STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1969 - 003

Local properties of analytic functions

by

H.G.J. Pijls

June 1969

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam, The Netherlands.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications; itis sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O) and the Central Organization for Applied Scientific Research in the Netherlands (T.N.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Local properties of analytic functions

1. The Weierstrass Preparation Theorem.
   (Properties of the local ring A_n).
2. The Oka Theorem.
   (Coherence of the sheaf $\mathcal{A}_{n,\Omega}$).
3. Analytic sets.
   (Summary of some theorems).

1. The Weierstrass Preparation Theorem.

(Properties of the local ring A_n).

Definition 1.1. If $z = (z_1,\dots,z_n) \in \mathbb{C}^n$ (or more generally z is in a complex manifold), let $A_{n,z}$ or A_z denote the set of equivalence classes of functions f which are analytic (holomorphic) in some neighborhood of z, under the equivalence relation: $f \sim g$ if $f=g$ in some neighborhood of z. If f is analytic in a neighborhood of z, we write $(f)_z$ for the residue class of f in A_z, which is called the germ of f at z. $A_{n,z}$ is called the ring of germs of analytic functions at the point $z \in \mathbb{C}^n$. Instead of $A_{n,0}$, we also write A_n. For the residue class $(f)_0$ we shall often write f, thus identifying f with its germ at the origin.

It is clear that A_n can be made into a ring with unit. Its elements can be identified with the set of all power series

$$\sum a_\nu z^\nu$$

which converge in some neighborhood of 0, that is, with the set of all arrays $\{a_\nu\}$ such that

$$\sum |a_\nu| r^{|\nu|} < \infty \qquad \text{for some } r>0.$$

The ring A_n is an integral domain; this follows from the identity theorem for analytic functions.

Furthermore, it is clear that $f \in A_n$ has an inverse in A_n iff $f(0) \neq 0$; this follows from the fact that, if f is analytic in a neighborhood of 0 and $f(0) \neq 0$, then $\frac{1}{f(z)}$ is also analytic in a neighborhood of 0. So the nonunits of A_n are precisely the germs of functions which vanish at the origin; and these obviously form an ideal in A_n. The ring A_n is therefore a local ring (i.e. a ring in which the nonunits form an ideal).

For further investigation of the properties of the ring A_n it is convenient to develop a technique which facilitates the induction step from A_{n-1} to A_n. Here we consider A_{n-1} as a subring of A_n (A_{n-1} consists of all germs of analytic functions f(z) which are independent of the variable z_n). The idea is now to effect the transition from A_{n-1} to A_n in two stages by introducing the ring $A_{n-1}[z_n]$.

Definition 1.2. $A_{n-1}[z_n]$ is the polynomial ring over A_{n-1} in the "indeterminate" z_n. So an element of $A_{n-1}[z_n]$ can be written in the form

$$a_0 + a_1 z_n + \ldots + a_k z_n^k$$

where $a_0, a_1, \ldots, a_k \in A_{n-1}$.

Then we have:

$$A_{n-1} \subset A_{n-1}[z_n] \subset A_n \ .$$

The polynomial extension $A_{n-1} \subset A_{n-1}[z_n]$ is handled by means of algebraic theorems (such as Gauss' theorem on unique factorization in a polynomial ring and the Hilbert basis theorem). The extension $A_{n-1}[z_n] \subset A_n$ is treated by the so-called Weierstrass preparation theorem. To formulate this theorem we need some further definitions.

Definition 1.3. An element $f \in A_n$, which is normalized in the z_n-direction (i.e. $f(0,z_n)$ does not vanish identically) is called regular or order p, if there exists a representative function for f (which we call also f) such that $f(0,z_n)$ has a zero of order p at $z_n = 0$. A Weierstrass polynomial π of degree p $(p \geq 1)$ in z_n is an element of $A_{n-1}[z_n]$ of the form

$$\pi = z_n^p + \sum_{\nu=1}^{p} a_\nu z_n^{p-\nu} \quad ,$$

where the coefficients $a_\nu \in A_{n-1}$ are nonunits.

Theorem 1.1. (Weierstrass Preparation Theorem). Let $f \in A_n$ be regular of order p in z_n. Then there exists a unique Weierstrass polynomial $\pi \in A_{n-1}[z_n]$ of degree p such that

$$f = u\pi$$

for some unit $u \in A_n$.

Furthermore, any $g \in A_n$ can be written in a unique manner in the form

$$g = af + b \ ,$$

where $a \in A_n$ and $b \in A_{n-1}[z_n]$ is a polynomial of degree < p.

We shall prove the theorem in a more general form which is expressed in terms of functions, rather than just in terms of germs of functions.

Theorem 1.2. (Weierstrass Preparation Theorem). Let f be analytic in a neighborhood of the closure of the set

$$\Delta = \{z \in \mathbb{C}^n \mid |z_1|<r_1,\dots,|z_n|<r_n\}$$

and suppose that $f(0,z_n)$ is zero only for $z_n = 0$ in $|z_n|\leq r$, and that the origin is a zero of order p ($p\geq 1$). Suppose moreover that

$$f(z) \neq 0 \quad \text{for } |z_j|\leq r_j \quad (j\leq n-1),\ |z_n| = r_n.$$

Then, for any function g analytic on Δ, there exist analytic functions

a on Δ', and

$b_1,\dots,b_p$ on Δ',

where $\Delta' = \{(z_1,\dots,z_{n-1})\in \mathbb{C}^{n-1} \mid |z_j|<r_j, j\leq n-1\}$,

such that

$$(1) \qquad g = af + \sum_{\nu=1}^{p} b_\nu z_n^{p-\nu} \quad \text{on } \Delta .$$

Also, there exist analytic functions

u on Δ', which is nowhere zero, and

$a_1,\dots,a_p$ on Δ', $a_\nu(0) = 0$ $(1\leq\nu\leq p)$,

such that

$$(2) \qquad f = u(z_n^p + \sum_{\nu=1}^{p} a_\nu z_n^{p-\nu}) .$$

Moreover, there exists a constant $M>0$, depending only on Δ and f, such that, in (1) we have

$$(3) \qquad \sup_{\Delta} |a| , \sup_{\Delta'} |b_\nu| \leq M \sup_{\Delta} |g|$$

The representation (1) is unique.

