stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 101/81 APRIL

A.M. COHEN

A CHARACTERIZATION BY ORDER OF THE GENERALIZED HEXAGONS AND A NEAR OCTAGON WHOSE LINES HAVE LENGTH THREE

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51E25

A characterization by order of the generalized hexagons and a near octagon whose lines have length three

by

Arjeh M. Cohen

ABSTRACT

A proof (indicated by J. Tits) is given of the well-known fact that any generalized hexagon of order (2,t) is isomorphic to one of two generalized hexagons associated with the group $G_2(2)$ if t=2 and to the classical hexagon associated with the group ${}^3D_4(2)$ if t=8.

Moreover, it is proved that the near octagon on 315 points associated with the Hall-Janko sporadic simple group HJ is uniquely determined by its parameters.

KEY WORDS & PHRASES: *generalized polygons, near n-gons, distance-regular graphs*

1. INTRODUCTION

The unicity of the generalized hexagons of order (2,2) - up to duality - and of order (2,8) seems to be well known and has been claimed by (at least) two authors: TIMMESFELD [5;(3.4)] and TITS [6;(11.5)]. Timmesfeld has sketched a proof of three steps. Two of these steps lack a proof and the third step is an application of group theoretic results obtained elsewhere in the paper. Tits did not publish any indication of a proof, but upon request provided an elegant determination of the line structure induced on the points of maximal distance to a given point for the order (2,2) case, i.e. the $G_2(2)$- hexagons.

The ideas exploited in Tits' determination turned out to be very useful in establishing complete proofs for both the order (2,2) and the order (2,8) case (the latter leading to the so-called ${}^3D_4(2)$- hexagon). Also the near octagon associated with the Hall-Janko sporadic simple group HJ (cf. [2]) could be shown to be unique of its order in much the same fashion. For this reason, but also because of nonexistence of full proofs in the literature (to the best of my knowledge), I have taken the liberty to include Tits' arguments. Perhaps needless to say, I am very grateful to J. Tits for his kind response to my question.

2. DEFINITIONS AND PRECISE STATEMENTS OF RESULTS

Graphs are undirected, without loops or multiple edges. Subsets of the point set of a graph are often identified with their induced subgraphs. For γ a point of the graph Γ and a nonnegative integer i, the set of points in Γ of distance i to γ is denoted by $\Gamma_i(\gamma)$. An *i-path* is a path of length i.

An *incidence system* $(\mathcal{P},\mathcal{L})$ is a set of *points* $\mathcal{P}$ and a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ whose elements are called *lines*. To such a system we associate the *collinearity graph* Γ whose vertices are the points and in which adjacency is collinearity.

The following notion appears in [4]. A *regular near* 2d-*gon of order* $(s,t;t_2,\ldots,t_{d-1})$ *and of diameter* d is an incidence system $(\mathcal{P},\mathcal{L})$ such that each line contains exactly s+1 points, each point is on exactly t+1 lines and such that there are integers $t_1,t_2,\ldots,t_d$ with $t_1=0$ and $t_d=t$ with the property that any two points $\alpha,\beta \in \mathcal{P}$ with $\alpha \in \Gamma_i(\beta)$ have precisely $1+t_i$ lines through α bearing a single point of $\Gamma_{i-1}(\beta)$ while the other $t-t_i$ lines through α have no points of $\Gamma_{i-1}(\beta)$ or $\Gamma_i(\beta)$ but α (here, $0 \le i \le d$). Moreover, Γ is assumed to be connected so that d is its diameter. A *generalized* 2d-*gon of order* (s,t) is a regular near 2d-gon of order (s,t;0,0,...,0) and of diameter d.

The collinearity graph Γ of a regular near 2d-gon of order and diameter as above is a *distance-regular graph with intersection array* $\{s(t+1),s(t-t_1),\ldots,s(t-t_{d-1});\ 1,1+t_2,\ldots,1+t_{d-1},1+t_d\}$ according to the definitions in [1]. This means that for any i ($0 \le i \le d$) and any two points $\alpha,\beta \in \Gamma$ of mutual distance i, there are exactly $1+t_i$ points in $\Gamma_{i-1}(\alpha) \cap \Gamma_1(\beta)$ and exactly $s(t-t_i)$ points of $\Gamma_{i+1}(\alpha) \cap \Gamma_1(\beta)$ (here, $t_0 = -1$).

We shall restrict attention to the case of regular near 2d-gons with line size 3 (i.e. s=2). It is easy to see that the above incidence system is completely determined by its collinearity graph if $t_2=0$. Therefore, the claims that the $G_2(2)$-, ${}^3D_4(2)$-hexagons and the HJ-octagon are uniquely determined by their orders (again, up to duality for $G_2(2)$), can be alternately formulated as follows. Given an intersection array i, denote by $\kappa(i)$ the number of isomorphism classes of distance-regular graphs with intersection array i.

<u>THEOREM 1</u>. (Tits) $\kappa(\{6,4,4;1,1,3\}) = 2$.

<u>THEOREM 2</u>. $\kappa(\{18,16,16;1,1,9\}) = 1$.

<u>THEOREM 3</u>. $\kappa(\{10,8,8,2;1,1,4,5\}) = 1$.

In a distance-regular graph Γ whose maximal cliques have size 3, *lines* are by definition maximal cliques. Note that this definition coincides with the one for any near 2d-gon whose collinearity graph is Γ.

From now on Γ is the collinearity graph of a near 2d-gon without 4-circuits whose lines have size 3. If γ,δ are two collinear points of Γ, we shall denote by $\gamma\delta$ the line through γ,δ and by $\gamma*\delta$ the third point of $\gamma\delta$. If γ,δ are points of mutual distance 3, then there are distinct $\gamma_i \in \Gamma_1(\gamma) \cap \Gamma_2(\delta)$ and $\delta_i \in \Gamma_1(\delta) \cap \Gamma_1(\gamma_i)$ ($i=1,2,\ldots,1+t_3$). Denote by $\widetilde{\gamma\delta}$ the set $\bigcap_{i=1}^{1+t_3} \Gamma_1(\gamma_i*\delta_i)$. Clearly $|\widetilde{\gamma\delta}| \le 1$. If $|\widetilde{\gamma\delta}| = 1$ for each pair

γ,δ with $\gamma \in \Gamma_3(\delta)$, we say that Γ has the *regulus condition*. This notion is taken from [3].

