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TRANSCENDENCE PROPERTIES OF THE CARLITZ-BESSELFUNCTIONS

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Transcendence-properties of the Carlitz-Besselfunctions.

1. Introduction.

In 1935 L. Carlitz [1] introduced the function

$$\psi(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^r}}{F_r},$$

where

$$F_r = \prod_{j=0}^{r-1} (x^{q^r} - x^{q^j}), \quad r = 1,2,\ldots\ldots ; \qquad (1.1)$$

$$F_0 = 1.$$

It furnishes an explicit example of an entire function in an algebraically closed field with non-archimedian valuation [7]. Let $\mathbb{F}_q$ denote the field of q elements where $q = p^n$ for some prime-number p and natural number n.

We can give $E \in \mathbb{F}_q[x]$ the non-archimedian valuation

$$|E| = q^{\text{dg}\, E},$$

where dg E denotes the degree of E and dg 0 = $-\infty$.

The quotientfield will be denoted by $\mathbb{F}_q\{x\}$, the completion with respect to $|\,|$ by $\mathbb{F}_q((x^{-1}))$, and the algebraic closure of $\mathbb{F}_q((x^{-1}))$ by Φ. The valuation $|\,|$ can be extended to Φ in a unique way (see [9], §78).

An element $\alpha \in \Phi$ is a root of a polynomial with coefficients in $\mathbb{F}_q[x]$ and α is said to be an algebraic element. In [5] L.I. Wade proved that for algebraic $\alpha \neq 0$ $\psi(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$.

The function $\psi(t)$ can also be written as the product

$$\psi(t) = t \prod_{E} \left(1 - \frac{t}{E\xi}\right),$$

where E runs through all non-zero elements of $\mathbb{F}_q[x]$ and ξ is given by

$$\xi = \lim_{k\to\infty} \frac{(x^q - x)^{\frac{q^k}{q-1}}}{\prod_{j=1}^{k} (x^{q^j} - x)}$$

Let $\lambda(t)$ be the inverse function of ψ, hence

$$\psi(\lambda(t)) = \lambda(\psi(t)) = t \ ;$$

$\lambda(t)$ is determined mod ξ.

In [5] and [6] L.I. Wade proved the transcendence of ξ and in [8] he proved an analogue of the theorem of Gelfond-Schneider:

- If $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q\{x\}$, then at least one of the three quantities α, β, $\psi(\beta\lambda(\alpha))$ is transcendental. If $\alpha = 0$ and $\lambda(0) = E\xi \neq 0$ then the statement still holds -

In 1960 Carlitz [2] introduced the function

$$J_n(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^{n+r}}}{F_{n+r} F_r^{q^n}} .$$

For all linear functions f, i.e. functions with the properties

$$\begin{cases} f(t+u) = f(t) + f(u) \\ f(ct) \ = cf(t) \qquad \text{for } c \in \mathbb{F}_q, \end{cases}$$

the Δ-operator is defined by

$$\Delta f(t) = f(xt) - xf(t).$$

In this report we shall prove the following

Theorem: let $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q\{x\}$ and n be an arbitrary integer then at least one of the elements of the set

$$\{\alpha,\ \beta,\ J_n(\alpha),\ \Delta J_n(\alpha),\ J_n(\alpha\beta),\ \Delta J_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$.

2. We shall use several propositions of the previous papers and recall them here without proofs.

<u>Definition 2.1</u> The function $f(t)$ is called <u>entire</u> if $f(t)$ converges for all $t \in \Phi$.

<u>Definition 2.2</u> An element $\alpha \in \Phi$ is called an <u>algebraic integer</u> if α is a root of a monic polynomial over $\mathbb{F}_q[x]$.

<u>Definition 2.3</u> Let f be a linear function then we define the operators $\Delta^r (r = 1, 2, \ldots)$ by

$$\Delta f(t) = f(xt) - xf(t)$$

$$\Delta^r f(t) = \Delta^{r-1} f(xt) - x^{q^{r-1}} \Delta^{r-1} f(t), \ (r \geq 2).$$

We shall sometimes denote $f(t)$ by $\Delta^0 f(t)$.

<u>Definition 2.4</u> (see [1]).

The linear polynomial $\psi_k(t)$, $(k = 0, 1, 2, \ldots)$ is defined by

$$\psi_k(t) = \prod_{dg\ E<k} (t-E) = \sum_{j=0}^{k} (-1)^{k-j} \frac{F_k}{F_j\, L_{k-j}^{q^j}} t^{q^j},$$

where F_j is defined by (1.1),

$$L_k = \prod_{j=1}^{k} (x^{q^j} - x), \ (k = 1, 2, \ldots),$$

$L_0 = 1$ and

E runs through all polynomials (including 0) of degree $< k$.