Proof.

We begin by proving the existence of u and a_ν. Let $z' = (z_1,\ldots,z_{n-1}) \in \Delta'$, and set

$$\sigma_0(z') = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{\partial f(z',\zeta)}{\partial\zeta}\frac{1}{f(z',\zeta)}\,d\zeta .$$

Then, $\sigma_0(z')$ is the number of zeros of $f(z',z_n)$ in $|z_n|<r_n$; so $\sigma_0(0) = p$. Further, $\sigma_0(z')$ is clearly a continuous function of z'; hence $\sigma_0(z') = p$ for $z'\in\Delta'$. Let $\zeta_1(z'),\ldots,\zeta_p(z')$ denote the zeros of $f(z',\zeta)$ in $|\zeta|<r_n$. Then set

$$\pi(z',z_n) = \prod_{j=1}^{p} (z_n - \zeta_j(z')) .$$

We have, for $k\geq 0$,

$$\sigma_k(z') \overset{\text{def}}{=} \sum_{j=1}^{p} \{\zeta_j(z')\}^p = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{\partial f(z',\zeta)}{\partial\zeta}\frac{\zeta^k}{f(z',\zeta)}\,d\zeta ,$$

(cf. [A, p. 124])

so that $\sigma_k(z')$ is analytic on Δ'. If

$$a_\nu(z') = (-1)^\nu \sum_{1\leq j_1<\ldots<j_\nu\leq p} \zeta_{j_1}(z')\ldots\zeta_{j_p}(z')$$

is the ν-th elementary symmetric functions of the ζ_j, then a_ν is a polynomial in $\sigma_1,\ldots,\sigma_p$ (with rational coefficients), so that a_ν is analytic on Δ', and

$$\pi(z',z_n) = z_n^p + \sum_{\nu=1}^{p} a_\nu(z')z_n^{p-\nu} .$$

For any $z'\in\Delta'$, f and π have the same zeros. Hence

$$\frac{f(z',z_n)}{\pi(z',z_n)} = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{f(z',\zeta)}{\pi(z',\zeta)}\frac{1}{\zeta-z_n}\,d\zeta$$

is analytic on Δ. Similarly for $\frac{\pi(z',z_n)}{f(z',z_n)}$.

Hence $f = u\pi$ where u is analytic and nowhere zero on Δ;

Further, it is clear that we may replace Δ by a slightly larger polydisc. without altering the conditions of the theorem. Hence u and u^{-1} are bounded on Δ, and so are the coefficients of π.

To prove the division algorithm and inequality (3) we may replace f by π. If g is analytic on Δ, let

$$a(z',z_n) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\frac{1}{\zeta-z_n}\,d\zeta ,$$

where $0<r<r_n$. Clearly the integral is independent of r if $|z_n|<r$, so that a is analytic on Δ. Then, if $|z_n|<r$,

$$g(z',z_n) - a(z',z_n)\pi(z',z_n) = \frac{1}{2\pi i}\int_{|\zeta|=r} g(z',\zeta)\left\{1 - \frac{\pi(z',z_n)}{\pi(z',\zeta)}\right\}\frac{1}{\zeta-z_n}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\left\{\frac{\zeta^p - z_n^p + \sum_{\nu=1}^{p} a_\nu(z')\,(\zeta^{p-\nu} - z_n^{p-\nu})}{\zeta - z_n}\right\}d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\sum_{\nu=1}^{p} c_\nu(z',\zeta)\quad z_n^{p-\nu}\,d\zeta ,$$

where $c_\nu(z',\zeta)$ is a polynomial in ζ of degree < p whose coefficients are linear combinations of the a_ν. If we set

$$b_\nu(z') = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\,c_\nu(z',\zeta)\,d\zeta ,$$

we obtain

$$g(z) - a(z)\pi(z) = \sum_{\pi=1}^{p} b_\nu(z')\,z_n^{p-\nu} .$$

Moreover the a_ν are bounded on Δ and $\pi(z',\zeta)$ is bounded away from zero on the set $|z_j|\le r_j$, $j\le n-1$, $|\zeta| = r_n$. Hence

$$\sup_{\Delta'} |b_\nu| \leq M \sup_{\Delta'} |g| ,$$

where M depends only on the a_ν and π. Hence, since

$$g - a\pi = \sum_{=1}^{p} b_\nu z_n^{p-\nu} ,$$

we deduce that for fixed $z' \in \Delta'$, we have

$$|a(z',z_n)| \leq \frac{M' \sup_{\Delta} |g|}{\inf_{|\zeta|=r_n} |\pi(z',\zeta)|} ,$$

so that

$$\limsup_{|z_n| \to r_n} |a(z',z_n)| \leq M'' \sup_{\Delta} |g| .$$

By the maximum principle, this implies that

$$\sup_{\Delta} |a| \leq \sup_{\Delta} |g| .$$

To demonstrate the uniqueness of (1), suppose that we have two such expressions:

$$g = af + \sum_{\nu=1}^{p} b_\nu z_n^{p-\nu} = a'f + \sum_{\nu=1}^{p} b'_\nu z_n^{p-\nu} .$$

Then

$$(a - a')f = \sum_{\nu=1}^{p} (b'_\nu - b_\nu) z_n^{p-\nu} .$$

This implies that for fixed $z' \in \Delta'$, the polynomial

$$\sum_{\nu=1}^{p} (b'_\nu - b_\nu) z_n^{p-\nu}$$

of degree $\leq p-1$ has at least p zeros in $|z_n| < r_n$; this implies $b_\nu = b'_\nu$ and it follows that $a = a'$.

