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AFDELING ZUIVERE WISKUNDE ZN 95/80 MAART
(DEPARTMENT OF PURE MATHEMATICS)

A.M. COHEN

FINITE GROUPS OF REAL OCTAVE AUTOMORPHISMS

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Finite groups of real octave automorphisms

by

Arjeh M. Cohen

ABSTRACT

The finite subgroups of the real connected semi-simple Lie group of type G_2 are determined by use of the 7-dimensional irreducible representation of G_2, which is actually an isomorphism onto the automorphism group of the real (nonsplit) octaves. The result depends on D.B. Wales' classification of primitive unimodular linear groups of degree 7. In the case where the finite subgroup H is such that the restriction to H of the above-mentioned representation of the Lie group is primitive, the result implies that H must be isomorphic to one of the groups $PG\ell_2(7)$, $PS\ell_2(8)$, $PS\ell_2(13)$, $U_3(3)$, $G_2(2)$.

KEY WORDS & PHRASES: *Real octaves, Automorphisms, finite groups, Lie groups*

0. INTRODUCTION

The octaves or real Cayley numbers have been studied fairly intensively since their appearance in the literature (see [2] for an overview and further references).

Although the automorphism group of the real octaves has been determined (it is the Lie group $G_2(\mathbb{R})$ as demonstrated in for instance [6] and in [9]), no list of conjugacy classes of its finite subgroups, known to the author, is present in the literature.

In this paper these finite groups of octave automorphisms are classified by use of a theorem of D.B. Wales appearing in [11] that lists all finite complex quasi-primitive unimodular linear groups of degree 7 up to conjugacy.

As any finite subgroup of the complex Lie group $G_2(\mathbb{C})$ is contained in a maximal compact subgroup of $G_2(\mathbb{C})$ and hence conjugate to a subgroup of $G_2(\mathbb{R})$ (cf. [10]), the classification of finite automorphism groups of the real octaves coincides with the classification of the finite subgroups of $G_2(\mathbb{C})$.

To conclude this introduction I want to express my gratitude to T.A. Springer, whose help has been crucial for the start as well as the outcome of this work.

1. FUNDAMENTAL NOTIONS AND NOTATIONS

We let $\mathbb{O}$ be the real division algebra of the octaves. This algebra is nonassociative and has a basis $e_0 = 1, e_1, e_2, \ldots, e_7$ over $\mathbb{R}$ such that its multiplication is determined by the rules $e_i^2 = -1$ and $e_i e_j = e_k$ whenever (ijk) is one of the 3-cycles $(1+r, 2+r, 4+r)$ (i,j,k,r running through the integers mod 7 and all values taken in $\{1,2,\ldots,7\}$). For details concerning $\mathbb{O}$ the reader is referred to [7] or [9].

We need the anisotropic nondegenerate quadratic form Q on $\mathbb{O}$ given by

$$Q\Big(\sum_{i=0}^{7} e_i\alpha_i\Big) = \sum_{i=0}^{7} \alpha_i^2 \qquad (\alpha_i \in \mathbb{R}).$$

This form is multiplicative: $Q(xy) = Q(x)Q(y)$ $(x,y \in \mathbb{O})$. The bilinear form

$(.|.)$ corresponding to Q is given by $2(.|.) = Q(x+y) - Q(x) - Q(y)$. It is used to define the involutory anti-automorphism $x \mapsto \tilde{x}$ on $\mathbb{O}$ in the following way: $\tilde{x} = (x|1)1 - x$. The field $\mathbb{R}$ is often regarded upon as a subfield of $\mathbb{O}$ by means of the natural embedding $\mathbb{R} \to \mathbb{R}.1$.