<u>Lemma 2.1</u> (expansionformula)

Let f be an entire linear function over Φ then for $M \in \mathbb{F}_q[x]$ with degree $\leq m$ we have

$$f(Mt) = \sum_{k=0}^{m} \frac{\psi_k(M)}{F_k} \Delta^k f(t).$$

Proof: see [1], §4.

Definition 2.5 Let f be the power-series defined by

$$f(t) = a_h t^h + a_{h+1} t^{h+1} + \dots \tag{2.1}$$

$(a_i \in \Phi,\ h \in \mathbb{N},\ a_h \neq 0)$,

then define

$$r_1 = \min_{i>h} \left|\frac{a_h}{a_i}\right|^{\frac{1}{i-h}} \quad \text{if this minimum exists}$$

and

$$i_1 = \max \{i \mid \left|\frac{a_h}{a_i}\right|^{\frac{1}{i-h}} = r_1\} \text{ if this maximum exists.}$$

Furthermore inductively

$$r_k = \min_{i>i_{k-1}} \left|\frac{a_{i_{k-1}}}{a_i}\right|^{\frac{1}{i-i_{k-1}}} \quad \text{if this minimum exists}$$

and

$$i_k = \max \{i \mid \left|\frac{a_{i_{k-1}}}{a_i}\right|^{\frac{1}{i-i_{k-1}}} = r_k\} \text{ if this maximum exists.}$$

We now have the following

Lemma 2.2 The power series f(t) of (2.1) has i_1-h zeros t in Φ with $|t| = r_1$, and $i_k - i_{k-1}$ zeros t in Φ with $|t| = r_k$ $(k \geq 2)$ and 0 is a zero of multiplicity h. These are the only zeros of f(t).
Proof: see [7], theorem 1 and [4], II §3.

Theorem 2.2 (maximum - modulus theorem)
Let f be defined by (2.1) and let f(t) be convergent for all t with $|t| < R$, then for all r, $0 < r < R$

$$\max_{\substack{a\in\Phi \\ |a|\leq r}} |f(a)| \text{ exists and}$$

$$\max_{\substack{a\in\Phi \\ |a|\leq r}} |f(a)| = \max_{n\geq h} |a_n|\, r^n.$$

Proof: see [4], II §2.

Lemma 2.3 If the function f of (2.1) is an entire function then f is either a polynomial or there exists an infinite sequence of different zeros $b_i (i = 1, 2, \dots)$, $b_i \neq 0$ such that f can be written in the form

$$f(t) = a_h\, t^h \prod_{i=1}^{\infty} \left(1 - \frac{t}{b_i}\right)^{j_i},$$

where j_i denotes the multiplicity of the zero b_i.

Proof: See [4], III (22).

Lemma 2.4 An entire function of the form (2.1) is either a polynomial or a transcendental function.

Proof: see [7], theorem 5.

Corollary 2.5 An entire transcendental function is not identically zero.

Proof: f can be written as

$$f(t) = a_h\, t^h \prod_{i=1}^{\infty} \left(1 - \frac{t}{b_i}\right)^{j_i}.$$

Let $r > 0$ be such that $|b_i| > r$ for all i, then for all t with $|t| = r$ we have

$$|f(t)| = |a_h|\, r^h > 0.$$

Hence $f \not\equiv 0$.

3. Properties of $J_n(t)$

The definition of the function

$$J_n(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^{n+r}}}{F_{n+r}\, F_r^{\,q^n}},$$

which is initially defined for all non-negative rational entires n, $t \in \Phi$, can be extended to all $n \in \mathbf{Z}$ if we define

$$\frac{1}{F_{-n}} = 0 \qquad (n = 1, 2, \dots).$$

It follows immediately that

$$J_{-n}(t) = (-1)^n \{J_n(t)\}^{q^{-n}} \tag{3.1}$$

Furthermore we have

$$\Delta^r J_n(t) = J_{n-r}^{q^r}(t) \tag{3.2}$$

for all $r \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Hence the expansionformula for $J_n(t)$ $(n \in \mathbb{Z})$ becomes

$$J_n(Mt) = \sum_{r=0}^{m} \frac{1}{F_r} \psi_r(M) J_{n-r}^{q^r}(t) \tag{3.3}$$

From $\quad J_n(xt) - x J_n(t) = J_{n-1}^{q}(t)$

and

$$J_n(xt) - x^{q^n} J_n(t) = - J_{n+1}(t)$$

for all $n \in \mathbb{Z}$ we get the recurrence formula

$$J_{n+1}(t) - (x^{q^n} - x) J_n(t) + J_{n-1}^{q}(t) = 0. \tag{3.4}$$

We also have

$$J_n(x^2t) - (x^{q^n} + x) J_n(xt) + x^{q^n+1} J_n(t) = - J_n^q(t) \tag{3.5}$$

<u>Lemma 3.1</u> $J_{2n}(t)$ and $J_{2n+1}(t)$ are linear polynomials in $J_0(t)$ and $\Delta J_0(t)$ of degree q^n with coefficients in $\mathbb{F}_q[x]$ of degree $< q^{2n}$ resp. $< q^{2n+1}$.