3. AUXILIARY RESULTS ON COVERS OF GRAPHS

Given two graphs Γ,Δ, we call the map $\pi:\Gamma \to \Delta$ sending points to points and edges to edges a *cover* of Δ whenever its restriction to $\Gamma_1(\gamma)$ is a bijection between the points of $\Gamma_1(\gamma)$ and the points of $\Delta_1(\pi(\gamma))$ for each point γ of Γ. Note that the cardinality of $\pi^{-1}(\delta)$ only depends on the connected component of δ. We call π an m-*cover* of Δ if $|\pi^{-1}(\delta)| = m$ for each γ of Γ. Two covers $\pi_1: \Gamma_1 \to \Delta_1$ and $\pi_2: \Gamma_2 \to \Delta_2$ are called isomorphic if there are graph isomorphisms $\phi: \Gamma_1 \to \Gamma_2$ and $\psi: \Delta_1 \to \Delta_2$ such that $\pi_2\phi = \psi\pi_1$. Unicity of covers is always meant up to isomorphism. We shall often refer to Γ as a cover of Δ when in fact we have a map $\pi: \Gamma \to \Delta$ in mind.

Denote by $H(n,2)$ the n-cube over $\mathbb{F}_2$, i.e. the graph whose points are the vectors in $\mathbb{F}_2^n$ and where two points are adjacent if they differ in exactly one coordinate. Denote by $H^o(n,2)$ for n odd the graph on the 2^{n-1} vectors in $\mathbb{F}_2^n$ of even weight (the *weight* of a vector being the number of nonzero coordinates) in which two points are adjacent whenever their vector sum has weight $n-1$.

<u>LEMMA 1</u>. $H(n,2)$ *for* $n \geq 2$ *has a unique* 2-*cover without* 4-*circuits. This cover is connected and bipartite. It admits an involutory automorphism interchanging the points whose images coincide.*

<u>PROOF</u>. By induction. For $n=2$, the lemma reflects that the 8-circuit is the unique 2-cover of the 4-circuit without 4-circuits. Suppose $n > 2$. Let H_0,H_1 be the induced subgraph of $H(n,2)$ on the points whose last coordinates are 0,1 respectively, and let $\tilde{H}_0,\tilde{H}_1$ be their unique 2-covers (they are unique by induction).

Consider $\gamma_0 \in \tilde{H}_0$, $\gamma_1 \in \tilde{H}_1$ such that their images in H_0 and H_1 differ in the last coordinate only. By symmetry (applying the involutory automorphism to $\tilde{H}_1$ if necessary), we may assume that γ_0 and γ_1 are adjacent. If $\delta_0 \in \tilde{H}_0$, $\delta_1 \in \tilde{H}_1$ are adjacent to γ_0, γ_1 respectively and have images in H_0, H_1 differing in the last coordinate only, they cannot be adjacent in any 2-cover without 4-circuits for otherwise $\gamma_0\ \delta_0\ \delta_1\ \gamma_1\ \gamma_0$ would be a 4-circuit.

This implies that δ_0 must be adjacent to the unique point $\delta_1^1 \in \tilde{H}_1$ distinct from δ_1 with the same image as δ_1 in H_1. It is readily verified that proceeding with any neighboring points as for δ_0, one can construct the 2-cover in a unique fashion. □

LEMMA 2. (i) $H(3,2)$ *is the unique* 2-*cover of* $H^o(3,2)$ *without* 3-*circuits.*
(ii) $H^o(n,2)$ *for* n *odd,* $n \geq 5$, *has at most one* 2-*cover without* 4-*circuits. If it exists, it is connected.*

PROOF. (i) Left to the reader.
(ii) Consider the partial subgraph H (not induced!) of $H^o(n,2)$ on the points of $H^o(n,2)$ in which two vectors are adjacent if and only if exactly n-2 of the first n-1 coordinates are distinct. Addition of the vector (111....10) to all vectors in H with an odd number of nonzero coordinates among the n-1 first coordinates, shows that H is actually isomorphic to $H(n-1,2)$. By the above lemma, H has a unique 2-cover $\tilde{H}$ without 4-circuits. Now let π: $\Gamma \to H^o(n,2)$ be a 2-cover containing $\tilde{H}$ as a partial subgraph such that the restriction $\pi|_{\tilde{H}}$ is this 2-cover of H. Consider $\gamma_0, \gamma_1 \in \Gamma$ with $\pi(\gamma_0) = 00...0$ and $\pi(\gamma_1) = 11...10$. The involutory automorphism of $H^o(n,2)$ interchanging the first two coordinates of $\mathbb{F}_2^n$ and fixing the others lifts to a unique automorphism ϕ of Γ fixing the points whose π-images have 1 in the first two coordinates and interchanging the points whose π-images have o in the first two coordinates. Application of ϕ to γ_0, γ_1 if necessary, yields that up to isomorphism we may assume γ_0, γ_1 to be adjacent.

An argument similar to the one in the previous lemma, using the connectivity of $\tilde{H}$, yields that the edges in $H^o(n,2)$ outside H lift in at most one way (up to isomorphy) to pairs of edges, completing $\tilde{H}$ to a 2-cover of $H^o(n,2)$ without 4-circuits. □

Suppose Γ is the collinearity graph of a regular near 2d-gon of order $(2,t;0,t_2,\ldots,t_{d-1})$ and fix a point ω in Γ. The 1+t lines through ω are labelled 1,2,...,1+t and the two points in $\Gamma_1(\omega)$ of line j are labeled 0_j and 1_j. Thus each point in $\Gamma_1(\omega)$ is uniquely determined by its label. Points of $\Gamma_1(\omega)$ are identified with their labels.

We shall also adhere labels to points in $\Gamma_d(\omega)$. Let γ be such a point. Lable γ by the vector in $\mathbb{Z}_2^{1+t}$, whose j-th coordinate $(1 \leq j \leq 1+t)$ is 0,1 according as 0_j or 1_j is the nearest point on line j through ω. Two points of $\Gamma_d(\omega)$ may have the same label. Nevertheless, this labeling is useful as is indicated by the following lemma.