Any automorphism of $\mathbb{O}$ preserves the bilinear form $(.|.)$ as well as the alternating trilinear form f given by $f(x,y,z) = (xy|z)$ $(x,y,z, \in \mathbb{O})$. As it must fix 1, such an automorphism may be viewed as an orthogonal transformation on $\mathbb{R}^{\perp}$ stabilizing f. On the other hand, it is easily derived from the nondegeneracy of Q that any orthogonal transformation g on $\mathbb{R}^{\perp}$ stabilizing f can be extended uniquely to an automorphism of $\mathbb{O}$ by $g(\alpha+x) = \alpha+g(x)$ $(\alpha \in \mathbb{R}, x \in \mathbb{R}^{\perp})$. Thus the automorphism group $\mathrm{Aut}(\mathbb{O})$ of $\mathbb{O}$ can be identified with the subgroup of the 7-dimensional linear orthogonal group $\mathcal{O}(\mathbb{R}^{\perp})$ stabilizing a particular alternating trilinear form.

There is another interpretation of $\mathrm{Aut}(\mathbb{O})$ that we shall frequently employ, namely the above-mentioned isomorphism with $G_2(\mathbb{R})$. This leads for instance to the existence of a maximal torus T consisting of all automorphisms $t_{\theta,\eta}$ $(\theta,\eta \in \mathbb{R})$ where $t_{\theta,\eta}$ has matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\eta & 0 & 0 & 0 & \sin\eta \\ 0 & -\sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta+\eta) & -\sin(\theta+\eta) & 0 \\ 0 & 0 & 0 & 0 & \sin(\theta+\eta) & \cos(\theta+\eta) & 0 \\ 0 & 0 & -\sin\eta & 0 & 0 & 0 & \cos\eta \end{pmatrix}$$

with respect to $e_1,\dots,e_7$.

Moreover, the normalizer N of T inside $\mathrm{Aut}(\mathbb{O})$ is the semi-direct product of T and the dihedral group W of order 12 generated by the automorphisms

$$a = \delta^{-}_{\{1,4,6,7\}}\,(235)(476)$$

and

$$s = \delta^{-}_{\{2,3,4,5,6,7\}}\,(23)(47).$$

Here, the notation is the following: a permutation π on 7 letters stands for the linear transformation determined by $e_i \mapsto e_{\pi(i)}$ $(1 \le i \le 7)$, and δ_K^- for K a subset of $\{1,2,\dots,7\}$ stands for the linear map sending e_i to $-e_i$ whenever $i \in K$ and fixing e_i if $i \notin K$.

Extending scalars to the complex numbers, we obtain the complexified algebra $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$ for which extensions of Q, $(.|.)$, ~, and f can be defined without difficulty. We shall not distinguish the notations for these operators and their extensions. Note that Q is still nondegenerate, but no longer anisotropic. Defining complex conjugation $x \mapsto \bar{x}$ on $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$ as usual by

$$\sum_{i=0}^{7} e_i\alpha_i \mapsto \sum_{i=0}^{7} e_i\bar{\alpha}_i \qquad (\alpha_i \in \mathbb{C}),$$

we regain $\mathbb{O}$ from $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$ as the set of fixed points with respect to complex conjugation.

Similarly to what we have seen for Aut($\mathbb{O}$), the group Aut($\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$) can be identified with both $G_2(\mathbb{C})$ and the group of transformations on $\mathbb{C}^{\perp}$, orthogonal with respect to Q and preserving f. Clearly, Aut($\mathbb{O}$) is the subgroup of Aut($\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$) consisting of all real transformations.

2. SOME PROPERTIES OF FINITE SUBGROUPS OF AUT($\mathbb{O}$)

The above indicated interpretations of Aut($\mathbb{O}$) lead to a number of properties stated in the next three lemmas. The notation is as in the preceding section. Moreover C_m^{ℓ} for $\ell, m \in \mathbb{N}$ denotes the direct product of ℓ copies of the cyclic group C_m of order m.

LEMMA 1. (Borel-Serre). *If* G *is a finite nilpotent subgroup of* Aut($\mathbb{O}$), *then*

(i) G *is conjugate to a subgroup of the normalizer* N *of* T;

(ii) *If* $G \cong C_p^{\ell}$ *for some prime* p *and a natural number* ℓ, *then either* $\ell \le 2$ *and* $p > 2$, *or* $\ell = 3$.

