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Normal operators in finite-dimensional Hilbert spaces

There are many problems in analysis which can be reduced to questions about normal operators in Hilbert spaces. It is therefore very important to get a thorough understanding of the structure of these operators. The essential features are the same in both the finite and the infinite-dimensional case. Unfortunately this fact is often obscured by treating these cases differently. In the finite-dimensional case geometrical ideas are dominant, operators are replaced by their matrices, determinants are used heavily and the main goal is to transform the given matrix into a diagonal matrix. In the general case matrices are almost useless, determinants do not exist and representations in diagonal form make no sense.

This does not mean that nothing can be said in this case, it only means that these methods and notions are not the proper ones to treat this problem.

The real meaning of the fact that a normal matrix can be transformed into diagonal form by unitary matrices is that the corresponding linear operator can be represented as a multiplication operator on some function space isomorphic to the given Hilbert space. In this formulation the theorem remains true for infinite-dimensional spaces.

I shall try to give you some idea of how for finite dimensions this theorem may be proved without using determinants, etc., and what modifications are to be expected in the infinite-dimensional case. In order to stress the main ideas I shall restrict myself to typical special cases and make no attempt to give proofs of well-known facts.

Let us begin with some definitions!
Let C^k denote the set of all k-tuples $x = (\xi_0, \xi_1, \ldots, \xi_{k-1})$ of complex numbers ξ_i. Define addition and multiplication with complex numbers in the usual way: $x + y = (\xi_0 + \eta_0, \ldots, \xi_{k-1} + \eta_{k-1})$, $\alpha x = (\alpha\xi_0, \ldots, \alpha\xi_{k-1})$.
This set is called (k-dimensional) Hilbert space if to every pair of elements x,y is associated a complex number (x,y) satisfying the following conditions:

$(x,x) \geq 0$, $(x,x) = 0$ if and only if $x = 0$.
$(x,y) = \overline{(y,x)}$
$(\alpha x,y) = \alpha(x,y)$
$(x_1+x_2,y) = (x_1,y) + (x_2,y)$

The number (x,y) is called the inner product of x and y.
Different inner products define different Hilbert spaces. There really exist different inner products, e.g. for every choice of real numbers $\mu_i > 0$
$(x,y) = \sum_{i=0}^{k-1} \xi_i \overline{\eta_i} \mu_i$ satisfies all requirements.

Let H^k be the Hilbert space formed by the set C^k together with the inner product $(x,y) = \xi_0 \overline{\eta_0} + \ldots + \xi_{k-1} \overline{\eta_{k-1}}$.

Denote by $||x|| = \sqrt{(x,x)}$ the norm of x. Then we have:

$||x|| \geq 0$, $||x|| = 0$ if and only if $x = 0$.

$||\alpha x|| = |\alpha| \; ||x||$

$||x+y|| \leq ||x|| + ||y||$ (triangle inequality).

$|(x,y)| \leq ||x|| \; ||y||$ (Cauchy-Schwarz inequality).

A linear operator A on H^k is a mapping A of H^k into itself satisfying

$A(x_1+x_2) = Ax_1 + Ax_2$

$A(\alpha x) = \alpha Ax$

for every $x \in H^k$.

There is a one-to-one correspondence between linear operators A and complex k x k - matrices $\tilde{A} = (a_{ij})$ given by

$$A(\xi_0, \xi_1, \ldots, \xi_{k-1}) = \left(\sum_{j=0}^{k-1} a_{0j} \xi_j, \sum_{j=0}^{k-1} a_{1j} \xi_j, \ldots, \sum_{j=0}^{k-1} a_{k-1,j} \xi_j \right).$$

For any pair of linear operators A,B in H^k we define $A + B$, αA, AB by $(A+B)x = Ax + Bx$, $(\alpha A)x = \alpha Ax$, $(AB)x = A(Bx)$ for all $x \in H^k$.

Let 0 be the null operator ($0x \equiv 0$) and E the identity ($Ex = x$ for all x).

The operator A^* corresponding to the matrix $\overline{(a_{ji})}$ is called the adjoint of A. It is uniquely determined by the equation $(Ax,y) = (x,A^*y)$ for all $x,y \in H^k$ and has the following properties:

$A^{**} = A$, $(A+B)^* = A^* + B^*$, $(\alpha A)^* = \overline{\alpha} A^*$, $(AB)^* = B^* A^*$.

For every operator A a norm $||A||$ may be defined by $||A|| = \sup_{||x||=1} ||Ax||$.