Proof: this is an immediate consequence of the recurrence-formula (3.4) for $J_n(t)$.

<u>Remark</u> From the linearity of $J_0(t)$ and the fact that $J_0'(t) \equiv 1$ we get: $J_0(t)$ has only singular zeros and if t_0 is a zero of $J_0(t)$ then so is ct_0 with $c \in \mathbb{F}_q$.

From (3.1) we see that we can write

$$J_n(t) = \{G_n(t)\}^{q^n}, \ (n \geq 0),$$

where $\quad G_n(t)$ is a linear function with

$G_n'(t) \equiv$ non-zero constant.

Hence all zeros of $G_n(t)$ are single and therefore all zeros of $J_n(t)$, $(n \geq 0)$ have multiplicity q^n.

Let us denote $\mathrm{dg}\ \alpha = {}^q\log |\alpha|$ for all $\alpha \in \Phi$.

As a consequence of lemma 2.2 we have

Lemma 3.2 $J_n(t)$, $(n \geq 0)$ has a zero of order q^n in 0 and $q^{n+k} - q^{n+k-1}$ zeros of degree $n + 2(k-1) + \frac{2q}{q-1}$, $(k = 0, 1, 2, \ldots)$.

Remark From lemma 3.2 we can deduce that if t_0 is a zero of $J_n(t)$ for some $n > 0$ then t_0 is neither a zero of $J_{n-1}(t)$ nor a zero of $J_{n+1}(t)$.

4. Transcendence properties of $J_0(t)$

In his book "Einführung in die Transzendenten Zahlen", Schneider discussed some transcendence properties of the Besselfunctions. Here we can use the method of the proof of the analogue of the Gelfond-Schneider theorem to prove the transcendence of at least one of the elements $\{\alpha, \beta, J_0(\alpha), \Delta J_0(\alpha), J_0(\alpha\beta), \Delta J_0(\alpha\beta) \,|\, \alpha \neq 0, \beta \notin \mathbb{F}_q\{x\}\}$.

Definition 4.1 Let $\alpha \in \Phi$ be algebraic over $\mathbb{F}_q\{x\}$ of degree s. Then by $\alpha = \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(s)}$ we denote the conjugate elements of α. Let $K(\alpha)$ denote the extension of $\mathbb{F}_q((x^{-1}))$ in which we have the extended valuation $|\,|$, where

$$\mathrm{dg}\ \alpha = {}^q\log|\alpha|.$$

Define

$$d^*(\alpha) = \max_{j=1,\ldots,s} \mathrm{dg}(\alpha^{(j)}).$$

Lemma 4.1 Let $m, n \in \mathbb{N}$ with $0 < m < n$; the system of linear equations

$$\sum_{i=1}^{n} A_{ki}\, X_i = 0\,, \quad (k = 1, \ldots, m) \tag{4.1}$$

where $A_{ki} \in \mathbb{F}_q[x]$ and $\max\limits_{k,i} dg(A_{ki}) \leq a$ $(a \in \mathbb{N})$, has a non-trivial solution $X_1,\ldots,X_n$ with

$$X_i \in F_q[x],$$

such that

$$dg\ X_i < [\frac{ma}{n-m} + 1]\ , \qquad (i = 1, \ldots, n).$$

Proof: Define

$$Y_k\ := \sum_{i=1}^{n} A_{ki}\ X_i, \qquad (k = 1, \ldots, m)$$

then for $X_i \in \mathbb{F}_q[x]$, Y_k is a polynomial. Let U be an arbitrary natural number. The cube $\{(X_i)_{i=1}^n \mid dg\ X_i < U\}$ contains q^{Un} grating points. If $dg\ X_i < U$ $(i = 1, \ldots, n)$ then

$$dg\ Y_k \leq \max_i\ (dg\ A_{ki} + dg\ X_i) < a + U,\ (k = 1, \ldots, m).$$

Every grating point $(X_i)_i$ corresponds with a grating-point of the cube $\{(Y_k)_{k=1}^m \mid dg\ Y_k < a + U\}$ which contains $(q^{a+U})^m$ points.

If we choose

$$U = [\frac{ma}{n-m} + 1]$$

then at least two different points $(X_i^{(1)})_i$ and $(X_i^{(2)})_i$ induce the same point $(Y_k)_k$.