PROOF. See [4] for the proof of (i) as well as for the proof of $\ell \le 3$, and $\ell = 2$ if $p \ne 2,3$ in case $G \cong C_p^{\ell}$ as in (ii). It remains to establish that $G \cong C_3^3$ does not occur inside $G_2(\mathbb{R})$. If such a subgroup G exists, then up to conjugacy we may assume that G is in N and that G contains $a^2 = (253)(467)$. Now $t_{\theta,\eta}$ is centralized by a^2 if and only if $t_{\theta,\eta} = t_{\eta,-\theta-\eta} = t_{-\theta-\eta,\theta}$.

Thus $t_{\theta,\eta}$ belongs to G whenever $t_{\theta,\eta}$ is a power of $t_{2\pi/3,2\pi/3}$. It results that the number $|G/G\cap T|$ is a multiple of 3^2, which contradicts that $|G/G\cap T|$ divides $|N/T| = 2^2.3$. This finishes the proof. □

LEMMA 2. *Let* G *be a finite subgroup of* Aut($\mathbb{O}$) *and let* χ *be the character of* G *on* $\mathbb{R}^\perp$ *(so* $\chi(g) = \mathrm{trace}(g|_{\mathbb{R}^\perp})$ *for* $g \in G$*). The following holds:*

(i) *If* χ *is irreducible, then* $v(\chi) = 1$, *where*

$$v(\chi) = \frac{1}{|G|} \sum_{g\in G} \chi(g^2);$$

(ii) $(\chi_a^3|1) \geq 1$, *where*

$$\chi_a^3(g) = \frac{\chi(g)^3+2\chi(g^3)-3\chi(g^2)\chi(g)}{6} \qquad (g \in G);$$

(iii) *If* p *is a prime* $\neq 2,3$ *dividing* $|G|$ *and* S *is a* p-*Sylow subgroup of* G, *then* S *is abelian, and conjugate to a subgroup of* T;

(iv) *If* $g \in G$ *is of order* m, *then there are* $k,\ell \in \mathbb{Z}$ *such that*

$$\chi(g) = 1+2\cos\frac{2\pi k}{m} + 2\cos\frac{2\pi\ell}{m} + 2\cos\frac{2\pi(k+\ell)}{m};$$

(v) *If* $g \in G$ *is of order* 2, *then* $\chi(g) = -1$;

(vi) *If* $g \in G$, *then* $\det(g) = 1$ *(i.e.* G *is unimodular)*;

(vii) *If* χ *is irreducible, then the centre* Z(G) *of* G *is trivial.*

PROOF.

(i) follows from the fact that the restriction to $\mathbb{R}^\perp$ is a real representation of G.

(ii) expresses the fact that G preserves the alternating trilinear form f restricted to $\mathbb{R}^\perp$.

(iii) is a direct consequence of lemma 1(i), as S is nilpotent.

(iv) It is well-known from the theory of Lie groups that any $g \in G$ (being semi-simple) is conjugate to an element of T (see for instance [4]).

(v),(vi) are obtained by application of (iv) to g.

(vii) As a consequence of the irreducibility of G, any element of Z(G) has only one eigenvalue (disregarding multiplicities). By (iv), this eigenvalue must always be 1. □

<u>LEMMA 3</u>. *Let* G,χ *be as in lemma 2. If* $(\chi_a^3|1) = 1$, *then the conjugacy class of* G *within* Aut($\mathbb{O}$) *coincides with the intersection of* Aut($\mathbb{O}$) *and the conjugacy class of* G *within the full linear group of* $\mathbb{C}^{\perp}$.