LEMMA 3. *Suppose Γ is the collinearity graph of a generalized 2d-gon of order $(2,2^a)$ and ω is a point of Γ. Then $\Gamma_d(\omega)$ is a m-cover of $H^o(2^a+1,2)$, where $m = 2^{(ad-a+d-2^a)}$.*

PROOF. If two points in $\Gamma_d(\omega)$ are adjacent their labelings differ by precisely 2^a coordinates. Moreover, the labels of two distinct neighbors in $\Gamma_d(\omega)$ of a given point γ in $\Gamma_d(\omega)$ coincide with the label of γ in distinct coordinates. These observations are direct consequences of the generalized 2d-gon axioms. Without loss of generality, we may assume that $\Gamma_d(\omega)$ contains a point labeled 00 .... 0. The lemma is now immediate since $H^o(2^a+1,2)$ and $\Gamma_d(\omega)$ have the same valency and $H^o(2^a+1,2)$ is connected. □

In the sequel, we shall frequently use the above labelings for $\Gamma_1(\omega)$ and $\Gamma_d(\omega)$.

4. PROOF OF THEOREM 1.

Γ is the collinearity graph of a generalized hexagon of order (2,2). Fix a point ω of Γ. The idea is to show first that $\Gamma_3(\omega)$ is one of two possible graphs and second that $\Gamma_3(\omega)$ determines Γ uniquely.

The points of $\Gamma_1(\omega)$ and $\Gamma_3(\omega)$ are labeled as in Section 3. By Lemma 3, the labeling of $\Gamma_3(\omega)$ is an 8-cover of $H^o(3,2)$, the complete graph on 4 points. To obtain more information on $\Gamma_3(\omega)$, the edges are labeled, too: an edge $\{\gamma,\delta\}$ of $\Gamma_3(\omega)$ is labeled i_j whenever $\gamma*\delta \in \Gamma_1(i_j)$, where $i_j \in \Gamma_1(\omega)$. In this case, we say that $\{\gamma,\delta\}$ is of *type* j. Thus the type of an edge in $\Gamma_3(\omega)$ is the line through ω to which the edge is nearest. Clearly, two adjacent edges are of distinct type.

The proof consists of 15 steps.

(1) *If $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ are distinguished edges in $\Gamma_3(\omega)$ with the same label then α,γ have mutual distance ≥ 2 inside $\Gamma_3(\omega)$.*

Clearly, the two edges are not incident. Moreover, $\gamma \in \Gamma_1(\alpha)$ would imply the pentagon $\alpha\ \gamma(\gamma*\delta)i_j(\alpha*\beta)\alpha$ where i_j is the common label of $\{\alpha,\beta\}$.

(2) *If in the path* $\alpha\ \beta\ \gamma\ \delta\ \varepsilon$ *of* $\Gamma_3(\omega)$ *without repetitions, the edges* $\{\alpha,\beta\}$ *and* $\{\delta,\varepsilon\}$ *are of the same type, they have the same label.*

Let 0_j be the label of $\{\alpha,\beta\}$ and let $\zeta \in \Gamma_1(\gamma) \cap \Gamma_3(\omega)$ be distinct from β,δ. Then $\{\gamma,\zeta\}$ has label 1_j by (1), so $\{\delta,\varepsilon\}$ has label 0_j, again by (1).

(3) *In* $\Gamma_3(\omega)$, *a path* $\alpha\ \beta\ \gamma\ \delta\ \varepsilon$ *without repetitions whose edges have types* j,k,l,j *respectively, extends to a single hexagon inside* $\Gamma_3(\omega)$ *whose edge types are either* j,k,l,j,k,l *or* j,k,l,j,l,k.

In Γ there is a point $\eta \in \alpha\beta$ of distance 2 to ε. Let $\zeta \in \Gamma$ be such that $\{\zeta\} = \Gamma_1(\varepsilon) \cap \Gamma_1(\eta)$. Clearly $\eta \neq \beta$. If $\eta \neq \alpha$, then $\eta\ \zeta\ \varepsilon(\delta{*}\varepsilon)i_j\ \eta$, where i_j is the common label of $\{\alpha,\beta\}$ and $\{\delta,\varepsilon\}$ (see (2)), would be a pentagon. Thus $\eta = \alpha$. Suppose $\zeta \in \Gamma_2(\omega)$. Then $\{\alpha,\alpha{*}\zeta\}$ and $\{\varepsilon,\varepsilon{*}\zeta\}$ are edges of $\Gamma_3(\omega)$ with the same label. Without loss of generality we may take this label to be 0_k. By (2), the label of $\{\beta,\gamma\}$ must then be 0_k, but by (1) it must be 1_k. This is a contradiction, whence $\zeta \in \Gamma_3(\omega)$. Step (3) now readily follows.

(4) *In* $\Gamma_3(\omega)$ *each hexagon has two edges of each type* 1,2,3.

By (1), no type could occur three times.

A hexagon in $\Gamma_3(\omega)$ is said to be *periodic* if its edges have types, j,k,l,j,k,l and *aperiodic* otherwise, i.e. if its edges have types j,k,l,j,l,k.

(5) *If* $\Gamma_3(\omega)$ *contains no periodic hexagons, it consists of two components each of which is isomorphic to the unique 2-cover of* $H(3,2)$ *without 4-circuits.*

Let Ω be a connected component of $\Gamma_3(\omega)$. By (3), we may assume that Ω contains a hexagon $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ with edge types 1,2,3,1,3,2. For each $i(1 \le i \le 6)$, let α_i^1 be the third point of $\Gamma_3(\omega) \cap \Gamma_1(\alpha_i)$. Applying (3) to $\alpha_5\alpha_6\alpha_1\alpha_2\alpha_2^1$ and to $\alpha_4\alpha_3\alpha_2\alpha_1\alpha_1^1$ respectively one finds that $\{\alpha_5^1,\alpha_2^1\}$ and $\{\alpha_1^1,\alpha_4^1\}$ are edges of type 1 in $\Gamma_3(\omega)$. Applying (3) to $\alpha_3^1\alpha_3\alpha_2\alpha_2^1\alpha_5^1$, $\alpha_3^1\alpha_3\alpha_4\alpha_4^1\alpha_1^1$, $\alpha_6^1\alpha_6\alpha_1\alpha_1^1\alpha_4^1$, $\alpha_6^1\alpha_6\alpha_5\alpha_5^1\alpha_2^1$ respectively, one gets four new points $\eta,\eta^1\theta,\theta^1$ of $\Gamma_3(\omega)$ such that $\alpha_5^1\eta\alpha_3^1$, $\alpha_1^1\eta^1\alpha_3^1$, $\alpha_4^1\theta\alpha_6^1$, $\alpha_2^1\theta^1\alpha_6^1$ are paths in $\Gamma_3(\omega)$. Finally (3) applied to $\eta\alpha_3^1\alpha_3\alpha_4\alpha_4^1$ and $\eta^1\alpha_3^1\alpha_3\alpha_2\alpha_2^1$ shows that $\{\theta,\eta\}$ and $\{\theta^1,\eta^1\}$ are edges of $\Gamma_3(\omega)$. Now Ω has exactly 16 points and is as claimed.