Hence $(X_i^{(1)} - X_i^{(2)})_i$ is a solution of (4.1) and

$$dg\ (X_i^{(1)} - X_i^{(2)}) \leq \max\ (dg\ X_i^{(1)},\ dg\ X_i^{(2)}) < [\frac{ma}{n-m} + 1]\ ,$$

$$(i = 1, \ldots, n).$$

<u>Lemma 4.2</u> Let K be a separable extension of $\mathbb{F}_q\{x\}$ of degree σ. Let $r, s \in \mathbb{N}$ with $0 < r < s$. Then the system of linear equations

$$\sum_{i=1}^{s} \alpha_{ki}\ \xi_i = 0, \qquad (k = 1, \ldots, r) \qquad (4.2)$$

where α_{ki} are algebraic integers in K and $a = \max\limits_{k,i} d^*(\alpha_{ki})$ has a non-trivial solution $(\xi_i)_{i=1}^{r}$ with

$$\xi_i \in \mathbb{F}_q[x],$$

such that

$$d^*(\xi_i) < \frac{cs + ra}{s - r}, \qquad (i = 1, \ldots, s)$$

where c is a positive constant only depending on the field K.
Proof: Let $\beta_1, \ldots, \beta_\sigma$ be a base of algebraic integers for K over $\mathbb{F}_q\{x\}$, then

$$\xi_i = \sum_{j=1}^{\sigma} X_{ij}\, \beta_j, \qquad (i = 1, \ldots, s) \tag{4.3}$$

where $X_{ij} \in \mathbb{F}_q[x]$.
Substituting (4.3) in (4.2) we get

$$\sum_{i=1}^{s} \alpha_{ki}\, \xi_i = \sum_{i=1}^{s} \sum_{j=1}^{\sigma} \alpha_{ki}\, \beta_j\, X_{ij} = 0, \quad (k = 1, \ldots, r). \tag{4.4}$$

Here $\alpha_{ki}\, \beta_j$ are algebraic integers, hence

$$\alpha_{ki}\, \beta_j = \sum_{l=1}^{\sigma} M_{kijl}\, \beta_l, \quad (k=1,\ldots,r;\ i=1,\ldots,s;\ j=1,\ldots,\sigma) \tag{4.5}$$

with $M_{kijl} \in \mathbb{F}_q[x]$.
Substituting (4.5) in (4.4) we get

$$\sum_{i=1}^{s} \sum_{j=1}^{\sigma} \sum_{l=1}^{\sigma} M_{kijl}\, \beta_l\, X_{ij} = 0, \quad (k = 1,\ldots, r). \tag{4.6}$$

The $(\beta_l)_{l=1}^{\sigma}$ form a base over $\mathbb{F}_q\{x\}$ and therefore (4.6) becomes

$$\sum_{i=1}^{s} \sum_{j=1}^{\sigma} M_{kijl}\, X_{ij} = 0, \ (k=1,\ldots, r;\ l=1,\ldots, \sigma). \tag{4.7}$$

This is a system of $r\sigma$ linear equations in $s\sigma$ variables with polynomial coefficiënts. Considering the conjugated forms of (4.5):

$$(\alpha_{ki}\, \beta_j)^{(\nu)} = \sum_{j=1}^{\sigma} M_{kijl}\, \beta_l^{(\nu)}, \quad (\nu = 1,\ldots, \sigma)$$

we can express M_{kijl} as a linear combination of $(\alpha_{ki}\ \beta_j)^{(\nu)}$ with coefficients that only depend on the field K and therefore

$$\operatorname{dg} M_{kijl} < c_1 + \max_{i,j,k} d^*(\alpha_{ki}\ \beta_j) < c_2 + a,$$

where c_1, c_2 are positive constants only depending on K. We can choose c_2 such that $c_2 + a \in \mathbb{N}$.
Now we can use lemma 4.1 and (4.7) has a solution in polynomials $(X_{ij})_{i,j}$, $(i = 1, \ldots, s;\ j = 1, \ldots, \sigma)$ such that

$$\operatorname{dg} X_{ij} < [\ \frac{r\sigma(c_2 + a)}{s\sigma - r\sigma} + 1\ ] = [\ \frac{r(c_2 + a)}{s - r} + 1\ ].$$

Hence from (4.3) we deduce that the system (4.2) has a non-trivial solution $\xi_1, \ldots, \xi_s$ such that

ξ_i is an algebraic integer and

$$d^*(\xi_i) \leq \max_{\substack{j=1,\ldots,\sigma \\ i=1,\ldots,s}} d^*(X_{ij}\ \beta_j) <$$

$$< c_3 + [\ \frac{r(c_2 + a)}{s - r} + 1] < \frac{cs + ar}{s - r}\ ,$$

where $c > 0$ only depends on K.