<u>PROOF</u>. Suppose G,H are subgroups of Aut($\mathbb{O}$) and the linear transformation t on $\mathbb{C}^{\perp}$ satisfies $tGt^{-1} = H$. Then a standard argument shows that we may assume that t is orthogonal and has determinant 1. Now $(xy|z)$ and $((tx)(ty)|tz)$ are both trilinear alternating forms on $\mathbb{C}^{\perp}$, so by the hypothesis $(\chi_a^3|1) = 1$ the one is a scalar multiple, say α, of the other:

$$((tx)(ty)|tz) = \alpha(xy|z).$$

Extending t to $\mathbb{O}$ by prescribing t1 = 1, we get $(\alpha xy|z) = ((tx)(ty)|tz) = (t^{-1}((tx)(ty))|z)$, and by the nondegeneracy of Q,

$$\alpha t(xy) = (tx)(ty).$$

Substitution of x = y = 1 yields $\alpha = 1$ and $t \in \mathrm{Aut}(\mathbb{O})$. □

Whenever in the sequel the terms *(ir)reducibility* and *(im)primitivity* are used with respect to a subgroup G of Aut($\mathbb{O}$), they are meant to pertain to G viewed as a linear group on $\mathbb{C}^{\perp}$.

The study of subgroups G of Aut($\mathbb{O}$) is divided into three cases, according as G is reducible, imprimitive (and irreducible), primitive.

3. REDUCIBLE GROUPS

We shall now outline the structure of a finite group G of automorphisms of $\mathbb{O}$ stabilizing in its action on $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$ a nontrivial linear subspace of $\mathbb{C}^{\perp}$ of dimension, say m $(1 \le m < 7)$. If $m \ge 4$, there must be another linear subspace of $\mathbb{C}^{\perp}$ of dimension ≤ 3 left invariant by G, so without loss of generality we may assume $m \le 3$.

We contend that $\mathbb{C}^{\perp}$ *contains a* G-*invariant linear subspace of dimension* 3. This is obviously true if all irreducible constituents of G have degree 1. Suppose that U is a linear subspace of dimension 2 on which G acts irreducibly. Then the $\mathbb{C}^{\perp}$-part of the subalgebra spanned by U is G-invariant. If $U = \bar{U}$ (i.e. U is real), then this subalgebra is 3-dimensional. As $U \cap \bar{U}$ is a G-invariant subspace of U, we are left with $U \cap \bar{U} = \{0\}$. Then $U + \bar{U}$ is 4-dimensional and G-invariant, so its orthoplement $(U+\bar{U}+\mathbb{C})^{\perp}$ is as required.

Let U be a 3-dimensional G-invariant linear subspace of $\mathbb{C}^{\perp}$. Assume, first of all, that $U+\mathbb{C}$ is a quaternion subalgebra of $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$. From [9], it results that G is a finite subgroup of the semidirect product of Aut(U) with the norm-1 subgroup $U_1(U)$ of the multiplicative group of U consisting of all elements $x \in U$ with $Q(x) = 1$. Here Aut(U) acts on $U_1(U)$ in its natural representation.

We next claim that *if* G *does not stabilize a real quarternion subalgebra, it must fix a* 1-*dimensional real subspace of* $\mathbb{R}^{\perp}$.

Suppose not; then, as G stabilizes $U \cap \bar{U}$, the dimension of $U \cap \bar{U}$ cannot be 1. So if $U \cap \bar{U}$ is a nontrivial subspace of $U \cap \bar{U}$, its dimension is 2. But then $U \cap \bar{U}$ generates a real quaternion subalgebra invariant under G. Hence either $U \cap \bar{U} = \{0\}$ or $U = \bar{U}$. Define V to be $U+\bar{U}$ in the first case and $U + \sum_{x,y \in U} xy\mathbb{C}$ in the second case. Then V is a 6-dimensional real vector space invariant under G and thus admits a 1-dimensional real orthoplement in $\mathbb{C}^{\perp}$. This establishes the claim.