Let $\Gamma_3(\omega)$ be as in (5). For $\gamma \in \Gamma_3(\omega)$, there is a unique point of distance 4 to γ within $\Gamma_3(\omega)$ with the same label as γ. Denote this point by γ^σ.

(6) *Let* $\Gamma_3(\omega)$ *be without periodic hexagons and let* $\gamma \in \Gamma_3(\omega)$. *If* $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$ *then* $\gamma*\delta$ *is collinear with* $\gamma^\sigma*\delta^\sigma$.

Any $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$ has distance 3 to γ^σ inside $\Gamma_3(\omega)$ and is therefore in $\Gamma_3(\gamma^\sigma)$. However, there are two minimal paths from δ to γ^σ within $\Gamma_3(\omega)$; the third must be $\delta(\gamma*\delta)(\gamma^\sigma*\delta^\sigma)\gamma$.

(7) *Let* $\Gamma_3(\omega)$ *be as in* (5) *and let* γ_0, γ_1 *be points of* $\Gamma_3(\omega)$ *from distinct connected components. Then there is at most one path of length* 3 *from* γ_0 *to* γ_1 *containing an edge of* $\Gamma_2(\omega)$.

Suppose $\gamma_0\delta\ \varepsilon\ \gamma_1$ is a 3-path with $\delta, \varepsilon \in \Gamma_2(\omega)$. Then γ_0^σ is adjacent to ε by (6) so that the label of γ_0 (being that of γ_0^σ) has at least one coordinate in common with the label of γ_1. There are two more 3-paths from γ_0 to γ_1. If any of these two would contain an edge of $\Gamma_2(\omega)$, there would be a second 2-path from γ_0^σ to γ_1 by (6), which is absurd.

(8) *Let* $\Gamma_3(\omega)$ *be as in* (5). *For each* $\delta \in \Gamma_2(\omega)$ *and each component* Ω *of* $\Gamma_3(\omega)$, *we have* $|\Gamma_1(\delta) \cap \Omega| = 2$.

Each edge of Ω consists of two neighbors of a point in $\Gamma_2(\omega)$. The statement follows from the observation that no two edges of Ω could span lines incident in $\Gamma_2(\omega)$.

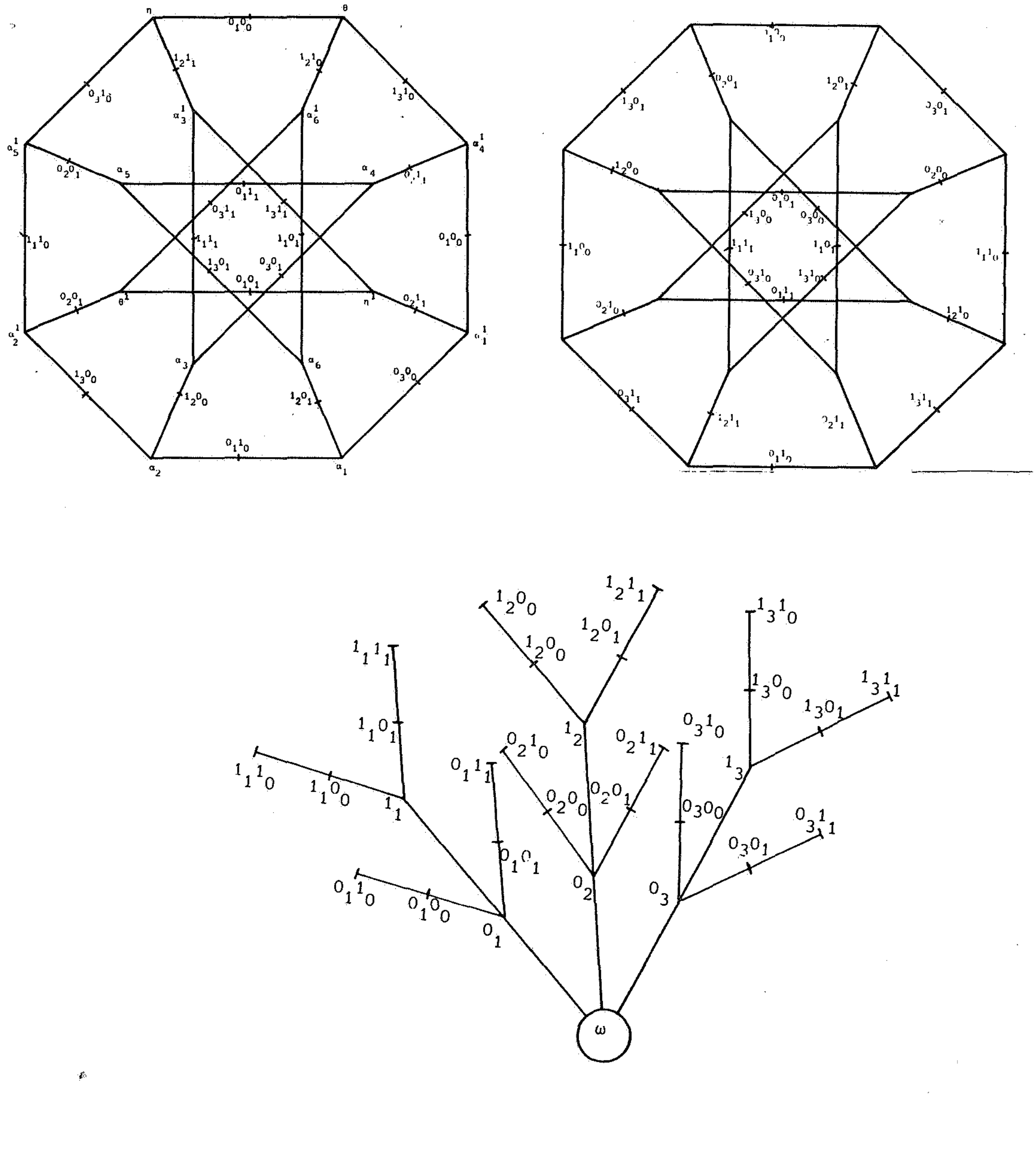
(9) *Let* $\Gamma_3(\omega)$ *be as in* (5). *The labels of points from distinct components of* $\Gamma_3(\omega)$ *have weights of distinct parity.*