<u>Theorem 4.3</u> Let $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q\{x\}$, then at least one of the elements of the set

$$V = \{\alpha,\ \beta,\ J_0(\alpha),\ \Delta J_0(\alpha),\ J_0(\alpha\beta),\ \Delta J_0(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$.
Proof: Suppose all elements of the set V are algebraic over $\mathbb{F}_q\{x\}$, then they generate an algebraic extension of $\mathbb{F}_q((x^{-1}))$ of exponent e. Let K be the separable extension of $\mathbb{F}_q((x^{-1}))$ generated by the p^e-th powers of the elements of V and let $[K : \mathbb{F}_q((x^{-1}))] = s$. Also the

q^e-th powers of the elements of V are elements of K and there exists a polynomial $\Gamma \in \mathbb{F}_q[x]$ of degree c_0 such that

$$\Gamma\alpha^{q^e},\ \Gamma\beta^{q^e},\ \Gamma\{J_0(\alpha)\}^{q^e},\ \Gamma\{\Delta J_0(\alpha)\}^{q^e},\ \Gamma\{J_0(\alpha\beta)\}^{q^e},\ \Gamma\{\Delta J_0(\alpha\beta)\}^{q^e}$$

are algebraic integers of K.

Let k and l be natural numbers which will be determined later. Define the function

$$L(t) := P_1(t) + P_2(t)\, J_0^{q^e}(t\alpha) + \ldots + P_{q^{2k}}(t)\, \{J_0(t\alpha)\}^{q^e(q^{2k}-1)},$$

where

$$P_i(t) = \sum_{j=0}^{q^{2l}-1} X_{ij}\, t^{jq^e},\ (i = 1, \ldots, q^{2k}). \qquad (4.8)$$

We now proceed in several steps.

Denote $m := k + l - 1$ and $k < \frac{1}{3} l$. (4.9)

Step 1; Assertion: we can determine the coefficients X_{ij} ($0 \leq j \leq q^{2l}-1$; $1 \leq i \leq q^{2k}$) of the polynomials P_i such that

(1) all X_{ij} are algebraic integers, not all zero

(2) for all $A, B \in \mathbb{F}_q[x]$ with $dg\, A < m$, $dg\, B < m$

$$L(A + \beta B) = 0.$$

Proof: Since $\alpha \neq 0$ $J_0(t\alpha) \not\equiv 0$. Hence substituting $t = A + \beta B$ ($dg\, A < m$, $dg\, B < m$) in (4.8) we get a non-trivial system of q^{2m} equations in $q^{2(k+l)}$ variables X_{ij}:

$$L(A + \beta B) = \sum_{i=0}^{q^{2k}-1} \sum_{j=0}^{q^{2l}-1} (A + \beta B)^{jq^e}\, J_0(\alpha(A + \beta B))^{iq^e}\, X_{ij} = 0,$$

$$(dg\, A < m,\ dg\, B < m). \qquad (4.10)$$

Since $J_0(t)$ is a linear function we have

$$J_0(\alpha(A + \beta B)) = J_0(\alpha A) + J_0(\alpha\beta B).$$

Using the expansionformula (3.3) and the formulae (3.1) and (3.2) we obtain

$$J_0(\alpha A) = \sum_{\mu=0}^{m} \frac{(-1)^\mu}{F_\mu} J_\mu(\alpha)\, \psi_\mu(A).$$

Notice that $\frac{\psi_\mu(A)}{F_\mu}$ is the polynomial $AL_\mu \prod'_{dg\, E<\mu} (A^{q-1} - E^{q-1})$,

where the product is taken over all primary polynomials E and L_μ is defined in def. 2.4.

According to lemma 3.1 $J_\mu(\alpha)$ is a polynomial of degree $q^{[\frac{\mu}{2}]}$ in $J_0(\alpha)$ and $\Delta J_0(\alpha)$ with coefficients of degree $< q^\mu$ in $\mathbb{F}_q[x]$. Hence

$$dg\ J_0(\alpha A) \le mq^m + \max_\mu dg\ J_\mu(\alpha) \le$$

$$\le mq^m + q^m + 2q^{\frac{m}{2}} \max(dg\ J_0(\alpha),\ dg\ \Delta J_0(\alpha)).$$

The coefficients of X_{ij} in the linear equations (4.10) are polynomials in

β^{q^e} of degree $q^{2l}-1$

$J_0^{q^e}(\alpha)$, $(\Delta J_0(\alpha))^{q^e}$, $J_0^{q^e}(\alpha\beta)$, $(\Delta J_0(\alpha\beta))^{q^e}$ of degree $(q^{2k}-1)q^{[\frac{m}{2}]}$

with coefficients in $\mathbb{F}_q[x]$.

Since $q^{2l}-1 + 2(q^{2k}-1)q^{[\frac{m}{2}]} < q^{2l+2}$ by multiplying the equations (4.10) with

$\Gamma^{q^{2l+2}}$

we get a system of q^{2m} equations in $q^{2(k+l)}$ variables X_{ij}:

$$\Gamma^{q^{2l+2}} L(A + \beta B) = \sum_{i=0}^{q^{2k}-1} \sum_{j=0}^{q^{2l}-1} D_{ij}\, X_{ij} = 0,$$

$$(dg\ A < m,\ dg\ B < m), \quad (4.11)$$

were the D_{ij} are algebraic integers of K.