Since Aut($\mathbb{O}$) is transitive on the points of $\mathbb{R}^{\perp}$, we may assume without harming the generality that G leaves invariant $e_1\mathbb{R}$, so that G is contained in the stabilizer of $\{\pm e_1\}$ within Aut($\mathbb{O}$). It is well known, however, that the stabilizer of e_1 in Aut($\mathbb{O}$) is isomorphic to $SU_3(\mathbb{C})$ and easily derived that the stabilizer of $\{\pm e_1\}$ is isomorphic to $SU_3(\mathbb{C})$ extended (as a semi-direct product) by complex conjugation. The possible G_{e_1}, being isomorphic to subgroups of $SU_3(\mathbb{C})$, can be read off from [1]. It is straightforward to determine all possible extensions G of G_{e_1} of degree 2 within the semidirect product of $SU_3(\mathbb{C})$ and the group generated by complex conjugation. We shall not bother to write them down. The results of this section are summarized in the next theorem.

<u>THEOREM</u>. *Any finite subgroup* G *of* Aut($\mathbb{O}$) *that is reducible in* $\mathbb{C}^{\perp}$ *is isomorphic to a subgroup of either* (i) *the semidirect product of* $U_1(\mathbb{H})$ *by* Aut($\mathbb{H}$), *where* $U_1(\mathbb{H})$ *is the group of elements having norm* 1 *in the real quaternion division algebra* $\mathbb{H}$; *or* (ii) *the semidirect product of the special*

unitary group $SU_3(\mathbb{C})$ *by the group of order* 2 *generated by complex conjugation.*

4. IMPRIMITIVE GROUPS

Suppose for the duration of this section that G is a finite irreducible imprimitive subgroup of $\mathrm{Aut}(\mathbb{O})$.

The blocks of a system of imprimitivity for G must have dimensions 1, so the stabilizer H of G of all the blocks in such a system is abelian.

By lemma 1 this group H is up to conjugacy of G contained in the normalizer N of the torus T (notation of section 2). As N stabilizes $\{\pm e_1\}$, the complement K of the 2-Sylow subgroup of H has a nonzero fixed space. But K is normal in G, so this space is G-invariant and thus equal to the whole space $\mathbb{R}^{\perp}$ by the irreducibility of G. Thus K = 1 and H is a 2-group. Inspection of the character of H on $e_1\mathbb{R}$ yields that $H \cong C_2^m$ for some $m \in \mathbb{N}$. Lemma 1 yields that $m \le 3$, so that according to Cliffords theorem m = 0 or m = 3.

Suppose that $H \cong C_2^3$. If $g \in G$ would contralize H, then g would leave invariant all the blocks of the system of imprimitivity (because the characters of H on any two of these blocks differ) and g would belong to H. Therefore G/H is isomorphic to a subgroup of $\mathrm{Aut}(C_2^3) \cong PS\ell_2(7)$. On the other hand, if H = 1, then G is isomorphic to a transitive subgroup of the symmetric group sym(7) on 7 letters. It is a well-known fact that $PS\ell_2(7)$ is the only subgroup of Sym(7) having an irreducible representation over $\mathbb{C}$ of degree 7. This implies that $G \cong PS\ell_2(7)$. The conclusion is stated in the following theorem.

THEOREM. *If* G *is a finite irreducible imprimitive subgroup of* $\mathrm{Aut}(\mathbb{O})$, *then either* G *is isomorphic to* $PS\ell(7)$ *or* G *has a normal subgroup* H *isomorphic to* C_2^3 *such that* G/H *is isomorphic to a subgroup of* $PS\ell_2(7)$.

EXAMPLES. In fact both possibilities occur. H.S.M. COXETER [5] exhibited a nonsplit extension of C_2^3 by $PS\ell_2(7)$ inside $\mathrm{Aut}(\mathbb{O})$. In the notation of section 1 the generators for such a group may be taken to be (1234567), (124)(365), $\bar{\delta}_{\{1,2,4,7\}}(12)(36)$.