It has the following properties:

$||A|| \geq 0$, $||A|| = 0$ if and only if $A = 0$.

$||\alpha A|| = |\alpha| \; ||A||$

$||A+B|| \leq ||A|| + ||B||$

$||AB|| \leq ||A|| \; ||B||$

$||A^*|| = ||A||$

$||A^* A|| = ||A^*|| \; ||A||$

We can now define the class of operators which we are interested in:
An operator A is called normal if $AA^* = A^* A$.
The importance of this concept depends on two facts:
1) Normal operators are easy to treat.
2) Most operators which arise in applications have this property.
To get familiar with normal operators consider some special cases:
An operator A is called self-adjoint if $A^* = A$.
This implies that the corresponding matrix satisfies $(a_{ij}) = \overline{(a_{ji})}$.
Every normal operator A has a unique representation of the form
$A = A_1 + iA_2$, $A_1^* = A_1$, $A_2^* = A_2$, where

$$A_1 = \frac{A + A^*}{2}, \quad A_2 = \frac{A - A^*}{2i} \quad \text{and} \quad A_1A_2 = A_2A_1.$$

Normal operators show thus remarkable analogies to complex numbers, where self-adjoint operators correspond to real numbers. This analogy can be extended. Corresponding to complex numbers of absolute value one we define unitary operators to be those satisfying $U U^* = U^* U = E$.
The unitary operators have the following geometric meaning: They are just those operators which preserve the whole structure of H^k, i.e. they are characterized as linear operators leaving the inner product invariant:
$(Ux, Uy) = (x, y)$ for all $x, y \in H^k$. For every unitary U the inverse U^{-1} exists and satisfies $U^{-1} = U^*$.
In analogy to the representation $z = re^{i\phi}$ of complex numbers every normal operator A has a representation in the form $A = RU = UR$ with unitary U and self-adjoint R. This is an easy corollary of more general theorems given later.

The structure theory of normal operators begins with the observation that the underlying Hilbert space may be decomposed into the orthogonal sum of "cyclic" subspaces. We don't carry out this decomposition here, but assume from the beginning that the space H^k is already "cyclic" with respect to the given normal operator A. This means that there exists an element $x_0 \in H^k$, $||x_0|| = 1$, such that every other $x \in H^k$ has the form $x = p(A)x_0$ for some polynomial $p(A) = c_0E + c_1A + \dots c_nA^n$ of A.

If $P(A)$ denotes the set of all polynomials in A then our fundamental assumption may be expressed as $H^k = P(A)x_0$.
The first consequence of this assumption is that the correspondence $p(A) \to p(A)x_0$ is one-to-one, or with other words that $p_0(A)x_0 = 0$ implies $p_0(A) = 0$, i.e. $p_0(A)x = 0$ for all $x \in H^k$. But this follows immediately from the fact that $x = p(A)x_0$ for some $p(A)$ and therefore $p_0(A)x = p_0(A)\, p(A)x_0 = p(A)\, p_0(A)x_0 = 0$.
This implies that $p_1(A), \ldots, p_l(A) \in P(A)$ are linearly independent if and only if $p_1(A)x_0, \ldots, r_l(A)x_0$ are linearly independent in H^k. Therefore $P(A)$ is also k-dimensional. The elements $E, A, \ldots, A^{k-1}$ are linearly independent and are therefore a base of $P(A)$. For otherwise every polynomial would be expressible as a linear combination of $E, A, \ldots, A^l$ with $l \leq k-2$ and $P(A)$ would not be k-dimensional. This fact may also be stated in the following form: There exist complex numbers $c_i \neq 0$ such that $\pi_0(A) = c_0E + c_1A + \ldots + A^k = 0$.

For every polynomial $p(A)$ of degree less than k we have $p(A) \neq 0$.
We consider now the corresponding polynomial $\pi_0(\lambda) = c_0 + c_1\lambda + \ldots + c_{k-1}\lambda^{k-1} + \lambda^k$. The equation $\pi_0(\lambda) = 0$ has exactly k roots, say $\lambda_0, \ldots, \lambda_{k-1}$.
We have then $\pi_0(\lambda) = c_0 + c_1\lambda + \ldots + c_{k-1}\lambda^{k-1} + \lambda^k = (\lambda - \lambda_0) \ldots (\lambda - \lambda_{k-1})$. This equation means that the coefficients c_i are expressible in terms of the λ_j's and remains true if one replaces λ by A. Therefore we have

$$\pi_0(A) = (A - \lambda_0E)(A - \lambda_1E) \ldots (A - \lambda_{k-1}E).$$

This equation contains the whole structure of the operator A.

