROOF. c ⇒ b trivial; b ⇒ a (Theorem 5.1); a ⇒ c. Let $E \subseteq X$ be finite, with $|E| = n$. Then X^n is distal, so pointwise almost periodic. From Lemma 5.3 it follows that E is almost periodic in 2^X. □

Note that, if T is abelian, then X is pointwise almost periodic iff every finite subset of $\Gamma(x)$ is almost periodic in 2^X for every $x \in X$, so in particular, if X is minimal, then 2^X_f has a dense subset of almost periodic points (4.4(a) and 4.6).

<u>THEOREM 5.5</u> [KOO] ([4] Theorem 4.1). *Let* X *be compact* T_2. *Then the following statements are equivalent:*

a. X *is uniform almost periodic;*

b. 2^X *is pointwise almost periodic;*

c. 2^X *is uniform almost periodic.*

<u>PROOF</u>. a $\Rightarrow$ c (Theorem 4.2); c $\Rightarrow$ b (Remark 4.1). b $\Rightarrow$ a X is distal by Corollary 5.2 and pointwise locally almost periodic by Theorem 4.3, so X is uniform almost periodic by [2], 5.28. □

<u>THEOREM 5.6</u>. *Let* X *be a* T_3*-space (uniform space) and let* $\{x_1,\dots,x_n\}$ *be almost periodic in* 2^X_f (2^X_u). *Then for every* $A \in C\{x_1,\dots,x_n\}$ *we have* $|A| = n$.

<u>PROOF</u>. First observe that, for an arbitrary ttg (T,Y) and for every $y \in Y$ which is almost periodic and has local basis of closed nbhds, we have that C(y) is minimal. Let $A \in 2^X$ be a compact subset of X. It follows from the regularity of X, that A has a local basis of closed nbhds, both in 2^X_f and in 2^X_u ([5], 4.9.10). So if $A \in 2^X_f$ (2^X_u) is compact and almost periodic, then C(A) is minimal in 2^X_f (2^X_u). We show first that $|A| \le n$ for every $A \in C(\{x_1,\dots,x_n\})$. So let $A \in C(\{x_1,\dots,x_n\})$ and suppose $|A| > n$. Choose n+1 different points in A, $y_1,\dots,y_n$ say.

a. Let $V_1,\dots,V_{n+1}$ be pairwise disjoint open nbhds of $y_1,\dots,y_{n+1}$, respectively. Then $A \in \langle V_1,\dots,V_{n+1},X\rangle$. However, $\langle V_1,\dots,V_{n+1},X\rangle \cap \Gamma(\{x_1,\dots,x_n\}) = \phi$, otherwise there would be $t \in T$ and $j \in \{1,\dots,n\}$, with tx_j occurring in two different V_i's. It follows that $A \notin C(\{x_1,\dots,x_n\})$, a contradiction.

b. Choose a symmetric $\alpha \in \mathcal{U}$ such that $\{\alpha(y_i) \mid i \in \{1,\dots,n+1\}\}$ is pairwise disjoint. Similar to a we get a contradiction.

In the same way the assumption $|A| < n$ for some $A \in C(\{x_1,\dots,x_n\})$ leads to the conclusion that $\{x_1,\dots,x_n\} \notin C(A)$, which contradicts the minimality of $C(\{x_1,\dots,x_n\})$. □

We now want to prove a converse of Lemma 5.3, which in the case of compact T_2 spaces has been done by KOO ([4], Theorem 4.2). Our method is exactly the same, but a weaker condition turned out to be sufficient. Define the map f: $X^n \to 2^X$ by $f((x_1,\ldots,x_n)) = \{x_1,\ldots,x_n\}$. Then f is easily seen to be equivariant, i.e., $f(t(x_1,\ldots,x_n)) = tf((x_1,\ldots,x_n))$ for all $t \in T$. Also, f is continuous with respect to 2^X_f as well as 2^X_u. Indeed, let $<U_1,\ldots,U_m>$ $(\alpha^*(\{x_1,\ldots,x_n\}$ for a symmetric $\alpha \in \mathcal{U}$) be a nbhd of $\{x_1,\ldots,x_n\}$ in 2^X_f (2^X_u); then $f(V_1\times\ldots\times V_n) \subseteq <U_1,\ldots,U_m>$ $(f(\alpha(x_1)\times\ldots\times\alpha(x_n)) \subseteq \subseteq \alpha^*(\{x_1,\ldots,x_n\}))$ with $V_1\times\ldots\times V_n$ as in the proof of Lemma 5.3.

