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THE SET OF QUANTIFIERS OF AN ATOMIC BOOLEAN ALGEBRA

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The set of quantifiers of an atomic Boolean algebra[1]

by Daniel B. Demaree[2]

A <u>quantifier</u> on a Boolean algebra, $\mathfrak{A}$, is a mapping, E, from A into A such that (i) $E0 = 0$ (ii) $x \leq Ex$ (iii) $E(x \cdot Ey) = Ex \cdot Ey$. In the paper [1] of Baayen, a partial ordering of quantifiers on a given Boolean algebra, $\mathfrak{A}$, is defined by setting $E \leq E'$ iff for every $a \in A$, $Ea \leq E'a$. In that paper, Baayen asks the question: Does the set of all quantifiers on a Boolean algebra always form a lattice? In the case where $\mathfrak{A}$ is complete and atomic, the ordering in question is isomorphic to the ordering of all partitions on the atoms, and hence the answer is 'yes'.
In this report we give a characterization of the set of quantifiers on an atomic Boolean algebra, from which we contruct a counterexample to the question of Baayen. Indeed, quantifiers E, E' on a certain atomic Boolean algebra are found which have neither a least upper bound nor a greatest lower bound.

<u>Theorem</u> 1. Let $\mathfrak{A}$ be an atomic Boolean algebra, and let $\leq$ be the partial ordering of the set, Q, of all quantifiers on $\mathfrak{A}$. Let R be the set of all partitions, P, of At($\mathfrak{A}$) satisfying the condition

(*) $\Sigma\cup\{Pa : a \in At(\mathfrak{A}) \text{ and } a \leq x\}$ exists in $\mathfrak{A}$, for each $x \in A$

where Pa denotes the element of P containing a. Then $(Q, \leq)$ is isomorphic to $(R, \leq)$.

<u>Proof</u>: Let $B = At(\mathfrak{A})$. For each $E \in Q$ let $\bar{E}$ be the associated partition of B, such that $\bar{E}a = \{b \in B : Eb = Ea\}$. We claim $\bar{E}$ satisfies condition (*), in fact $\Sigma\cup\{\bar{E}a : a \in B \text{ and } a \leq x\} = Ex$ for every $x \in A$. For suppose $a \leq x$, $a \in B$, and $b \in \bar{E}a$. Using well-known properties of quantifiers we have $b \leq Eb = Ea \leq Ex$, which establishes Ex as an upper bound for $\cup\{\bar{E}a : a \in B \text{ and } a \leq x\}$. To see that Ex is the least upper bound, suppose $b \leq y$ whenever $b \in \bar{E}a$, $a \in B$, and $a \leq x$.

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By the fact that $\mathfrak{A}$ is atomic and E is completely additive (a property of quantifiers) we have

$x = \Sigma\{a: a \in B \text{ and } a \leq x\}$, and $Ex = \Sigma\{Ea: a \in B \text{ and } a \leq x\}$.
It is also not difficult to see that if a,b are atoms, then $b \leq Ea$ implies $Eb = Ea$. Thus if $b \in B$, $b \leq Ex$, then $b \leq Ea$ for some $a \in B$ with $a \leq x$, whence $b \in \bar{E}a$ for some $a \in B$ with $a \leq x$, and therefore $b \leq y$. This implies $Ex \leq y$ and consequently

$Ex = \Sigma\cup\{\bar{E}a: a \in B \text{ and } a \leq x\}$. Thus $\bar{E}$ satisfies $(^{*})$. We have thus seen that every element $E \in Q$ determines an element $\bar{E} \in R$. We claim that the function $F = \langle(E, \bar{E}) : E \in Q\rangle$ is the desired isomorphism.

It is not difficult to see that F is biunique. To see that the $RgF = R$, suppose $P \in R$. Defining $Ex = \Sigma\cup\{Pa: a \in B \text{ and } a \leq x\}$ for every $x \in A$, it is obvious that $E0 = 0$ and $x \leq Ex$. To see that $E(x \,.\, Ey) = Ex \,.\, Ey$, suppose $b \in B$ and $b \leq E(x \,.\, Ey)$. Then $b \in Pa$ for some $a \leq x \,.\, Ey$ and for some $c \leq y$, $Pa = Pc$. Hence $b \leq Ex \,.\, Ey$. Conversely, suppose $b \leq Ex \,.\, Ey$. Then $b \in Pa$ for some $a \leq x$ and $b \in Pc$ for some $c \leq y$. Thus $Pa = Pc$, $a \in Pc$, $a \leq Ey$, $a \leq x \,.\, Ey$, and $b \leq E(x \,.\, Ey)$. Thus for any $b \in B$, we have $b \leq E(x \,.\, Ey)$ iff $b \leq Ex \,.\, Ey$, and $\mathfrak{A}$ being atomic, this implies $E(x \,.\, Ey) = Ex \,.\, Ey$. Consequently, $E \in Q$. We claim that $\bar{E} = P$. Indeed, suppose $a \in B$. Then $\bar{E}a = \{b \in B: b \leq Ea\}$ and $Ea = \Sigma Pa$, and hence $\bar{E}a = Pa$.