If we put

$c_0 = dg\ \Gamma$

$c_1 = dg\ \beta$

$c_2 = \max\{dg\ J_0(\alpha),\ dg\ \Delta J_0(\alpha),\ dg\ J_0(\alpha\beta),\ dg\ \Delta J_0(\alpha\beta)\}$,

$\mathrm{dg}\, D_{ij} \leq c_0\, q^{2l+2} + q^{2l+e}(m + c_1) + q^{2k+e+m}(m + c_2).$

According to (4.9) this becomes

$$\mathrm{dg}\, D_{ij} \leq q^{2l+e}(2m + c_3),$$

and

$$d^* D_{ij} \leq q^{2l+e}(2m + c_4).$$

Now we can apply lemma 4.2 with $\sigma = s$, $r = q^{2m}$, $s = q^{2k+2l}$ and $a = \max_{i,j} d^* D_{ij}$ and we can determine a set $\{X_{ij};\ 0 \leq j \leq q^{2l}-1;\ 1 \leq i \leq q^{2k}\}$ such that (1) and (2) are satisfied and furthermore

$$d^* X_{ij} < (2m + c_5) q^{2l+e} \quad \text{for } l > l_0, \qquad (4.12)$$

where $c_5 > 0$ only depends on the field K.

Let $\mu \geq m$ be a natural number and define:

$$\eta = \mu - k + 1, \qquad (4.13)$$

hence $\eta > 1$; furthermore define

$$\mathcal{B}(\mu) := \{A + \beta B \mid \mathrm{dg}\, A < \mu,\ \mathrm{dg}\, B < \mu;\ A \text{ and } B \text{ not both } 0\}.$$

Step 2; Assertion: if $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$, then $L(t) = 0$ for all $t \in \mathcal{B}(\mu + 1)$.

Proof: Suppose $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$ and take

$$\xi \in \mathcal{B}(\mu + 1) \setminus \mathcal{B}(\mu)$$

then $d\xi = \mu + c_1$.

Let $l > l_1 \geq l_0$ be chosen such that $m > c_1$, then

$$d\xi < 2\mu.$$

From (4.8) and (4.10) we get

$$\max_{\mathrm{dg}t=2\mu} \mathrm{dg}\, L(t) \leq d^* X_{ij} + q^{2l+e} 2\mu + q^{2k+e} \max_{\mathrm{dg}t=2\mu} \mathrm{dg}\, J_0(\alpha t).$$

From the explicit formula for $J_0(\alpha t)$ and theorem 2.2 we obtain

$$\max_{dgt=2\mu} dg\, J_0(\alpha t) \leq \max_{r\geq 0} \{q^r(2\mu + dg\alpha - 2r)\} \leq c_6\, q^{\mu}.$$

Substituting this and using (4.12) and (4.13) we get

$$\max_{dgt=2\mu} dg\, L(t) \leq q^{2\eta+e} (4\mu + c_7 q^{4k}) \qquad (4.14)$$

where $c_7 > 0$ only depends on K.
Since L(t) is an entire function with zeros for all $t \in \mathcal{B}(\mu)$ the function

$$\frac{L(t)}{\prod_{\mathcal{B}(\mu)} (t - A - \beta B)}$$ is an entire function, hence we can

apply theorem 2.2 and therefore we have

$$dg \left(\frac{L(\xi)}{\prod_{\mathcal{B}(\mu)} (\xi - A - \beta B)}\right) \leq \max_{dgt=2\mu} dg \left(\frac{L(t)}{\prod_{\mathcal{B}(\mu)} (t - A - \beta B)}\right) .$$

Using (4.14) and substituting $d\xi = \mu + c_1$ we obtain

$$dg\, L(\xi) - q^{2\mu}(\mu + c_1) \leq q^{2\eta+e}(4\mu + c_7\, q^{4k}) - 2\mu q^{2\mu},$$

hence

$$dg\, L(\xi) \leq q^{2\eta+e} [4\mu + c_7 q^{4k} + (c_1 - \mu)q^{2k-2}]. \qquad (4.15)$$

Since we have chosen $\xi \in \mathcal{B}(\mu + 1)$ and since the X_{ij} are polynomials $L(\xi)$ is a polynomial in β^{q^e} of degree $q^{2l}-1$ and in $J_0(\alpha)^{q^e}$, $(\Delta J_0(\alpha))^{q^e}$, $J_0(\alpha\beta)^{q^e}$, $(\Delta J_0(\alpha\beta))^{q^e}$ of degree $(q^{2k}-1)\, q^{[\mu/2]}$.