This completes the proof Theorem 1.2.

As an application of the Weierstrass preparation theorem, we shall derive some further properties of the local ring A_n.

Definition 1.4. An element $f \in A_n$ (resp. $A_{n-1}[z_n]$) is called reducible over A_n (resp. $A_{n-1}[z_n]$) if it can be written as a product $f = g_1 g_2$, where g_1, g_2 are nonunits of A_n (resp. $A_{n-1}[z_n]$); elements without this property are called irreducible over A_n (resp. $A_{n-1}[z_n]$). A unique factorization domain is an integral domain with an identity element in which every nonunit can be written as a finite product of irreducible factors, and in which such a factorization is unique up to the order of its factors and units of the ring.

Lemma 1.1. Let $f = g\pi$, where f, g and $\pi \in A_n$. If $f \in A_{n-1}[z_n]$ and if $\pi \in A_{n-1}[z_n]$ is a Weierstrass polynomial, then $g \in A_{n-1}[z_n]$.
Proof. Since π is monic (the leading coefficient of π, as a polynomial in z_n, is equal to 1), we can make an algebraic division over A_{n-1}; then we obtain

$$f = g'\pi + r,$$

where g' and r belong to $A_{n-1}[z_n]$ and r is a polynomial of degree lower than that of π. But the uniqueness stated in the Weierstrass preparation theorem then implies $g' = g$ and $r = 0$.

Lemma 1.2. A Weierstrass polynomial $\pi \in A_{n-1}[z_n]$ is reducible over A_n iff it is reducible over $A_{n-1}[z_n]$. If π is reducible, then all of its factors are Weierstrass polynomials, modulo units of $A_{n-1}[z_n]$.
Proof. Suppose first $\pi = g_1 g_2$ where $g_j \in A_n$ are nonunits; since π is a Weierstrass polynomial, both g_1 and g_2 are regular in z_n. Applying the Weierstrass preparation theorem, write $g_j = u_j \pi_j$ $(j = 1,2)$, where $u_j \in A_n$ are units and $\pi_j \in A_{n-1}[z_n]$ are Weierstrass polynomials; thus $\pi = (u_1 u_2)(\pi_1 \pi_2)$. From the uniqueness it follows that $u_1 u_2 = 1$ and $\pi = \pi_1 \pi_2$.
Second, suppose that $\pi = g_1 g_2$ where $g_j \in A_{n-1}[z_n]$ are nonunits of that ring. If g_1 were a unit in A_n, then $g_2 = \frac{1}{g_1}\pi$. From lemma 1.1. it would follow that $\frac{1}{g_1} \in A_{n-1}[z_n]$; this is impossible, since g_1 is a nonunit of $A_{n-1}[z_n]$, and therefore g_1 is a nonunit of A_n.

Theorem 1.3. The local ring A_n is an unique factorization domain.
Proof. When $n = 0$ the theorem is trivial. Therefore assume that the theorem holds for A_{n-1}. We apply now Gauss' theorem:

If A is an unique factorization domain, then so is the polynomial ring $A[X]$ (cf. [W 1,p.70]).

So it follows that $A_{n-1}[z_n]$ is also an unique factorization domain. Consider any element $f \in A_n$; by a suitable linear change of coordinates we can make f regular in z_n. Then, by the Weierstrass preparation theorem, write $f = u\pi$ where $u \in A_n$ is an unit, $\pi \in A_{n-1}[z_n]$ is a Weierstrass polynomial. The polynomial π can be written uniquely, up to order and units in $A_{n-1}[z_n]$, as a product of irreducible polynomials; this provides in view of lemma 1.2. an unique factorization in A_n, up to order and units in A_n.

Definition 1.5. A commutative ring A with an identity element is called Noetherian if every ideal $I \subset A$ is finitely generated over A, that is, if there exist elements $f_1,\dots,f_k \in I$, so that every $f \in I$ can be written

$$f = \sum_{i=1}^{k} a_i f_i$$

for some $a_i \in A$.

Theorem 1.4. The local ring A_n is Noetherian.
Proof. For $n = 0$ the theorem is trivial. Assume that the theorem has already been proved for A_{n-1}. We apply the Hilbert basis theorem:

If A is a Noetherian ring, then so is the polynomial ring $A[X]$ (cf. [W 2,p.18]).

So $A_{n-1}[z_n]$ is a Noetherian ring. Consider any ideal $I \subset A_n$ which contains some non zero elements f. After a linear change of coordinates we may suppose that f is regular in z_n of order p. After multiplication by an unit we can further assume that $f \in I \cap A_{n-1}[z_n]$ is a Weierstrass polynomial. Since $A_{n-1}[z_n]$ is Noetherian the ideal $I \cap A_{n-1}[z_n]$ has a finite set of generators. Now $f, f_1,\dots,f_k$ generate the entire ideal I, which we see as follows. If $g \in I$, write $g = af + b$, where $b \in A_{n-1}[z_n]$; but b is clearly also in I, hence in $I \cap A_{n-1}[z_n]$, so that

$$b = \sum_{i=1}^{k} a_i f_i$$

for some elements $a_i \in A_{n-1}[z_n]$. Thus

$$g = af + \sum_{i=1}^{k} a_i f_i \ .$$

2. The Oka Theorem.

(Coherence of the sheaf $\mathcal{A}_{n,\Omega}$).