By (8) there are points γ_0, γ_1 from distinct components of $\Gamma_3(\omega)$ and $\delta \in \Gamma_2(\omega)$ such that $\gamma_0, \gamma_1 \in \Gamma_1(\delta)$. If the labels of γ_0 and γ_1 agree in an odd number of coordinates, we may assume (after interchanging γ_1 and $\gamma_1*\delta$ if necessary) that they coincide. On the other hand, (7) implies that the labels of γ_0^σ and γ_1 differ by at least one coordinate, in conflict with the labeling of γ_0 and γ_1. Thus the labels of γ_0 and γ_1 agree in an even number of coordinates.

(10) *If* $\Gamma_3(\omega)$ *has no periodic hexagon, then* Γ *is isomorphic to the graph described in Figure 1.*

FIGURE 1.

The dual classical generalized hexagon of order (2,2). Line segments represent lines of the generalized hexagon. The three vertices labeled $i_j k_l$ in the picture are identified in order to represent a point of $\Gamma_2(\omega)$. Each occurrence of the vertex $i_j k_l$ provides a line through the point $i_j k_l$.



Label the points of $\Gamma_2(\omega)$ by labels $i_j k_1$ $(i,k,l \in \mathbb{F}_2;\ j \in \{1,2,3\})$ such that the unique point of $\Gamma_2(\omega)$ with label $i_j k_1$ is adjacent to $i_j \in \Gamma_1(\omega)$ and to $i_j(k+1)_1 \in \Gamma_2(\omega)$. It is evident that each point of $\Gamma_2(\omega)$ may be identified with its label.

Let Ω_0, Ω_1 be the connected components of $\Gamma_3(\omega)$ whose points have labels of even and odd weights respectively (compare (9)). By (6) and (8) we may label the points $\gamma * \delta$ for $\gamma, \delta \in \Omega_0$ as indicated in the figure. It remains to show that the edges between points of $\Gamma_2(\omega)$ and of Ω_1 are as described.

Let $\gamma, \delta \in \Omega_1$ span an edge labeled 0_1. Then we may assume without harming generality that $\gamma * \delta = 0_1 0_0$ and that γ, δ have labels 001, 010 respectively. Let $\gamma_0, \delta_0 \in \Omega_0 \cap \Gamma_1(0_1 0_0)$ have labels 000, 011 respectively. Now $\gamma_0^\sigma \in \Gamma_3(\gamma)$ by (5), and by (7) there are two 3-paths from γ_0^σ to γ without edges in $\Gamma_2(\omega)$. These two paths must be $\gamma_0^\sigma(\gamma_0^\sigma * 0_2 0_0)(1_3 0_1)\gamma$ and $\gamma_0^\sigma(0_3 0_0)\theta\ \gamma$ where $\theta \in \Omega_1 \cap \Gamma_1(\gamma) \cap \Gamma_2(0_2)$.

Thus starting from $0_1 0_0$, all neighbors of $1_3 0_1$ and $0_3 0_0$ have been found. It is straightforward to continue in this way and determine all remaining edges.

(11) *A path $\alpha_1\alpha_2\alpha_3$ in $\Gamma_3(\omega)$ of length 3 contained in a periodic hexagon is not contained in a second hexagon.*

Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ be the periodic hexagon, with edges of type 1,2,3,1,2,3 say. Without loss of generality, we may assume a second hexagon containing $\alpha_1\alpha_2\alpha_3$ to be either $\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\alpha_1$ or $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\alpha_1$ for $\beta_i \neq \alpha_j$. Application of (4) yields a contradiction in the first case and determines the edge types of the second hexagon in the latter case. There are two incident edges then, both of type 3, so there is no such second hexagon.

(12) *Every edge of a periodic hexagon belongs to exactly one other hexagon, and this hexagon is periodic, too.*

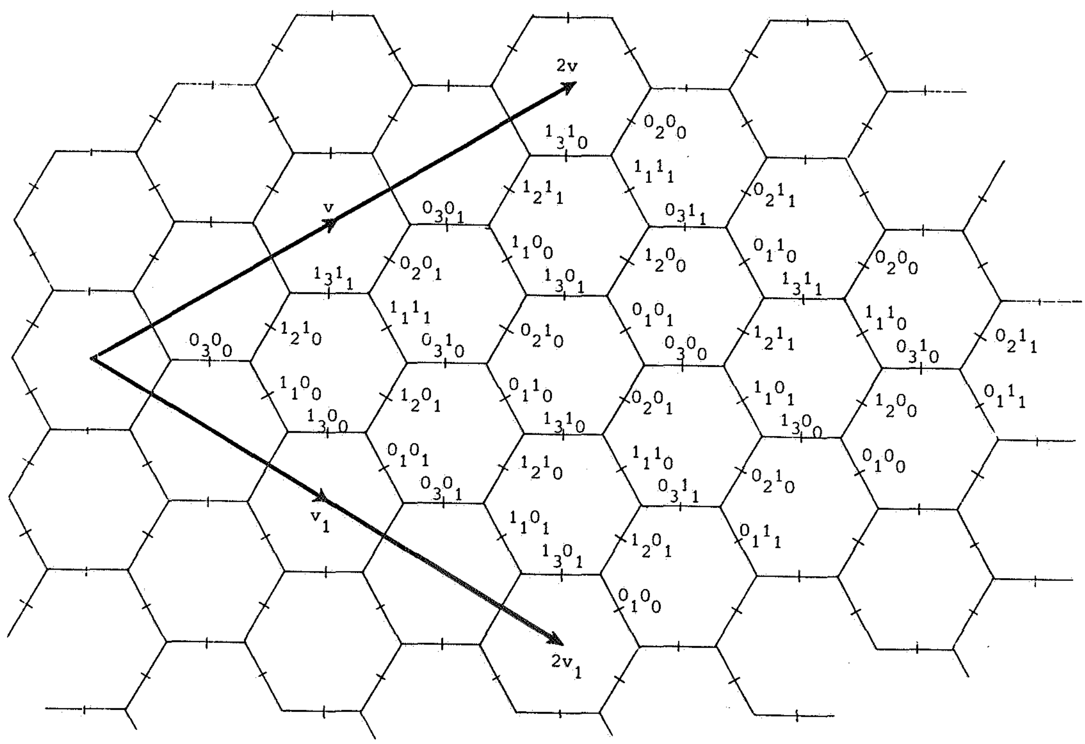
Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ be a hexagon with edge types i,j,k,i,j,k. From α_2 there starts a path $\alpha_2\beta_1\beta_2\beta_3$ without repetitions and of edge type k,j,i. Apply (3) to obtain a hexagon $\alpha_1\alpha_2\beta_1\beta_2\beta_3\beta_4\alpha_1$. Note that $\alpha_6 \neq \beta_4$ in view of (11) and that this hexagon is periodic. Finally any hexagon distinct from $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ but containing α_1, α_2 must contain $\beta_1\alpha_2\alpha_1\beta_4$, so coincides with the second hexagon by (11).