Since $q^{2l}-1 + 2(q^{2k}-1)q^{[\mu/2]} < q^{2\eta} + 2q^{2k+(\eta+k-1)}$

$$< 3q^{2\eta},$$

$\Gamma^{2\eta}$ $L(\xi)$ is an algebraic integer and hence if N denotes the norm of an element of K over $\mathbb{F}_q\{x\}$ we have:

$N(\Gamma^{2\eta}L(\xi)$ is a polynomial and therefore

$$\mathrm{dg}\ N(\Gamma^{2\eta}L(\xi)) \text{ is either } \geq 0 \text{ or } -\infty.$$

From (4.15) we have

$$\mathrm{dg}\ (N(\Gamma^{2\eta}L(\xi))) \leq s\ [q^{2\eta}c_0 + q^{2\eta+e}\{4\mu+c_7q^{4k}+(c_1-\mu)q^{2k-2}\}]$$

$$< sq^{2\eta+e}\{\mu(4-q^{2k-2}) + c_8q^{4k}\},$$

where $c_8 > 0$ if $k > k_0$.

Now choose $k > k_1 > k_0$ such that $4 - q^{2k-2} > 0$,
and afterwards $l > l_1 > l_0$ such that

$$\mu(4 - q^{2k-2}) + c_8q^{4k} < 0.$$

Then combining both inequalities for $\mathrm{dg}(N(\Gamma^{2\eta}L(\xi)))$ we get $L(\xi) = 0$.
This concludes the proof of step 2.

Step 3: Denote for arbitrary ν by π_ν the product $\prod_{\mathcal{B}(\nu)} (A + \beta B)$ then

$$\mathrm{dg}\ \pi_\nu < (\nu + c_1)q^{2\nu}.$$

From step 2 we conclude that for all A, B $\in \mathbb{F}_q[x]$ the $A + \beta B$ are zeros of $L(t)$. Since $\beta \notin \mathbb{F}_q\{x\}$ these zeros are all different and therefore the entire functions $L(t)$ has an infinite number of zeros. According to lemma 2.4 $L(t)$ is a transcendental function and hence (corr. 2.5)

$$L(t) \not\equiv 0.$$

Furthermore from lemma 2.3 we have:

$$L(t) = \alpha_h\ t^h \prod_{i=1}^{\infty} (1 - \frac{t}{b_i})^{j_i}$$

where the b_i are the zeros of $L(t)$ and j_i is the multiplicity of b_i.
From step 2 we conclude that every element of $\mathcal{B}(\nu)$ is a zero of $L(t)$.

and we have

$$L(t) = \alpha_h t^h \prod_{b_i \in \mathcal{B}(\nu)} (1 - \frac{t}{b_i})^{j_i} \prod_{b_i \notin \mathcal{B}(\nu)} (1 - \frac{t}{b_i})^{j_i}$$

and therefore

$$\max_{dgt=2\nu} dg\, L(t) \geq c_9 + 2h\nu + dg\,(\frac{t^{q^{2\nu}}-1}{\pi_\nu})_{dgt=2\nu}$$

$$\geq c_9 + 2\nu h + 2\nu(q^{2\nu}-1) - (\nu + c_1)q^{2\nu}$$

$$\geq q^{2\nu}(c_{10}\nu + c_{11}),$$

where $c_{10} > 0$.

On the other hand from the explicit formula of $L(t)$ we have

$$\max_{dgt=2\nu} dg\, L(t) < (2m + c_5)q^{2l+e} + 2\nu q^{2l+e} + c_6 q^{\nu+2k+e} .$$

Now let $l > l_1$ and $k > k_1$ be fixed.

Then for all ν we have

$$q^{2\nu}(c_{10}\nu + c_{11}) < 2\nu q^{\nu} C \tag{4.16}$$

where $c_{10} > 0$ and c_{11} are constants only depending on K and $C > 0$ is a fixed constant only depending on k and l. We can choose $\nu > \nu_0$ such that (4.16) is a contradiction. Hence our assumption that all the elements of V are algebraic is false, which proves the theorem.

In the proof of theorem 4.3 we use the formula

$$J_0(\alpha A) = \sum_{\mu=0}^{m} \frac{(-1)^\mu}{F_\mu} \psi_\mu(A)\, J_\mu(\alpha),$$

in which we can write $J_\mu(\alpha)$ as a linear polynomial in $J_0(\alpha)$ and $\Delta J_0(\alpha)$ with polynomials in x as coefficients. In the same way the expansionformula (3.3) gives

$$J_n(\alpha A) = \sum_{\mu=0}^{m} \frac{1}{F_\mu} \psi_\mu(A)\, J^{q^\mu}_{n-\mu}(\alpha) \tag{4.17}$$

Now we can prove the following

<u>Lemma 4.4</u> For all $r > 0$ $J^{q^r}_{n-r}(t)$ is a linear polynomial in $J_n(t)$ and $\Delta J_n(t)$ with polynomials in x as coefficients, i.e.

$$J^{q^r}_{n-r}(t) = \mathcal{P}(J_n(t), \Delta J_n(t)).$$

The degree of $\mathcal{P}$ in $J_n(t)$ and $\Delta J_n(t)$ is $\leq q^{[\frac{r}{2}]}$ and the coefficients of $\mathcal{P}$ have degree $\leq q^{[\frac{r}{2}]} \max(rq^n, q^r)$.