Up to now we have only considered the ring $A_{n,0}$, that is A_z for fixed $z \in \mathbb{C}^n$. In the present section we let z be variable; we shall prove a theorem (the Oka theorem) which goes beyond the Noetherian property of A_z.

Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{A}_{n,\Omega}$ the sheaf on Ω of germes of analytic functions.

Definition 2.1. Let $\mathcal{F}$ be an analytic sheaf over Ω (i.e. a sheaf of $\mathcal{A}$-modules over Ω). $\mathcal{F}$ is said to be locally finitely generated if for every $z \in \Omega$ there exists a neighborhood $U \subset \Omega$ of z and a finite number of sections $f_1, \ldots, f_q \in \Gamma(U, \mathcal{F})$ so that $\mathcal{F}_\zeta$ is generated by $(f_1)_\zeta, \ldots, (f_q)_\zeta$ as an A_ζ-module for every $\zeta \in U$.

Definition 2.2. Let $\mathcal{F}$ be an analytic sheaf over Ω. Let $f_1, \ldots, f_q \in \Gamma(U, \mathcal{F})$ where $U \subset \Omega$ is an open set and let $z \in U$. If there exists a tuple $(\alpha_1, \ldots, \alpha_q) \in A_z^q$ such that

$$\sum_{i=1}^{q} \alpha_i (f_i)_z = 0,$$

the tuple $(\alpha_1, \ldots, \alpha_q)$ is called a relation between $f_1, \ldots, f_q$ at z. The collection of all such relations forms an analytic sheaf over Ω (it is a subsheaf of $\mathcal{A}^p_{n,\Omega}$ since it is contained in $\mathcal{A}^p_{n,\Omega}$ as an open set). It is denoted by $\mathcal{R}(f_1, \ldots, f_q)$ and it is called the sheaf of relations between $f_1, \ldots, f_q$.

Definition 2.3. An analytic sheaf $\mathcal{F}$ over Ω is called coherent if

(i) $\mathcal{F}$ is locally finitely generated,

(ii) if $U \subset \Omega$ is an open set and $f_1, \ldots, f_q \in \Gamma(U, \mathcal{F})$ then the sheaf of relations $\mathcal{R}(f_1, \ldots, f_q)$ is locally finitely generated.

Lemma 2.1. A subsheaf $\mathcal{G}$ of a coherent sheaf $\mathcal{F}$ over Ω is coherent iff it is locally finitely generated.

Proof. Any section of $\mathcal{G}$ is a section of $\mathcal{F}$. Hence the sheaf of relations $\mathcal{R}$ of any finite number of sections of $\mathcal{G}$ is the sheaf of relations between these sections, considered as sections of $\mathcal{F}$. Since $\mathcal{F}$ is coherent, $\mathcal{R}$ is locally finitely generated.

We now come to our main theorem.

Theorem 2.1. The sheaf $\mathcal{A}^p$ $(= \mathcal{A}^p_{n,\Omega})$ is a coherent sheaf of rings.

Proof. $\mathcal{A}^p$ is locally finitely generated since the sections E_j, $1 \le j \le p$, defined by

$$E_j = (0,\dots,0,1,0,\dots,0) \quad \text{(1 in the j-th place)}$$

generate the stalk $\mathcal{A}^p_z$ at each point $z \in \Omega$. Hence the theorem reduces to the following one.

Theorem 2.2. (Oka Theorem). Let $F_1,\dots,F_q \in A(\Omega)^p$ $(= \Gamma(\Omega,\mathcal{A}^p))$, and let $\mathcal{R}$ be the sheaf of relations between $F_1,\dots,F_q$. Then $\mathcal{R}$ is locally finitely generated.

Remark. Since A_z is Noetherian, we know of course already that $\mathcal{R}_z$ is finitely generated for every $z \in \Omega$, but the important point in the theorem is that one can use "the same" generators for all ζ in a neighborhood of any given point z.

Proof. The proof consists of two parts.

(A) First we prove the theorem for $p>1$ assuming that it has already been proved for smaller values of p.

(B) Second we prove the case $p = 1$ assuming that the theorem has already been proved for every p in the $(n-1)$-dimensional case.

The theorem follows from (A) and (B).

(A)

Assume $z = 0 \in \Omega$. Then we have to construct a neighborhood U of 0 with the properties stated in Def. 2.1.

We shall use the following notations.
Let

$$F_1 = \begin{pmatrix} f_1^1 \\ \vdots \\ f_1^p \end{pmatrix}, \ \dots \ , F_q = \begin{pmatrix} f_q^1 \\ \vdots \\ f_q^p \end{pmatrix}$$

belong to $A(\Omega)^p$. The matrix with columns $F_1,\dots,F_q$ is denoted by F. Instead of $\mathcal{R}(F_1,\dots,F_q)$ we write also $\mathcal{R}(F)$.
Let

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} \in A_z^q \quad ;$$

then the element of $A_z{}^q$ that we obtain by applying the matrix $(F)_z$ (this is the matrix with columns $(F_1)_z,\dots,(F_q)_z$) to the vector α is denoted by

$$(F)_z\alpha \quad .$$

Then we have

(1) $$\alpha \in \mathcal{R}_z(F) \iff (F)_z\alpha = 0 \ .$$

Let $f^1 = (f_1^1,\dots,f_q^1)$ be the first row of F. Then by hypothesis we can find a neighborhood $U' \subset \Omega$ of 0 and finitely many elements

$$\alpha^{(\nu)} = \begin{pmatrix} \alpha_1^{(\nu)} \\ \vdots \\ \alpha_q^{(\nu)} \end{pmatrix} \in A(U')^q \qquad (\nu = 1,2,\dots,r)$$

so that $(\alpha^{(1)})_z,\dots,(\alpha^{(r)})_z$ generate the A_z-module $\mathcal{R}_z(f^1)$ $(=\mathcal{R}_z(f_1^1,\dots,f_q^1)$ for every $z \in U'$. This means that for any