(13) *If* $\Gamma_3(\omega)$ *contains a periodic hexagon, all its hexagons are periodic and* $\Gamma_3(\omega)$ *is the quotient of the graph in* Figure 2 *by the group of translations generated by* $2v$ *and* $2v^1$ (At this instant, disregard the labeled points in the figure).

FIGURE 2.

The classical generalized hexagon of order (2,2).

Conventions are as in Figure 1, with the exception that the constituent $\omega \cup \Gamma_1(\omega) \cup \Gamma_2(\omega)$ is not included as it is identical to the one in Figure 1.



Let Ω be a component of $\Gamma_3(\omega)$ containing a periodic hexagon. By (12), all hexagons of Ω are periodic and if one fills up each hexagon with a cell, it is clear that one gets a 2-dimensional manifold. The universal covering is paved with hexagons, each hexagon belonging to exactly three hexagons, hence the picture of Figure 2. Now Ω is the quotient of that picture by an automorphism group G. The automorphisms are the translations by vectors of the form $mv + m^1v^1$ $(m,m^1 \in \mathbb{Z})$ – they form a group G^1 – and the central symmetries with respect to the centers of the hexagons. But G cannot contain such a symmetry for otherwise the hexagon would become a triangle in Ω. Thus $G \subseteq G^1$. The quotient of the picture by G^1 has 8 vertices; therefore $[G^1:G] \leq 4$. The translations v, v^1, $v-v^1$ cannot belong to G, because Ω would have 4-circuits. Hence (13).

(14) *If* $\Gamma_3(\omega)$ *has a periodic hexagon,* Γ *is isomorphic to the graph described in Figure* 2.

The problem is to exhibit which edges of $\Gamma_3(\omega)$ span lines that have a point of $\Gamma_2(\omega)$ in common. Notice that to each point γ of $\Gamma_3(\omega)$ we can associate the unique point γ^σ of $\Gamma_3(\omega)$ of distance 5 to γ whose label coincides with the label of γ. Now consider an edge $\{\gamma,\delta\}$ of $\Gamma_3(\omega)$. There is a point δ^1 of the same label as δ in $\Gamma_3(\omega) \cap \Gamma_3(\delta) \cap \Gamma_1(\gamma^\sigma)$. As there are two 3-paths from δ to δ^1 inside $\Gamma_3(\omega)$, there must be a third path passing through $\gamma*\delta$ and $\delta^1*\gamma^\sigma$. Hence the latter two points are adjacent. The same argument with γ replaced by $(\delta^1)^\sigma$ and γ^1 by δ leads to adjacency of $\delta^1*\gamma^\sigma$ and $\gamma^1*(\delta^1)^\sigma$. The conclusion is that $\gamma*\delta$ and $\gamma^1*(\delta^1)^\sigma$ coincide. This completes the determination of $\Gamma_2(\omega) \cup \Gamma_3(\omega)$, and hence of Γ.

(15) *Either* (i) *or* (ii) *holds:*

(i) Γ *is isomorphic to the classical* $G_2(2)$*-hexagon, for each point* ω *of* Γ *the graph* $\Gamma_3(\omega)$ *is connected and contains only periodic hexagons.*

(ii) Γ *is isomorphic to the dual classical* $G_2(2)$*-hexagon, for each point* ω *of* Γ *the graph* $\Gamma_3(\omega)$ *has 2 connected components and contains only aperiodic hexagons.*

Notice that the graph of (10) cannot be 'classical' as it contains an ovoid: nine points of mutual distance 3. The statement is now a direct consequence of (10) and (14).

REMARK. In both cases (i) and (ii) of (15), a map σ on $\Gamma_3(\omega)$ has been defined. However $\sigma \in \mathrm{Aut}\,(\Gamma_3(\omega))$ only if (i) prevails; in that case σ can be extended to an involutory automorphism of Γ fixing $\{\omega\} \cup \Gamma_1(\omega)$ pointwise. This automorphism is used by Timmesfeld in [5].

5. PROOF OF THEOREM 2.

Γ is the collinearity graph of a generalized hexagon of order (2,8). Fix $\omega \in \Gamma$ and label the points of $\Gamma_1(\omega)$ and $\Gamma_3(\omega)$ as above. Application of Lemma 3 yields that $\Gamma_3(\omega)$ is a 2-cover of $H^O(9,2)$. In view of Lemma 2 this determines the (unlabeled) graph $\Gamma_3(\omega)$ uniquely. For $\gamma \in \Gamma_3(\omega)$, let γ^σ be the unique point in $\Gamma_3(\omega)$ distinct from γ with the same label as γ.

By [3], any generalized hexagon of order (2,8) in which the regulus condition holds, is the ${}^3D_4(2)$-hexagon. We shall therefore content ourselves with the following proof of the regulus condition in two steps.

(1) *For any* $\gamma \in \Gamma_3(\omega)$, *we have* $\gamma^\sigma \in \Gamma_3(\gamma)$. *Moreover, if* $\delta \in \Gamma_1(\gamma) \cap \Gamma_3(\omega)$, *then* $\delta \in \Gamma_3(\gamma^\sigma)$.