We need the following theorem, due to EISENBERG ([1]).

<u>THEOREM 5.7</u>. *Let* (T,X) *and* (T,Y) *be ttg's with compact* T_2 *phase spaces and let* Y *be minimal and* X *point transitive. Let* g: X → Y *be a continuous equivariant, locally one-to-one surjection. Then* X *is minimal.*

<u>LEMMA 5.8</u> ([4] Lemma 4.4). *Let* X *be a* T_2*-space and let* $(x_1,\ldots,x_n) \in X^n$ *be such that* $x_i \neq x_j$ *for* $i \neq j$. *Then* f *is locally one-to-one in* $(x_1,\ldots,x_n)$, *i.e.,* f *is one-to-one on some nbhd of* $(x_1,\ldots,x_n)$.

<u>LEMMA 5.9</u>. *Let* X *be* T_3 *(uniform) and let* $\{x_1,\ldots,x_n\}$ *be an almost periodic point in* 2^X_f (2^X_u); *then* $f' = f|_{C((x_1,\ldots,x_n))}$ *is a locally one-to-one map from* $C((x_1,\ldots,x_n))$ *onto* $C(\{x_1,\ldots,x_n\})$.

<u>PROOF</u>. Clearly $f(\Gamma((x_1,\ldots,x_n))) \subseteq \Gamma(\{x_1,\ldots,x_n\})$; the continuity of f implies that $f(C((x_1,\ldots,x_n))) \subseteq C(\{x_1,\ldots,x_n\})$. So by Theorem 5.6 we have for every $(y_1,\ldots,y_n) \in C((x_1,\ldots,x_n))$ that $y_i \neq y_j$ if $i \neq j$; hence f' is locally one-to-one by Lemma 5.8. We shall prove that $f(C((x_1,\ldots,x_n)))$ is closed in $C(\{x_1,\ldots,x_n\})$. Then the minimality of $C(\{x_1,\ldots,x_n\})$ (see the beginning of the proof of Theorem 5.6) implies that f' is surjective. Assume the existence of an $A \in C(\{x_1,\ldots,x_n\})\setminus f(C((x_1,\ldots,x_n)))$. It is clear from Theorem 5.6 that $|A| = n$, $A = \{y_1,\ldots,y_n\}$ say. Then for every permutation σ of $1,\ldots,n$ holds $(y_{\sigma(1)},\ldots,y_{\sigma(n)}) \notin C((x_1,\ldots,x_n))$, so we may choose pairwise disjoint open nbhds V_i^σ of y_i in X, such that $V^\sigma_{\sigma(1)}\times\ldots\times V^\sigma_{\sigma(n)} \cap \cap C((x_1,\ldots,x_n)) = \emptyset$. Define $V_i = \bigcap\{V_i^\sigma \mid \sigma$ permutation of $1,\ldots,n\}$. Then $<V_1,\ldots,V_n>$ is a nbhd of $\{y_1,\ldots,y_n\}$ in 2^X_f, with $<V_1,\ldots,V_n> \cap \cap f(C((x_1,\ldots,x_n))) = \emptyset$. Now $f(C(x_1,\ldots,x_n))$ is closed in $C(\{x_1,\ldots,x_n\})$. In the case of 2^X_u we choose suitable symmetric $\alpha^\sigma \in \mathcal{U}$ and similar to the

case of 2^X_f it follows that $f(C((x_1,\dots,x_n)))$ is closed in $C(\{x_1,\dots,x_n\})$. □

The following result is the converse of Lemma 5.3 and it slightly generalizes [4], Theorem 4.2.

THEOREM 5.10. *Let X be locally compact* T_2. *Then* $(x_1,\dots,x_n)$ *is an almost periodic point in* X^n, *iff* $\{x_1,\dots,x_n\}$ *is an almost periodic point in* 2^X_f (2^X_u) *and* $C((x_1,\dots,x_n))$ *is compact.*

PROOF. " ⇒ " X^n is locally compact T_2, so $C((x_1,\dots,x_n))$ is compact and by Lemma 5.3, $\{x_1,\dots,x_n\}$ is almost periodic.
" ⇐ ". By Lemma 5.9, $f': C((x_1,\dots,x_n)) \to C(\{x_1,\dots,x_n\})$ satisfies the conditions of Theorem 5.7. Since $C(\{x_1,\dots,x_n\})$ is minimal and $C((x_1,\dots,x_n))$ is point transitive, it follows from Theorem 5.7 that $C((x_1,\dots,x_n))$ is minimal, so $(x_1,\dots,x_n)$ is an almost periodic point in X^n. □

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