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} \in A_z{}^r$$

we have

$$(2) \qquad (f^1)_z \, (A)_z \, c = 0 \qquad (z \in U') ,$$

where A denotes the matrix with columns $\alpha^{(1)},\dots,\alpha^{(r)}$. Therefore,

$$(3) \qquad \mathcal{R}_z(F) \subset \{(A)_z \, c \mid c \in A_z\} \qquad (z \in U') .$$

We have, because of (1),

$$(4) \qquad (A)_z \, c \in \mathcal{R}_z(F) \iff (F)_z \, (A)_z \, c = 0 .$$

But, because of (2), $(F)_z \, (A)_z \, c = 0$ is equivalent to

$$(5) \qquad (F')_z \, (A)_z \, c = 0 \qquad (z \in U') ,$$

where F' is the matrix obtained from F by deleting the first row f^1. In view of (1) now (5) is equivalent to

$$c \in \mathcal{R}_z(F'A) \qquad (z \in U') .$$

So we have

$$(6) \qquad (A)_z \, c \in \mathcal{R}_z(F) \iff c \in \mathcal{R}_z(F'A) \qquad (z \in U') .$$

Since F'A has (p-1) rows it follows by the hypothesis that there is a neighborhood $U'' \subset U'$ of 0 and elements

$$\beta^{(\mu)} = \begin{pmatrix} \beta_1^{(\mu)} \\ \vdots \\ \beta_r^{(\mu)} \end{pmatrix} \in A(U'')^r \qquad (\mu = 1,2,\dots,s) ,$$

so that $(\beta^{(1)})_z,\dots,(\beta^{(s)})_z$ generate the A_z-module $\mathcal{R}_z(F'A)$ for every $z\in U''$. Let B be the matrix with columns $\beta^{(1)},\dots,\beta^{(s)}$ Then any $c\in\mathcal{R}_z(F'A)$ is of the form

(7) $$c = (B)_z d$$

where

$$d = \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix} \in A_z^{\,s} \qquad (z\in U'').$$

From (3), (6) and (7) it follows that the columns of the matrix $(AB)_z$ generate the A_z-module $\mathcal{R}_z(F)$ for every $z\in U''$.

(B)

Assume again that the given point is $z = 0\in\Omega$. We write $F_i = f_i$ $(1\le i\le q)$ and $f = (f_1,\dots,f_q)$. After a linear change of coordinates we may suppose that f_i $(1\le i\le q)$ satisfies the conditions of the Weierstrass preparation theorem. Since the assertion of the theorem is local and permits multiplication by units, we may suppose furthermore that

$$f_i = z_n^{\,p_i} + \sum_{\nu=0}^{p_i-1} a_\nu^{(i)}(z')\, z_n^\nu \qquad (1\le i\le q),$$

where $a_\nu^{(i)}$ is analytic in $\Omega'\subset\mathbb{C}^{n-1}$ $(0\in\Omega')$ and $a_\nu^{(i)}(0) = 0$.
We may suppose that $p = p_q = \max_{1\le i\le q} p_i$.
Let $\zeta = (\zeta',\zeta_n)\in\Omega$. We say that a relation $(\alpha_1,\dots,\alpha_q)\in\mathcal{R}_\zeta(f) = \mathcal{R}_\zeta$ is polynomial, if

$$\alpha_i\in A_{n-1,\zeta'}[z_n] \qquad (1\le i\le q).$$

We now prove the following.

(α) Let $\Omega = \Omega'\times D$ where $D = \{z_n\in\mathbb{C} \mid |z_n| < r_n\}$. Then, for any $\zeta = (\zeta',\zeta_n)\in\Omega$, $\mathcal{R}_\zeta$ is generated over $A_{n-1,\zeta'}$ by the polynomial relations in z_n of degree $\le p$.

Proof of (α).

Write

$$f_q(z',z_n) = u(z)\pi(z',z_n-\zeta_n) ,$$

where π is a Weierstrass polynomial in $z_n - \zeta_n$ with coefficients in $A_{n-1,\zeta'}$ (vanishing at ζ', except for the leading term) of degree $\rho\leq p$, and u is an unit. By lemma 1.1

$$u\in A_{n-1,\zeta'}[z_n]$$

and has degree $p - \rho$ ($\leq p$).

Clearly, for i>1, the element

$$s_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_q \\ 0 \\ \vdots \\ 0 \\ -f_i \end{pmatrix} \qquad (f_q \text{ is in the i-th place})$$

is a polynomial relation. If

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} \in \mathcal{R}_\zeta ,$$

we write

$$\alpha_i = c_i\pi + r_i \qquad (1\leq i\leq q-1),$$

where $c_i\in A_{n,\zeta}$ and $r_i\in A_{n-1,\zeta'}[z_n]$ and degree $(r_i) < \rho$. This can be written

$$\alpha_i = d_i f_q + r_i \qquad (1 \leq i \leq q-1),$$

where $d_i = c_i u^{-1} \in A_{n,\zeta}$. Hence

$$(*) \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} - d_1 s_1 - \ldots - d_{q-1} s_{q-1} = \begin{pmatrix} r_1 \\ \vdots \\ r_{q-1} \\ r_q \end{pmatrix},$$

where r_q is defined as

$$r_q = \alpha_q + \sum_{i=1}^{q-1} d_i f_i .$$

We must have $(r_1, \ldots, r_q) \in \mathcal{R}_\zeta$ since all other terms in the relation $(*)$ are in that module. So we have

$$r_q f_q = - \sum_{i=1}^{q-1} r_i f_i$$

is an element of $A_{n-1,\zeta'}[z_n]$ of degree $< p + \rho$. Hence by lemma 1.1.

$$r_q u \in A_{n-1,\zeta'}[z_n]$$

and has degree $< p$. Also

$$r_i u \in A_{n-1,\zeta'}[z_n] \qquad (1 \leq i \leq q-1)$$

and has degree $< p$. Thus

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} = d_1 s_1 + \ldots + d_{q-1} s_{q-1} + u^{-1} \begin{pmatrix} r_1 u \\ \vdots \\ r_q u \end{pmatrix}.$$

And since $s_1, \ldots, s_{q-1}$ and $(r_1 u, \ldots, r_q u)$ are polynomial relations of degree $\leq p$, the assertion (α) is proved.