The second statement is immediate. Since γ and γ^σ are not adjacent in $\Gamma_3(\omega)$, their mutual distance is ≥ 2. Suppose there exists $\delta \in \Gamma_1(\gamma) \cap \Gamma_1(\gamma^\sigma)$. Then necessarily $\delta \in \Gamma_2(\omega)$, so that $\gamma*\delta \in \Gamma_3(\omega)$. The latter point, being a neighbor of γ, has distance 3 to γ^σ in $\Gamma_3(\omega)$. Thus $\gamma^\sigma\delta(\delta*\gamma)$ can be completed to a pentagon, which is absurd. It follows that $\gamma^\sigma \notin \Gamma_2(\gamma)$, so that $\gamma^\sigma \in \Gamma_3(\gamma)$, whence the first statement.

(2) Γ *has the regulus condition.*

Let $\{\gamma,\delta\}$ be an edge of $\Gamma_3(\omega)$. Then $\{\gamma^\sigma,\delta^\sigma\}$ is an edge, too. By (1) the points $\gamma*\delta$ and $\gamma^\sigma*\delta^\sigma$ are distinct. Suppose they are nonadjacent. Then $\delta^\sigma(\gamma^\sigma*\delta^\sigma)i_j(\gamma*\delta)$, where $\{i_j\} = \Gamma_1(\omega) \cap \Gamma_1(\gamma*\delta)$, is a 3-path so that $\delta^\sigma \in \Gamma_3(\gamma*\delta)$. As $\delta^\sigma \in \Gamma_3(\gamma)$ by (1), we get $\delta^\sigma \in \Gamma_2(\delta)$, contradictory to (1). The conclusion is that $\gamma*\delta$ and $\gamma^\sigma*\delta^\sigma$ must be adjacent. Letting range δ over $\Gamma_1(\gamma) \cap \Gamma_3(\omega)$, we obtain $\gamma^\sigma \in \tilde{\omega\gamma}$ and we are done.

<u>REMARK</u>. Apart from ending the proof by referring to [3], one could also finish by observing that σ can be extended to an automorphism of Γ and applying Timmesfeld's Theorem [5;(3.3)] to the group generated by all σ for ω varying over the points of Γ.

<u>6. PROOF OF THEOREM 3.</u>

Γ is the collinearity graph of a regular near 8-gon of order (2,4;0,3). Fix $\omega \in \Gamma$ and label $\Gamma_1(\omega)$, $\Gamma_4(\omega)$ as above. We proceed in 7 steps.

(1) $\Gamma_4(\omega)$ *is the unique 2-cover of* $H(5,2)$ *without 4-circuits.*

The label of two adjacent points γ,δ of $\Gamma_4(\omega)$ differ in exactly one coordinate; they coincide in four coordinates as $\gamma*\delta$ has distance 2 to $1+t_3 = 4$ points on four distinct lines through ω, and the coordinates corresponding to the fifth line through ω must differ (for otherwise this line through ω would bear a point of distance 3 to γ,δ so that this point would be a fifth point of $\Gamma_1(\omega) \cap \Gamma_2(\gamma*\delta)$, conflicting $t_3=3$). Thus the labeling is a 2-cover of $H(5,2)$. Thanks to Lemma 1, we are done.

(2) *Put* $\Omega_i = \{\gamma \in \Gamma_3(\omega) | \Gamma_2(\gamma) \cap \{0_i,1_i\} = \emptyset\}$ *for* $1 \le i \le 5$. *Then* Γ *is the disjoint union of the* Ω_i *and each* Ω_i *is a 2-cover of* $H(4,2)$ *without 4-circuits.*

The points of Ω_i are labeled by vectors in $\mathbb{F}_2^4$ whose coordinates are indexed by the numbers $j(1 \le j \le 5;\ j \ne i)$ in such a way that the j-th coordinate of the label of $\gamma \in \Omega_i$ is 0,1 according as $0_j \in \Gamma_2(\gamma)$ or $1_j \in \Gamma_2(\gamma)$. Fundamental properties of near 2d-gons then yield that any edge $\{\gamma,\delta\}$ with $\gamma \in \Omega_i$ must belong to Ω_i, and imply that the labels of γ and δ differ in all but one coordinate. It follows that the labeling provides a cover of an isomorph of H(4,2) without 4-circuits. Each Ω_i, being such a cover, has at least 32 points. But $\Gamma_3(\omega)$ has 160 points in all, so each Ω_i is a 2-cover of H(4,2) as claimed.

For $\gamma \in \Gamma_3(\omega)$, let γ^σ be the unique second point of the connected component of γ whose label coincides with γ.

(3) *For any* $\gamma \in \Gamma_3(\omega)$, *we have* $\gamma^\sigma \in \Gamma_3(\gamma) \cup \Gamma_4(\gamma)$. *Moreover, if* $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$, *then* $\delta \in \Gamma_3(\gamma^\sigma)$.

(4) Γ *has the regulus condition.*

Proofs of (3), (4) will be omitted as they run parallel to those of (1), (2) for Theorem 2.

(5) *For each* $\gamma \in \Gamma_4(\omega)$ *there are two points* δ, δ^1 *of distance* 5 *to* γ *inside* $\Gamma_4(\omega)$. *Furthermore, we have either* $\Gamma_1(\gamma) \cap \Gamma_3(\delta) \subseteq \Gamma_3(\omega)$ *and* $\Gamma_1(\gamma) \cap \Gamma_3(\delta^1) \subseteq \Gamma_4(\omega)$, *or* $\Gamma_1(\gamma) \cap \Gamma_3(\delta^1) \subseteq \Gamma_3(\omega)$ *and* $\Gamma_1(\gamma) \cap \Gamma_3(\delta) \subseteq \Gamma_4(\omega)$.

The first statement is clear by (1). Let γ be labeled 00000 (this assumption does not harm the generality). Then δ, δ^1 are labeled 11111. Obviously $\delta, \delta^1 \in \Gamma_3(\gamma) \cup \Gamma_4(\gamma)$. Let $\gamma\ \gamma_1\gamma_2\gamma_3\delta$ be a path from γ to δ (possibly $\gamma_1 = \gamma_2$). If $\gamma_1, \gamma_3 \in \Gamma_3(\omega)$, then $\gamma_2 \in \Gamma_2(\omega) \cup \Gamma_3(\omega)$. But γ_1, γ_3 are labeled 0000 and 1111 respectively, so their mutual distance exceeds 2 in $\Gamma_3(\omega)$, and $\gamma_2 \in \Gamma_2(\omega)$. However, this contradicts the fact that the labels of γ_1 and γ_3 have no coordinate in common.