Consider now the polynomials $q_i(A) = \prod_{j \neq i} (A - \lambda_jE)$.
Then
$(A - \lambda_iE)q_i(A) = \pi_0(A) = 0$ and $q_i(A) \neq 0$ because this is a polynomial of degree k-1.

Using this result and the equality $||BB^*|| = ||B||^2$ holding for all operators B we get $(A^* - \overline{\lambda_i}E)q_i(A) = 0$, because

$||(A^* - \overline{\lambda_i}E)q_i(A)||^2 = ||((A^* - \overline{\lambda_i}E)q_i(A))\ ((A^* - \overline{\lambda_i}E)q_i(A))^*|| =$

$= ||(A^* - \overline{\lambda_i}E)\ (q_i(A))^* \circ ((A - \lambda_i E)q_i(A))|| = 0.$

We have therefore $A\ q_i(A) = \lambda_i\ q_i(A)$ and $A^*\ q_i(A) = \overline{\lambda_i}\ q_i(A)$.

Through successive applications of these equations we get

$$p(A)\ q_i(A) = p(\lambda_i)\ q_i(A)$$

and
$$(p(A))^* q_i(A) = \overline{p(\lambda_i)}\ q_i(A)$$

for arbitrary polynomials $p(A)$.

In particular we have

$||q_i(A)||^2 = ||(q_i(A))^*\ q_i(A)|| = ||\ \overline{q_i(\lambda_i)}\ q_i(A)|| =$

$= |q_i(\lambda_i)|\ ||q_i(A)||$ or $||q_i(A)|| = |q_i(\lambda_i)|$.

This implies $q_i(\lambda_i) \neq 0$ because $q_i(A) \neq 0$.

This in turn means that $\lambda_i \neq \lambda_j$ for $i \neq j$.

We may therefore consider the polynomials $p_i(A) = \dfrac{q_i(A)}{q_i(\lambda_i)}$.

Let us now summarize the results obtained so far:

<u>Theorem 1</u>:

Every $p(A) \in P(A)$ has a unique representation in the form

(1) $p(A) = p(\lambda_0)\ p_0(A) + \dots + p(\lambda_{k-1})\ p_{k-1}(A)$.

$P(A)$ contains with every $p(A)$ the adjoint $(p(A))^*$, which is given by

(1') $(p(A))^* = \overline{p(\lambda_0)}\ p_0(A) + \dots + \overline{p(\lambda_{k-1})}\ p_{k-1}(A)$.

<u>Proof</u>: We have $p_i(A)\ p_j(A) = p_i(\lambda_j)\ p_j(A) = 0,\ i \neq j$.

$(p_i(A))^2 = p_i(\lambda_i)\ p_i(A) = p_i(A)$

$(p_i(A))^* = p_i(\lambda_i)\ p_i(A) = \overline{p_i(\lambda_i)}\ p_i(A) = p_i(A)$

$p_0(A), \dots, p_{k-1}(A)$ are therefore linearly independent and form a base of $P(A)$.

If $p(A) = \sum c_i\ p_i(A)$, then $p(A)\ p_i(A) = c_i\ p_i(A)$, i.e. $p(\lambda_i)\ p_i(A) = c_i\ p_i(A)$ or $c_i = p(\lambda_i)$.

We introduce now the following notations: let $\sigma(A)$ denote the subset of the complex plane consisting of the elements $\{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$. Let $C(\sigma(A))$ be the set of all complex functions f on $\sigma(A)$. Every f on $\sigma(A)$ is of course uniquely determined by the values $f(\lambda_i)$. For every f we define a norm by

$$||f||_\infty = \sup_{z \in \sigma(A)} |f(z)| = \max_i |f(\lambda_i)|.$$

Then the following theorem holds:

Theorem 2:

The mapping V of P(A) into $C(\sigma(A))$ defined by $Vp(A) = p(z)$ is a one-to-one mapping of P(A) onto $C(\sigma(A))$ with the following properties:

1) $V(P(A) + q(A)) = Vp(A) + Vq(A)$

2) $V(\alpha p(A)) = \alpha Vp(A)$

3) $V(P(A)\ q(A)) = Vp(A) \circ Vq(A)$

4) $V((a(A))^*) = \overline{V(P(A))}$

5) $||V(p(A))||_\infty = ||p(A)||$.