To complete the proof of (B) we have therefore only to prove the following:

(β) There exist finitely many polynomial relations

$$\pi^{(\nu)} = \begin{pmatrix} \pi_1^{(\nu)} \\ \vdots \\ \pi_q^{(\nu)} \end{pmatrix}$$

of degree $\leq p$ in a neighborhood U of the origin such that any polynomial relation of degree $\leq p$ at $\zeta \in U$ is generated, over $A_{n-1,\zeta'}$, by the $\pi^{(\nu)}$

Proof of (β).

Let

$$\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_q \end{pmatrix} ,$$

$$\pi_i = \sum_{\nu=0}^{p} c_\nu^{(i)} (z') z_n^\nu \qquad (1 \leq i \leq q)$$

be any polynomial relation at $\zeta = (\zeta', \zeta_n)$ and write

$$f_i = \sum_{\nu=0}^{p} a_\nu^{(i)} (z') z_n^\nu \qquad (1 \leq i \leq q)$$

(note: $a_{p_i}^{(i)} = 1$, $a_\nu^{(i)} = 0$ if $\nu > p_i$).

Then π is a relation if and only if

$$\sum_{i=1}^{q} \sum_{k+l=\nu} a_k^{(i)} (z') c_l^{(i)} (z') = 0 \text{ in } A_{n-1,\zeta'} \quad \text{for } \nu = 0,1,\ldots,p.$$

This means that the element

$$(c_0^{(1)},\ldots,c_p^{(q)}) \in A_{n-1,\zeta'}^{(p+1)q}$$

is a relation between the $(p+1)q$ sections

$$s_{\nu}^{(i)}(z') = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_0^{(i)}(z') \\ \vdots \\ a_{p-\nu}^{(i)}(z') \end{pmatrix} \in \Gamma(U', \mathcal{A}_{n-1}^{p+1})$$

$(1 \leq i \leq q,\ 0 \leq \nu \leq p)$.

The statement (β) is now an immediate consequence of our induction hypothesis.

3. Analytic Sets.

Definition 3.1. An analytic set (analytic subvariety) in an open set $\Omega \subset \mathbb{C}^n$ is a subset V of Ω with the following property: for every $a \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of a and finitely many analytic functions $f_1, \ldots, f_r$ on U such that

$$V \cap U = \{z \in U \mid f_1(z) = \ldots = f_r(z) = 0\}.$$

Obviously, an analytic set V in Ω is closed in Ω. Furthermore, the interior of V in Ω is both open and closed in Ω. So, if Ω is connected and $V \neq \Omega$, then V is nowhere dense in Ω.

Theorem 3.1. Let V be an analytic set in an open set $\Omega \subset \mathbb{C}^n$. If Ω is connected, then $\Omega \setminus V$ is connected.
Proof. cf. [GR, p. 20].

Theorem 3.2. Let $\{f_i \mid i \in I\}$ be a collection of functions, analytic on an open set Ω. Then

$$\{z \in \Omega \mid f_i(z) = 0 \text{ for all } i \in \mathfrak{I}\}$$

is an analytic set in Ω.
Proof. cf. [GR, p. 86].

Definition 3.2. Let V_1 and V_2 be subsets of an open set $\Omega \subset \mathbb{C}^n$, and let $a \in \Omega$. The sets V_1 and V_2 are said to be equivalent at $a \in \Omega$ (V_1 and V_2 have the same germ at a) if there is a neighborhood U of a such that

$$V_1 \cap U = V_2 \cap U.$$

An equivalence class of this relation is called a germ of a set at a. The germ of a set at a is denoted by $\underline{V}_a$ (or: $\underline{V}$, when no confusion is possible).
The germ of an analytic set is called an analytic germ.

If $\underline{V}_1$ and $\underline{V}_2$ are germs of a set at a, then in an obvious way one defines the germs $\underline{V}_1 \cup \underline{V}_2$ and $\underline{V}_1 \cap \underline{V}_2$; also $\underline{V}_1 \subset \underline{V}_2$ has an obvious meaning.

Definition 3.3. Let $\underline{V}$ be the germ of a set at a; and let $f \in A_{n,a}$ (f is the germ of some function, also denoted by f, which is analytic in a neighborhood U of a). We say that f vanishes on $\underline{V}$ iff $\underline{V}$ is contained in the germ $\underline{V}(f)$ at a of the set

$$V(f) = \{z \mid z \in U,\ f(z) = 0\}.$$

The ideal of $\underline{V}$, id($\underline{V}$), is defined to be

$$id(\underline{V}) = \{f \mid f \in A_{n,a} \text{ and } f \text{ vanishes on } \underline{V}\} .$$

If $F \subset A_{n,a}$, then the locus of F, loc(F) is defined to be

$$loc(F) = \bigcap_{f \in F} \underline{V}(f)$$

Definition 3.4. If $\mathfrak{J}$ is an ideal in a ring R, we define the radical of $\mathfrak{J}$ to be the ideal

$$Rad(\mathfrak{J}) = \{x \in R \mid \exists\ n \in \mathbb{N} \text{ such that } x^n \in \mathfrak{J}\}.$$