Thus either γ_1 or γ_3 is a point of $\Gamma_4(\omega)$. Assume $\gamma_3 \in \Gamma_4(\omega)$, the reasoning for $\gamma_1 \in \Gamma_4(\omega)$ being similar. Then $\gamma_1, \gamma_2 \in \Gamma_3(\omega)$, and $\delta \in \Gamma_4(\gamma)$. Since $\gamma_1 \in \Gamma_3(\delta)$, there are four minimal paths from γ_1 to δ all of them containing an edge of $\Gamma_3(\omega)$. Each of these paths contains a point of $\Gamma_1(\delta) \cap \Gamma_4(\omega)$. From any such point ($\gamma_3$ is one of them) there start four paths of length 3 to γ containing an edge of $\Gamma_3(\omega)$. As any point of $\Gamma_1(\gamma) \cap \Gamma_3(\delta)$ is on such a path, it must be in $\Gamma_3(\omega)$. Finally, no minimal path from γ to δ^1 could pass through γ_1, for otherwise $|\Gamma_1(\gamma_2) \cap \Gamma_4(\omega)| = 4$. By the above argument, it results that $\Gamma_1(\gamma) \cap \Gamma_3(\delta^1) \subseteq \Gamma_4(\omega)$.

(6) $\Gamma_3(\omega) \cup \Gamma_4(\omega)$ *is unique.*

In view of (1) and (2), it remains to determine the edges between $\Gamma_3(\omega)$ and $\Gamma_4(\omega)$. Without harming generality, we may assume that γ is a point of $\Gamma_4(\omega)$ labeled 00000 and adjacent to a point γ_i in Ω_i $(1 \le i \le 5)$, while $\delta \in \Gamma_4(\omega)$ is labeled 11111 and satisfies $\Gamma_1(\gamma) \cap \Gamma_3(\delta) = \{\gamma_i \mid 1 \le i \le 5\}$ (see (5)). Now the four minimal paths from γ_i to δ determine the edges between neighbors of γ_i in $\Gamma_3(\omega)$ and points in $\Gamma_4(\omega)$.

Proceed as follows: Suppose $\{\zeta_1, \zeta_2\}$ is an edge of $\Gamma_3(\omega)$ for which the neighbors $\xi_i, \xi_i * \zeta_i \in \Gamma_4(\omega)$ $(i=1,2)$ are determined. We may assume ξ_1, ξ_2 to have mutual distance 4 inside $\Gamma_4(\omega)$ (interchange ξ_i and $\xi_i * \zeta_i$ if necessary), there is a unique point $\eta_1 \in \Gamma_1(\xi_2) \cap \Gamma_4(\omega)$ of distance 5 in $\Gamma_4(\omega)$ to ξ_1. The four paths of minimal length from ζ_1 to η_1 pass through distinct neighbors of ζ_1, thus determining all edges between $\Gamma_1(\zeta_1) \cap \Gamma_3(\omega)$ and $\Gamma_4(\omega)$.

Applying this argument along the edges of each component of $\Gamma_3(\omega)$, we obtain unicity of $\Gamma_3(\omega) \cup \Gamma_4(\omega)$.

(7) Γ *is unique.*

Of course $\{\omega\} \cup \Gamma_1(\omega) \cup \Gamma_2(\omega)$ is unique. In view of (6), it remains to show that the edges between $\Gamma_2(\omega)$ and $\Gamma_3(\omega)$ are uniquely determined. This again comes down to deciding which four edges of $\Gamma_3(\omega)$ span lines intersecting in a point of $\Gamma_2(\omega)$.

Consider two points γ, δ of $\Gamma_4(\omega)$ whose labels differ in exactly three coordinates. Obviously $\delta \in \Gamma_3(\gamma)$. Assume (to facilitate notations) the labels of γ, δ to be 00000 and 11100 respectively. There are three paths $\gamma\gamma_i\delta_i\delta$ $(i = 1,2,3)$ inside $\Gamma_4(\omega)$ and according to (6) there is a unique fourth of length 3 from γ to δ containing an edge $\{\zeta, \eta\}$ of $\Omega_4 \cup \Omega_5$. By (4), the regulus condition holds, so $\{\xi\} = \widetilde{\gamma\delta}$ for some $\xi \in \Gamma_2(\omega)$. Thus $\zeta * \eta$, $\gamma_i * \delta_i \in \Gamma_1(\xi)$ $(i = 1,2,3)$, and all lines through ξ bearing points of $\Gamma_3(\omega)$ are determined.

Now any edge of $\Gamma_3(\omega)$ is on a 3-path from γ to δ for certain γ, δ as before, so the above procedure describes how to determine all edges between $\Gamma_2(\omega)$ and $\Gamma_3(\omega)$. Consequently Γ is unique.

<u>REMARK</u>. The involutory map $\sigma \in \mathrm{Aut}(\Gamma_3(\omega))$ of (3) can be extended to an automorphism of Γ by decreeing that $\gamma^\sigma = \gamma$ for $\gamma \in \{\omega\} \cup \Gamma_1(\omega)$, $\gamma^\sigma = \delta * \gamma$ where $\{\delta\} = \Gamma_1(\omega) \cap \Gamma_1(\gamma)$ for $\gamma \in \Gamma_2(\omega)$, and γ^σ for $\gamma \in \Gamma_4(\omega)$ is the unique point distinct from γ in $\Gamma_4(\omega)$ whose label is the same as that of γ. The

Hall-Janko group arises as the group of automorphisms generated by the 315 involutions σ for ω ranging over the points of Γ.

REFERENCES

[1] BIGGS, N.L., *Algebraic Graph Theory,* Cambridge University Press, 1974.

[2] COHEN, A.M., *Geometries originating from certain distance-regular graphs,* to appear in Proceedings of Finite Geometries and Designs, LMS Lecture Note, Cambridge University Press, 1981.

[3] RONAN, M.A., *A note on the* $^3D_4(q)$ *generalized hexagons,* JCT,A,29 (1980), 249-250.

[4] SHULT, E.E. & YANUSHKA, A., *Near n-gons and line systems.* Geometriae Dedicata 9(1980) 1-72.

[5] TIMMESFELD, F.G., *A characterization of the Chevalley- and Steinberg-groups over* $\mathbb{F}_2$, Geometriae Dedicata 1 (1973) 269-321.

[6] TITS, J., *Sur la trialité et certains groupes qui s'enduisent,* Publ. Math. I.H.E.S. 2 (1959) 14-60.