Proof: It follows immediately from the definition of V that it is one-to-one and satisfies 1) - 4).

Every $f \in C(\sigma(A))$ is the image of some p(A), e.g.

$p(A) = f(\lambda_0)p_0(A) + \ldots + f(\lambda_{k-1})p_{k-1}(A)$ fulfills $Vp(A) = f$.

It remains to show that $||p(A)|| = ||p(z)||_\infty = \sup_i |p(\lambda_i)|$.

From

$|p(\lambda_i)| = ||p(\lambda_i)\ p_i(A)|| = ||p(A)\ p_i(A)|| \leq ||p(A)||\ ||p_i(A)|| = ||p(A)||$

we see at once $\sup_i |p(\lambda_i)| \leq ||p(A)||$.

To show the reverse inequality remark first that

$||p(A)|| = ||\sum p(\lambda_i)\ p_i(A)|| \leq \sum |p(\lambda_i)|\ ||p_i(A)|| \leq$
$\leq \sup_i |p(\lambda_i)| \circ \sum ||p_i(A)|| = k \circ \sup_i |p(\lambda_i)|$.

Therefore

$||p(A)||^2 = ||p(A)\ (p(A))^*|| \leq k \circ \sup_i |p(\lambda_i)|^2$ and more general

$||p(A)||^{2^n} \leq k \sup_i |p(\lambda_i)|^{2^n}$.

This implies $||p(A)|| \leq \sqrt[2^n]{k} \cdot \sup_i |p(\lambda_i)|$, $n = 1, 2, 3, \ldots$

In the limit for $n \to \infty$ we have therefore

$||p(A)|| \leq \sup_i |p(\lambda_i)|$.

From theorem 2 we can conclude the following facts:

1) The set $\sigma(A)$ is the spectrum of A, i.e. the set of all complex numbers μ such that $(A - \mu E)^{-1}$ does not exist.
$(A - \lambda_i E)^{-1}$ cannot exist because otherwise we would have $q_i(A) = (A - \lambda_i E)^{-1} (A - \lambda_i E)\, q_i(A) = (A - \lambda_i E)^{-1}\, \pi_0(A) = 0$, a contradiction.
If $\mu \neq \lambda_i$ then the function $f(z) = z - \mu$ has an inverse on $\sigma(A)$, namely $\frac{1}{f(z)} = \frac{1}{z - \mu}$ and therefore $(A - \mu E)^{-1}$ exists in $P(A)$.

2) A is selfadjoint if and only if $\sigma(A)$ is a subset of the real line.

Proof:

$$A = A^* \Longleftrightarrow z = \bar{z} \quad \text{on} \quad \sigma(A) \Longleftrightarrow \lambda_i = \bar{\lambda}_i .$$

3) A is unitary if and only if $\sigma(A)$ is a subset of the circle $|z| = 1$.

Proof:

$$AA^* = 1 \Longleftrightarrow z\bar{z} = 1 \quad \text{on} \quad \sigma(A) \Longleftrightarrow |\lambda_i| = 1.$$

4) Every normal operator A has a representation of the form $A = RU = UR$, R selfadjoint and U unitary.

Proof:

The function z on $\sigma(A)$ may be written in the form $z = re^{i\phi}$, from which the statement follows using 2) and 3).

We are now in the position to construct a Hilbert space isomorphic to H^k on which the operator A acts as a multiplication operator. We merely have to define an inner product on $C(\sigma(A))$. To this end define numbers μ_i by $\mu_i = (p_i(A)x_0, x_0)$.

Then $\mu_i > 0$ and $\sum_{i=0}^{k-1} \mu_i = 1$.

This follows at once from

$\mu_i = (p_i(A)x_0, x_0) = ((p_i(A))^* p_i(A)x_0, x_0) = (p_i(A)x_0, p_i(A)x_0) > 0$

and $\sum_i (p_i(A)x_0, x_0) = ((\sum_i p_i(A)x_0, x_0) = (x_0, x_0) = ||x_0||^2 = 1$.

Denote now by $L^2_\mu\ (\sigma(A))$ the Hilbert space consisting of the set $C(\sigma(A))$ with the usual algebraic operations and the inner product

$$(f,g)_\mu = \sum_{\lambda_i \in \sigma(A)} f(\lambda_i)\ \overline{g(\lambda_i)}\ \mu_i.$$

Then the following theorem holds:

Theorem 3:

The mapping T of H^k into $L^2_\mu\ (\sigma(A))$ defined by $T(p(A)x_0) = p(z)$ is a one-to-one mapping of H^k onto $L^2_\mu\ (\sigma(A))$ and has the following properties:

1) $T(x_1+x_2) = Tx_1 + Tx_2$

2) $T(\alpha x) = \alpha\ Tx$

3) $(Tx_1, Tx_2)_\mu = (x_1, x_2)$, in particular $||Tx||_\mu = ||x||$.