Theorem 3.3. Let $\underline{V}_1$, $\underline{V}_2$ and $\underline{V}$ be analytic germs (at 0), and let $\mathfrak{J}_1$, $\mathfrak{J}_2$ and $\mathfrak{J}$ be ideals in A_n. Then the following relations hold:

(i) $\underline{V}_1 \supset \underline{V}_2$ implies $id(\underline{V}_1) \subset id(\underline{V}_2)$,

(ii) $\mathfrak{J}_1 \supset \mathfrak{J}_2$ implies $loc(\mathfrak{J}_1) \subset loc(\mathfrak{J}_2)$,

(iii) $id(\underline{V}_1) \cap id(\underline{V}_2) = id(\underline{V}_1 \cup \underline{V}_2)$,

(iv) $loc(\mathfrak{J}_1) \cup loc(\mathfrak{J}_2) = loc(\mathfrak{J}_1 \cap \mathfrak{J}_2)$,

(v) $id(\underline{V}) = Rad\ id(\underline{V})$,

(vi) $loc(\mathfrak{J}) = loc(Rad\ \mathfrak{J})$.

Theorem 3.4. Let $\underline{V}$ be an analytic germ, and let $\mathfrak{J}$ be an ideal in A_n. Then

(i) $loc\ id\ (\underline{V}) = \underline{V}$,

(ii) $id\ loc\ (\mathfrak{J}) \supset Rad\ \mathfrak{J}$.

The proofs of Th. 3.3 and Th. 3.4 are simple (cf. [GR, p. 88]).

Definition 3.5. An analytic germ $\underline{V}$ is said to be irreducible if $\underline{V} = \underline{V}_1 \cup \underline{V}_2$, where $\underline{V}_1$ and $\underline{V}_2$ are analytic germs, implies $\underline{V} = \underline{V}_1$ or $\underline{V} = \underline{V}_2$.

Theorem 3.4. An analytic germ $\underline{V}$ is irreducible iff id($\underline{V}$) is prime.
Proof. cf. [GR, p. 89].

Theorem 3.5. Any analytic germ can be written as a finite union

$$\underline{V} = \bigcup_{i=1}^{k} \underline{V}_i$$

of irreducible analytic germs such that

$$\underline{V}_j \not\subset \bigcup_{i \neq j} \underline{V}_i$$

Proof. We need the following theorem:

> Theorem. Every ideal $\mathfrak{J}$ in a Noetherian ring is the intersection of finitely many primary ideals $\mathfrak{q}_i$ $(i = 1,\dots,k)$,
>
> $$\mathfrak{J} = \bigcap_{i=1}^{k} \mathfrak{q}_i$$
>
> where the $\mathfrak{P}_i$ (= Rad $\mathfrak{q}_i$) are unique.
> (For a proof, see [W2, §93]).

A decomposition of $\underline{V}$ is obtained as follows:

$$\begin{aligned} \underline{V} = \text{loc id } (\underline{V}) = \text{loc } \left(\bigcap_{i=1}^{k} \mathfrak{q}_i \right) &= \bigcup_{i=1}^{k} (\text{loc } \mathfrak{q}_i) \\ &= \bigcup_{i=1}^{k} (\text{loc } \mathfrak{P}_i) \\ &= \bigcup_{i=1}^{k} \underline{V}_i \ ; \end{aligned}$$

here the $\mathfrak{q}_i$ $(i = 1,\dots,k)$ are primary, hence the $\mathfrak{P}_i$ = Rad $\mathfrak{q}_i$ $(i = 1,\dots,k)$ are prime ideals.

Definition 3.6. The germs $\underline{V}_i$ ($i = 1,\dots,k$) introduced by the decomposition of Th. 3.5 are called the irreducible components of $\underline{V}$.

Now we come to a deep theorem.

Theorem 3.6. (Hilbert Nullstellensatz for analytic functions).
Let $\mathfrak{J}$ be any ideal of A_n. Then

$$\text{id loc } \mathfrak{J} = \text{Rad } \mathfrak{J}.$$

Proof.
The hard step is to prove the theorem for prime ideals; this requires a detailed investigation of the locus of a prime ideal (an irreducible variety); we refer to [GR, p. 93-97].

Then one easily finishes the proof as follows.
For any ideal $\mathfrak{J}$ of A_n we write:

$$\mathfrak{J} = \bigcap_{i=1}^{k} \mathfrak{q}_i ,$$

where the ideals $\mathfrak{q}_i$ are primary, and the ideals Rad $\mathfrak{q}_i = \mathcal{P}_i$ are prime.
The prime ideals $\mathcal{P}_1,\dots,\mathcal{P}_k$ are uniquely determined. Since loc $\mathfrak{q}_i$ = loc $\mathcal{P}_i$, then

$$\text{loc } \mathfrak{J} = \bigcup_{i=1}^{k} \text{loc } \mathcal{P}_i$$

and

$$\text{id loc } \mathfrak{J} = \bigcap_{i=1}^{k} \text{id loc } \mathcal{P}_i .$$

Since for prime ideals the theorem is supposed to be true, we have

$$\text{id loc } \mathfrak{J} = \bigcap_{i=1}^{k} \mathcal{P}_i = \text{Rad } \mathfrak{J}.$$

Definition 3.7. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. A point $a \in V$ is called a regular point of V of dimension p if there is a neighborhood U of a, $U \subset \Omega$, such that $V \cap U$ is a complex submanifold of dimension p of U. A point $a \in V$ is called singular if it is not regular.