It remains to show that the correspondence, F, preserves $\leq$. Suppose $a \in B$. Then $Ea \leq E'a$ iff $\bar{E}a \subseteq \bar{E}'a$. Thus $E \leq E'$ implies $\bar{E} \leq \bar{E}'$. Conversely $\bar{E} \leq \bar{E}'$ implies $Ea \leq E'a$ for $a \in B$, so by complete additivity of E, E', we have $E \leq E'$.

<u>Corollary 2</u>. There exists an atomic Boolean algebra, $\mathfrak{A}$, such that the set of all quantifiers on $\mathfrak{A}$ is not a lattice (under $\leq$). In fact, there exist a pair of elements which have no l.u.b. and no g.l.b.

<u>Proof</u>. Let ω denote the natural numbers, E the even numbers, and D the odd numbers. Let $\mathfrak{A}$ be the Boolean algebra of all subsets X of ω having the property that (i) either $E \cap X$ or $E \sim X$ is finite, and (ii) either $D \cap X$ or $D \sim X$ is finite. Clearly $\mathfrak{A}$ is an atomic Boolean algebra with atoms $\{k\}$ for $k \in \omega$. As a notational simplification we will use k to denote the atom $\{k\}$.

From Theorem 1 it suffices to define partitions P, T on the set of atoms of $\mathfrak{A}$, such that P, T satisfy $(^*)$, but such that there is no least element W satisfying $(^*)$ with $P \leq W$ and $T \leq W$, and also no greatest element V satisfying $(^*)$, such that $V \leq P$ and $V \leq T$.
We define P and T as follows:

$$P = 1,0,2 \mid 5,6,10 \mid 9,14,18 \mid 13,22,26 \mid \ldots \mid 3,4 \mid 7,8 \mid 11,12 \mid \ldots$$

$$T = 0 \mid 1,2,6 \mid 5,10,14 \mid 9,18,22 \mid 13,26,30 \mid \ldots 3,4 \mid 7,8 \mid 11,12 \mid \ldots$$

We note that P and T satisfy $(^*)$. To see that P and T have no l.u.b. suppose W is an upper bound for P and T, satisfying $(^*)$. Such upper bounds exist since the one-element partition satisfies $(^*)$.
Then $P \leq W$ and $T \leq W$, which implies

$$W0 \supseteq \{0,1,2,5,6,9,10,13,14,\ldots\}$$

Since W satisfies $(^*)$, $\Sigma W0$ must exist in $\mathfrak{A}$. Now W0 contains infinitely many even as well as odd numbers, hence $\omega \sim W0$ is finite. Thus there must exist an integer, k, such that $\{4k-1, 4k\} \subseteq W0$.
Let W' be defined by

$$W'0 = W0 \sim \{4k-1, 4k\}$$

$$W'4k = W4k-1 = \{4k-1, 4k\}$$

$$W'm = Wm \text{ for } m \in \omega \sim W0$$

Then W' satisfies $(^*)$ since W does, and $P \leq W'$, $T \leq W'$, but $W' < W$.
Thus a l.u.b. for P and T does not exist.

Finally, suppose V is a lower bound for P, T, satisfying $(^*)$. Such lower bounds exist, since the identity partition satisfies $(^*)$. Thus we have $Vk \subseteq Pk \cap Tk$ for every $k \in \omega$, and hence for every $k \in \omega$

$V0 = \{0\}$	$V8k \subseteq \{8k-1, 8k\}$
$V1, V2 \subseteq \{1,2\}$	$V8k+1 \subseteq \{8k+1, 16k+2\}$
$V3, V4 \subseteq \{3,4\}$	$V8k+2 \subseteq \{4k+1, 8k+2\}$
$V5 \subseteq \{5,10\}$	$V8k+3 \subseteq \{8k+3, 8k+4\}$
$V6 \subseteq \{6\}$	$V8k+4 \subseteq \{8k+3, 8k+4\}$
$V7 \subseteq \{7,8\}$	$V8k+5 \subseteq \{8k+5, 16k+10\}$
...	$V8k+6 \subseteq \{8k+6\}$
	$V8k+7 \subseteq \{8k+7, 8k+8\}$

We claim that if V satisfies (*), there can be at most a finite number of atoms k such that $|Vk| = 2$. For suppose $|Vk| = 2$ for infinitely many k. Then $\bigcup_{k \leq D} Vk$ contains infinitely many even atoms, and excludes all atoms of the form 8k+6. of which there are infinitely many. Thus $\Sigma\cup\{Va: a \leq D\}$ does not exist, and V does not satisfy (*).

So $|Vk| = 1$ for all but a finite number of atoms k. Hence there exist integers 4k-1, 4k such that $V4k-1 = \{4k-1\}$ and $V4k = \{4k\}$. Let V' bedefined by

$$V'4k-1 = V'4k = \{4k-1, 4k\}$$

$$V'm = Vm \text{ for } m \neq 4k-1, 4k$$

Then $V < V'$ and $V' \leq P, T$. Thus P and T have no g.l.b.

Reference

[1] P.C. Baayen, Partial ordering of quantifiers and of clopen equivalence relations. Math. Centrum Amsterdam Afd. Zuivere Wiskunde. ZW 1962--025 (1962), 15 pp.