4) $T(Ax) = z \cdot Tx$, i.e. the operator A corresponds to the operator "multiplication with the function z on $\sigma(A)$".

Proof: 1) and 2) are trivial.

3) follows from

$(p_1(A)x_0, p_2(A)x_0) = (\sum_i p_1(\lambda_i)\ p_i(A)x_0, \sum_j p_2(\lambda_j)\ p_j(A)x_0) =$

$= \sum_i \sum_j p_1(\lambda_i)\ \overline{p_2(\lambda_j)}\ ((p_j(A))^*\ p_i(A)x_0, x_0) =$

$= \sum_i p_1(\lambda_i)\ \overline{p_2(\lambda_i)}\ \mu_i = (p_1, p_2)_\mu$.

4) is a consequence of the equation

$T(A\ p(A)x_0) = zp(z) = z \cdot T(p(A)x_0)$.

<u>Remark</u>: It is easy to obtain from theorem 3 the usual formulation in terms of matrices: Denote by S the mapping from $L^2_\mu(\sigma(A))$ into H^k defined by $S(p(z)) = (\sqrt{\mu_0}\, p(\lambda_0), \ldots, \sqrt{\mu_{k-1}}\, p(\lambda_{k-1}))$
Then $(p(z), q(z))_\mu = (Sp, Sq)$ and
$S(zp(z)) = D_\lambda \circ S(p(Z))$, where D_λ denotes the diagonal matrix $D_\lambda = (d_{ii})$, $d_{ii} = \lambda_i$.
Set now U = ST. Then U is a linear mapping of H^k into itself satisfying $(Ux_1, Ux_2) = (STx_1, STx_2) = (Tx_1, Tx_2) = (x_1, x_2)$.
This means that U is unitary.
Moreover we have $U(Ax) = ST(Ax) = S(z(Tx)) = D_\lambda STx = D_\lambda Ux$, i.e. $UA = D_\lambda U$ or $UAU^{-1} = D_\lambda$.
This is usually expressed in words as: "A can be transformed into diagonal form by a unitary transformation" or "A is unitary equivalent to a diagonal matrix".

Let us now give some examples to illustrate the theory!
Consider first the Hilbert space H^k and the operator A defined by
$A(\xi_0, \xi_1, \ldots, \xi_{k-1}) = (\xi_{k-1}, \xi_0, \ldots, \xi_{k-2})$.
It is easy to see that $A^* = A^{-1}$, i.e. A is unitary.
Let $x_0 = (1, 0, \ldots, 0)$. Then $||x_0|| = 1$ and every $x = (\xi_0, \xi_1, \ldots, \xi_{k-1})$ can be written in the form $x = (\xi_0 E + \xi_1 A + \ldots + \xi_{k-1} A^{k-1}) x_0$.
Therefore $E, A, \ldots, A^{k-1}$ are linearly independent and $A^k = E$.
This implies $\pi_0(A) = A^k - E$. The roots of the equation $z^k - 1 = 0$ are precisely the k-th roots of unity 1, , ..., q^{k-1}, where
e.g. $q = e^{\frac{2\pi i}{k}}$.

Therefore $\pi_0(A) = \prod_{i=0}^{k-1} (A - \xi^i E)$.

This implies $q_0(A) = E + A + \ldots + A^{k-1}$ and therefore
$p_0(A) = \frac{1}{k}(E + A + \ldots + A^{k-1})$.
Because of $p_i(A) = p_0(\xi^{-i} a) = \frac{1}{k}(E + \xi^{-i} A + \ldots + \xi^{-i(k-1)} A^{k-1})$
we have
$\mu_i = (p_i(A)x_0, x_0) = \frac{1}{k}$, $i = 0, 1, \ldots, k-1$.