Theorem 3.7. Let V be an analytic set in an open set Ω of $\mathbb{C}^n$. The set of regular points of V is dense in V.
Proof. cf. [GR, p. 111].

Definition 3.8. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. A function f on V is said to be analytic at $a \in V$ if there is a neighborhood U of a in Ω and an analytic function $\tilde{f}$ in U with

$$\tilde{f} \mid U \cap V = f \mid U \cap V.$$

Theorem 3.8. (maximum principle). Let V be an analytic set in Ω and let f be an analytic function on Ω. Let $\underline{V}_a$ be irreducible and suppose that f is not constant on V in any neighborhood of a. Then

$$f(a) \in (f(V))^{\circ}.$$

Corollary. A compact analytic set in $\mathbb{C}^n$ consists of a finite number of points.
Proof. cf. [GR, p. 104-106].

Definition 3.9. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. For $z \in \Omega$, let $\mathfrak{J}_z(V)$ the ideal $id(\underline{V}_z)$ of A_z (the ideal of A_z of all germs of analytic functions vanishing on $\underline{V}_z$) (if $z \notin V$, then $\mathfrak{J}_z(V) = A_z$). Then

$$\mathfrak{J}(V) = \bigcup_{z \in \Omega} \mathfrak{J}_z(V)$$

defines a subsheaf of $\mathcal{A}(\Omega)$ ($= \mathcal{A}_{n,\Omega}$) (note that $\mathfrak{J}(V)$ is an open subset of $\mathcal{A}(\Omega)$). $\mathfrak{J}(V)$ is called the sheaf of ideals of the analytic set V.

Theorem 3.9. If V is an analytic set in an open set $\Omega \subset \mathbb{C}^n$, then $\mathfrak{J}(V)$ is a coherent analytic sheaf on Ω
Proof. cf. [GR, p. 138-141].

From this coherence theorem one can deduce the following interesting fact.

Theorem 3.10. Let V be an analytic set in an open set $\Omega\subset\mathbb{C}^n$. Then the set of singular points of V is again an analytic set in Ω.
Corollary. Any analytic set in an open set in $\mathbb{C}^n$ can be written as an union of complex manifolds.
Proof. cf. [GR, p. 141-142].

Let V be an analytic set in an open set $\Omega\subset\mathbb{C}^n$. Let us introduce the quotientsheaf

$$\tilde{\mathcal{H}} = \mathcal{A}(\Omega)/\mathcal{J}(V)$$

(observe that $\tilde{\mathcal{H}}_z = 0$ if $z\in\Omega\setminus V$).
Let $\mathcal{H}$ be the restriction of $\tilde{\mathcal{H}}$ to V. Then $\mathcal{H}$ can be identified with the sheaf of germs of analytic functions on V.

We now state special cases of Cartan's Theorem A and B.
Theorem 3.11. Let Ω be a domain of holomorphy in C^n and let $\mathcal{F}$ be a coherent analytic sheaf on Ω. Then the following hold:
A. $\Gamma(\Omega,\mathcal{F})$ generates $\mathcal{F}_z$ for all $z\in\Omega$,
B. $H^1(\Omega,\mathcal{F}) = 0$.

We shall give two applications of this theorem.

Theorem 3.12. Let Ω be a domain of holomorphy and let V be an analytic set in Ω. Then

$$V = \{z\in\Omega \mid f_i(z) = 0 \ \forall\, i\in I\} ,$$

where the f_i are analytic in Ω and I is some index set.
(This theorem says that a local simultaneous zero set).
Proof. The sheaf $\mathcal{J}(V)$ on Ω is coherent (Th. 3.9).
Hence (Th. 3.11. A) the global sections of $\mathcal{J}(V)$ generate the stalk of $\mathcal{J}(V)$ at each point $z\in\Omega$. For $z\in\Omega\setminus V$, the stalk $\mathcal{J}_z(V)$ contains the germ 1_z. Hence there are $f_j\in\Gamma(\Omega,\mathcal{J}(V))$ and $g_j\in A_z$ $(j = 1,\dots,k)$ such that

$$1_z = \sum g_j(f_j)_z.$$

Thus some $f_j(z) \neq 0$.

<u>Theorem 3.13</u>. Let Ω be a domain of holomorphy and let V be an analytic set in Ω. Then every function analytic on V is the restriction of a function analytic on Ω.

Proof. We have the following exact sequence of sheaves:

$$0 \to \mathfrak{J}(V) \to \mathcal{A}(\Omega) \to \tilde{\mathcal{H}} \to 0 \;;$$

and so the exact cohomology sequence:

$$\ldots \to \Gamma(\Omega,\mathcal{A}) \to \Gamma(\Omega,\tilde{\mathcal{H}}) \to H^1(\Omega,\mathfrak{J}(V)) \to \ldots$$

Now $\Gamma(\Omega,\tilde{\mathcal{H}}) = \Gamma(V,\mathcal{H})$.

Furthermore (Th. 3.11.B): $H^1(\Omega,\mathfrak{J}(V)) = 0$.

So the theorem follows.

References.

[A] Ahlfors, L., Complex Analysis. McGraw-Hill, New York, 1966.

[GR] Gunning, R.C, and Rossi, H., Analytic functions of several complex variables. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965

[H] Hörmander, L., An introduction to complex analysis in several variables. D. Van Nostrand Company, Inc., Princeton, N.J., 1966.

[N] Narasimhan, R., Introduction to the theory of analytic spaces. Lecture Notes in Mathematics No. 25. Springer, Berlin etc., 1966.

[W1] Van der Waerden, B.L., Moderne Algebra I. Springer, Berlin etc., 1953.

[W2] Van der Waerden, B.L., Moderne Algebra II. Springer, Berlin etc., 1950.