The existence of an irreducible subgroup of $\mathrm{Aut}(\mathbb{O})$ isomorphic to $PS\ell_2(7)$

will follow from the result in the next section, where the group $PG\ell_2(7)$ containing $PS\ell_2(7)$ is proved to be isomorphic to an irreducible subgroup of Aut($\mathbb{O}$). Note that the group isomorphic to $PS\ell_2(7)$ must be irreducible as the index of $PS\ell_2(7)$ in $PG\ell_2(7)$ is 2.

5. PRIMITIVE GROUPS

The relevant theorem, quoted from [11] without proof and slightly adapted reads as follows.

THEOREM. (D.B. Wales) *Let* G *be an irreducible finite subgroup of* $G\ell_7(\mathbb{C})$ *which is primitive and unimodular. Suppose that* G *has an abelian 7-Sylow subgroup. Then there are subgroup* G_1 *and* Z *such that* $G = G_1 \times Z$, *where* Z *has order* 1 *or* 7. *Moreover,* G_1 *is isomorphic to one of the following groups* (I) $PS\ell_2(13)$; (II) $PS\ell_2(8)$ *or an extension of* $PS\ell_2(8)$ *by a field automorphism of order* 3; (III) Alt(8) *or* Sym(8); (IV) $PG\ell_2(7)$; (V) $U_3(3)$ *or* $G_2(2)$; (VI) $S_6(2)$.

It should be remarked that the irreducible representation of $PS\ell_2(7)$ of degree 7 is not primitive (it is present in [11] under (IV) because it is quasiprimitive).

The above theorem enables us to prove the following result.

THEOREM. *There are precisely* 5 *conjugacy classes of finite primitive subgroups of* Aut($\mathbb{O}$). *Any such subgroup is isomorphic to* $PS\ell_2(13)$, $PS\ell_2(8)$, $PG\ell_2(7)$, $U_3(3)$ *or* $G_2(2)$.

PROOF. Let G be a finite primitive subgroup of Aut($\mathbb{O}$). By lemma 2, this group satisfies the hypotheses of Wales' theorem and has trivial centre. Let χ the character of G on $\mathbb{R}^{\perp}$. We shall now analyze the different cases of Wales' theorem.

In case (I) there is only one character of degree 7, so that $G\ell_7(\mathbb{C})$ contains only one conjugacy class of irreducible subgroups isomorphic to $PS\ell_2(13)$. Verification yields that $(\chi_a^3|1) = 1$, so that lemma 3 may be applied to infer that Aut($\mathbb{O}$) contains at most one conjugacy class of primitive subgroups isomorphic to $PS\ell_2(13)$.

As for case (II), there are two distinct conjugacy classes of primitive

subgroups of $G\ell_7(\mathbb{C})$ isomorphic to $S\ell_2(8)$. For a representative group in one of them, the character values are all rational; if G were taken from this conjugacy class, then $(\chi_a^3|1) = 0$ would hold, which is absurd because of lemma 2. So the other conjugacy class remains, and if G is taken from this class, we have $(\chi_a^3|1) = 1$, so that lemma 3 may be invoked again. Finally, note that the extension of $S\ell_2(8)$ by the field automorphism of order 3 does not occur as its subgroup $S\ell_2(8)$ is from the 'wrong' conjugacy class.
In (III), Alt(8) contains an element of order 2 with character 3. Thus Alt(8) and Sym(8) cancel by lemma 2(v). Similarly, in (VI), the group $S_6(2)$ has involutions with trace 5.
Concerning (V), it should be remarked that $U_3(3)$ has only one irreducible character 7 with real values.

Reasoning as before, the fact that $(\chi_a^3|1) = 1$ for χ the only remaining character in each of the cases $G \cong PG\ell_2(7)$, $U_3(3)$ or $G_2(2)$, yields that there are at most 5 conjugacy classes of primitive subgroups of Aut $\mathbb{O}$ as well as the second statement of the theorem.