Proof: From the recurrenceformula (3.4) we get

$J^{q^r}_{n-r}(t)$ is a linear polynomial in $J_n(t)$ and $J_{n+1}(t)$ of degree $q^{[\frac{r}{2}]}$ with polynomial coefficients with

$dg \leq \max(rq^n, q^r)$.

$$J_{n+1}(t) = (x^{q^n} - x)\, J_n(t) - \Delta J_n(t).$$

Therefore $J^{q^r}_{n-r}$ is a linear polynomial in $J_n(t)$ and $\Delta J_n(t)$ of degree $q^{[\frac{r}{2}]}$ with polynomial coefficients with $dg \leq q^{[\frac{r}{2}]} \max(rq^n, q^r)$.

This lemma gives us the following generalization of theorem 4.3:

<u>Theorem 4.5</u> Let $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q\{x\}$, then at least one of the elements of the set

$$W = \{\alpha, \beta, J_n(\alpha), \Delta J_n(\alpha), J_n(\alpha\beta), \Delta J_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$.

Proof: We proceed in the same way as in theorem 4.3 except that we replace J_0 by J_n and instead of (4.9) we have

$$m := k + l - 1 \text{ and } k < \frac{1}{6} l. \qquad (4.18)$$

Hence we obtain formula (4.10) with J_n instead of J_0.
According to (4.17) and lemma 4.4 $J_n(\alpha A)$ is a linear polynomial in $J_n(\alpha)$ and $\Delta J_n(\alpha)$ with since $m \to \infty$:

$$\begin{aligned} \operatorname{dg} J_n(\alpha A) &\leq mq^m + \max_{0\leq\mu\leq m} \operatorname{dg} J^{q^\mu}_{n-\mu}(\alpha) \\ &\leq mq^m + q^{\frac{3}{2}m} + 2q^{\frac{m}{2}} \max(\operatorname{dg} J_n(\alpha), \operatorname{dg} \Delta J_n(\alpha)). \end{aligned}$$

If we multiply (4.10^*) with $\Gamma^{q^{2l+2}}$ using (4.18) we find that the coefficients D_{ij} satisfy

$$\operatorname{dg} D_{ij} \leq q^{2l+e} (m + c_4),$$

and therefore

$$d^* X_{ij} < (2m + c_5)q^{2l+e} \quad \text{for } l > l_0.$$

Since $\max\limits_{\operatorname{dg}t=2\mu} \operatorname{dg} J_n(\alpha t) \leq \max\limits_r q^{n+r} (2\mu + \operatorname{dg} \alpha - n - 2r) \leq c_6 q^{n+\mu}$

(4.14) is replaced by

$$\max_{\operatorname{dg}t=2\mu} \operatorname{dg} L(t) \leq q^{2\eta+e} (4\mu + c_7^* q^{4k}) \qquad (4.14^*)$$

In the same way as in theorem 4.3 we can conclude that for all $\xi \in \mathfrak{B}(\mu)$, where μ is an arbitrary natural number

$$L(\xi) = 0,$$

and furthermore $L(t) \not\equiv 0$.
Similarly we obtain

$$\max_{\operatorname{dg}t=2\nu} L(t) \geq q^{2\nu}(c_{10}^*\nu + c_{11}^*) \text{ where } c_{10}^* > 0$$

and

$$\max_{\operatorname{dg}t=2\nu} L(t) < (2m + c_5)q^{2l+e} + 2\nu q^{2l+e} + c_6 q^{n+\nu+2k+e}$$

which if l and k are chosen suitable for big ν leads to a contradiction. Hence at least one of the elements of W is transcendental.

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Abstract.

In this report the following statement on transcendence-properties of the Carlitz-Besselfunctions $J_n(t)$ on fields with characteristic p is proved:
Let $\mathbb{F}_q$ denote the field of q elements where q is a power of the primenumber p and let $\mathbb{F}_q\{x\}$ be the quotientfield of the polynomial-ring $\mathbb{F}_q[x]$. Let $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q\{x\}$; then for an arbitrary integer n at least one of the elements of the set

$$\{\alpha, \beta, J_n(\alpha), \Delta J_n(\alpha), J_n(\alpha\beta), \Delta J_n(\alpha\beta)\},$$

where $\Delta J_n(t) = J_n(xt) - xJ_n(t)$, is transcendental over $\mathbb{F}_q\{x\}$.
The proof is based on the method of Schneider's proof of the Gelfond-Schneider theorem.