$L^2_\mu(\sigma(A))$ is thus the Hilbert space of all complex functions on the k-th roots of unity with inner product

$$(f,g)_\mu = \frac{1}{k} \sum_{i=0}^{k-1} f(\xi^i)\,\overline{g(\xi^i)} .$$

The mapping T of H^k onto $L^2_\mu(\sigma(A))$ is given by $Tx = T(\xi_0, \xi_1, \ldots, \xi_{k-1}) =$
$= \xi_0 + \xi_1 z + \ldots + \xi_{k-1} z^{k-1}$, and $T(Ax) = T(\xi_{k-1}, \xi_0, \ldots, \xi_{k-2}) =$
$= \xi_{k-1} + \xi_0 z + \ldots + \xi_{k-2} z^{k-1} = z(\xi_0 + \xi_1 z + \ldots + \xi_{k-1} z^{k-1})$
because $z^k \equiv 1$ on $\sigma(A)$.

At the end of this talk I shall try to give you some impression of the corresponding situation in the infinite-dimensional case. As I already have stated it differs only in technical details. Again the given Hilbert space can be decomposed into the orthogonal sum of "cyclic" subspaces, so that it suffices to consider only this case. But here "cyclic" has a somewhat different meaning: Denote by $P^*(A)$ the set of all polynomials in A and A^*, which coincides with P(A) in the finite-dimensional case. Then H is called "cyclic" if there exists an $x_0 \in H$, $||x_0|| = 1$, such that $P^*(A)x_0$ is dense in H.
The spectrum $\sigma(A)$, i.e. the set of complex numbers λ such that $(A - \lambda E)^{-1}$ does not exist, is always a closed bounded subset of the complex plane.
$P^*(A)$ is isometrically isomorphic to the set of all polynomials in z and $\bar{z}$ on $\sigma(A)$. This set is dense in the space $C(\sigma(A))$ of all continuous functions on $\sigma(A)$. On $\sigma(A)$ there exists a probability measure μ such that H is isomorphic to $L^2_\mu(\sigma(A))$, the set of all square-μ-integrable functions on $\sigma(A)$.
$C(\sigma(A))$ is a dense subspace of $L^2_\mu(\sigma(A))$, where the inner product is defined by $(f,g)_\mu = \int_{\sigma(A)} f(z)\,\overline{g(z)}\,d\mu(z)$.

The operator A corresponds to the operator "multiplication with z".

To make things as clear as possible without going into much detail let us consider the following example which can be considered as the infinite-dimensional analogue of the one just given.
Consider the Hilbert space $L^2(Z)$ of all twosided infinite sequences $x = (\xi_n) = (\ldots \xi_{-2}, \xi_{-1}, \xi_0, \xi_1 \ldots)$ of complex numbers ξ_n with $\sum_{-\infty}^{\infty} |\xi_n|^2 < \infty$ and the inner product $(x,y) = \sum_{-\infty}^{\infty} \xi_n \overline{\eta_n}$.
Let A be the operator defined by $A(\xi_n) = (\xi_{n-1})$. Then A is unitary and $\sigma(A)$ consists of the set of all complex numbers z with $|z| = 1$.
Let $x_0 = (\ldots 0, 0, 1, 0, 0, \ldots)$.
Then $P^*(A)x_0$ consists of the set of all sequences $x = (\xi_n)$ such that only finitely many ξ_i are different from zero.
$P^*(A)$ is isometrically isomorphic with the set of all functions of the form $\sum_{-\infty}^{\infty} c_n z^n$ on $|z| = 1$, where $c_n \neq 0$ only for finitely many n. Writing each z in the form $z = e^{2\pi it}$ this is just the set of all trigonometric polynomials $\sum_{-\infty}^{\infty} c_n e^{2\pi int}$ on $[0,1]$.
We define an inner product on this set by $(f(t), g(t)) = \int_0^1 f(t)\, \overline{g(t)}\, dt$.
It is easy to see that for $f(t) = \sum c_n e^{2\pi int}$, $g(t) = \sum d_n e^{2\pi int}$ we have $(f(t), g(t)) = \sum c_n \overline{d_n}$.

We have therefore a one-to-one correspondence between the set $P^*(A)x_0$ with the inner product of $L^2(Z)$ and the set of all trigonometric polynomials with the inner product $(f,g) = \int_0^1 f(t)\, \overline{g(t)}\, dt$, which is a dense subset of the Hilbert space $L^2([0,1])$ of all square-integrable functions on $[0,1]$.
This correspondence can be uniquely extended to a one-to-one mapping of $L^2(Z)$ onto $L^2([0,1])$ having all properties indicated in theorem 3.
In particular the operator A corresponds to multiplication with the function $e^{2\pi it}$.
The exact analysis of these things forms the subject matter of the theory of Fourier series.

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