It is left to prove that each of the groups listed in the theorem does indeed occur as a subgroup of Aut($\mathbb{O}$). Now B. Cooperstein kindly pointed out to me that this follows from the fact that up to conjugacy within $\mathbb{O}_7(\mathbb{R})$, the group Aut($\mathbb{O}$) is the unique group that stabilizes a trilinear alternating form on $\mathbb{R}^7$ and is irreducible in its natural representation on $\mathbb{R}^7$, this result being a direct consequence of the classification by J.A. Schouten of alternating trilinear forms on $\mathbb{R}^7$ given in [8]. We shall however furnish a proof by direct verification, presenting $PS\ell_2(8)$ and $PS\ell(13)$ as groups generated by explicitly given automorphisms of $\mathbb{O}$. As to $PG\ell_2(7)$, $U_3(3)$ and $G_2(2)$, it suffices to state that they are all three contained in $G_2(2)$, while the latter is known to be the full automorphism group of the integral octaves (cf. [3]).

Generators for $S\ell_2(8)$ are (in the notation of section 1)

$$\delta^-_{\{1,2,4,6\}}(1437526),$$

s, $t_{\pi,0}$, $t_{0,\pi}$ and the transformation whose matrix with respect to

$e_1,\dots,e_7$ is

$$\frac{1}{4}\begin{pmatrix}
\sigma^2+\sigma-2 & 1-\sigma^2 & \sigma^2-2 & 1+\sigma & 1 & 3-\sigma-\sigma^2 & \sigma \\
1-\sigma^2 & -\sigma & 2-\sigma-\sigma^2 & 3-\sigma-\sigma^2 & \sigma^2-2 & 1 & 1+\sigma \\
\sigma^2-2 & 2-\sigma-\sigma^2 & -1 & -\sigma & \sigma^2+\sigma-3 & -1-\sigma & 1-\sigma^2 \\
1+\sigma & 3-\sigma-\sigma^2 & -\sigma & 2-\sigma^2 & 1-\sigma^2 & \sigma^2+\sigma-2 & -1 \\
1 & \sigma^2-2 & \sigma^2+\sigma-3 & 1-\sigma^2 & -1-\sigma & -\sigma & \sigma^2+\sigma-2 \\
3-\sigma-\sigma^2 & 1 & -1-\sigma & \sigma^2+\sigma-2 & -\sigma & 1-\sigma^2 & 2-\sigma^2 \\
\sigma & 1+\sigma & 1-\sigma^2 & -1 & \sigma^2+\sigma-2 & 2-\sigma^2 & \sigma^2+\sigma-3
\end{pmatrix}$$

where $\sigma = 2\cos\frac{2\pi}{9}$.

The generators for $PS\ell_2(13)$ are

$$\delta^-_{\{1,4,6,7\}}(235)(476),$$

$t_{\frac{2\pi}{13},\frac{6\pi}{13}}$ and the linear transformation whose matrix with respect to $e_1,\dots,e_7$ is

$$\frac{1}{\sqrt{13}}\begin{pmatrix}
1 & 0 & 0 & -2 & 0 & -2 & -2 \\
0 & c_0 & c_1 & 0 & c_2 & 0 & 0 \\
0 & c_1 & c_2 & 0 & c_0 & 0 & 0 \\
-2 & 0 & 0 & u & 0 & v & w \\
0 & c_2 & c_0 & 0 & c_1 & 0 & 0 \\
-2 & 0 & 0 & v & 0 & w & u \\
-2 & 0 & 0 & w & 0 & u & v
\end{pmatrix}$$

where

$$c_r = \frac{-7+\sqrt{13}+4\cos\left(\frac{2\pi 4^r}{13}\right)+(3-\sqrt{13})\cos^2\left(\frac{2\pi 4^r}{23}\right)}{2} \qquad (r = 0,1,2)$$

and

$$u = (c_0+2c_2-2c_1)/\sqrt{13},$$

$$v = (c_2+2c_1-2c_0)/\sqrt{13},$$

$$w = (c_1+2c_0-2c_2)/\sqrt{13}\ . \qquad \square$$

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